

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 183

Editorial Board

B. BOLLOBÁS, W. FULTON, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

DERIVED CATEGORIES

There have been remarkably few systematic expositions of the theory of derived categories since its inception in the work of Grothendieck and Verdier in the 1960s. This book is the first in-depth treatment of this important component of homological algebra. It carefully explains the foundations in detail before moving on to key applications in commutative and noncommutative algebra, many otherwise unavailable outside of research articles. These include commutative and noncommutative dualizing complexes, perfect DG modules and tilting DG bimodules.

Written with graduate students in mind, the emphasis here is on explicit constructions (with many examples and exercises) as opposed to axiomatics, with the goal of demystifying this difficult subject. Beyond serving as a thorough introduction for students, it will serve as an important reference for researchers in algebra, geometry and mathematical physics.

Amnon Yekutieli is Professor of Mathematics at Ben-Gurion University of the Negev, Israel. His research interests are in algebraic geometry, ring theory, derived categories and deformation quantization. He has taught several graduate-level courses on derived categories. This is his fourth book.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit www.cambridge.org/mathematics.

Already Published

- 145 M. Viana *Lectures on Lyapunov Exponents*
 146 J.-H. Evertse & K. Györy *Unit Equations in Diophantine Number Theory*
 147 A. Prasad *Representation Theory*
 148 S. R. Garcia, J. Mashreghi & W. T. Ross *Introduction to Model Spaces and Their Operators*
 149 C. Godsil & K. Meagher *Erdős–Ko–Rado Theorems: Algebraic Approaches*
 150 P. Mattila *Fourier Analysis and Hausdorff Dimension*
 151 M. Viana & K. Oliveira *Foundations of Ergodic Theory*
 152 V. I. Paulsen & M. Raghupathi *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*
 153 R. Beals & R. Wong *Special Functions and Orthogonal Polynomials*
 154 V. Jurdjevic *Optimal Control and Geometry: Integrable Systems*
 155 G. Pisier *Martingales in Banach Spaces*
 156 C. T. C. Wall *Differential Topology*
 157 J. C. Robinson, J. L. Rodrigo & W. Sadowski *The Three-Dimensional NavierStokes Equations*
 158 D. Huybrechts *Lectures on K3 Surfaces*
 159 H. Matsumoto & S. Taniguchi *Stochastic Analysis*
 160 A. Borodin & G. Olshanski *Representations of the Infinite Symmetric Group*
 161 P. Webb *Finite Group Representations for the Pure Mathematician*
 162 C. J. Bishop & Y. Peres *Fractals in Probability and Analysis*
 163 A. Bovier *Gaussian Processes on Trees*
 164 P. Schneider *Galois Representations and (φ, Γ) -Modules*
 165 P. Gille & T. Szamuely *Central Simple Algebras and Galois Cohomology (2nd Edition)*
 166 D. Li & H. Queffelec *Introduction to Banach Spaces, I*
 167 D. Li & H. Queffelec *Introduction to Banach Spaces, II*
 168 J. Carlson, S. Müller-Stach & C. Peters *Period Mappings and Period Domains (2nd Edition)*
 169 J. M. Landsberg *Geometry and Complexity Theory*
 170 J. S. Milne *Algebraic Groups*
 171 J. Gough & J. Kupsch *Quantum Fields and Processes*
 172 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Discrete Harmonic Analysis*
 173 P. Garrett *Modern Analysis of Automorphic Forms by Example, I*
 174 P. Garrett *Modern Analysis of Automorphic Forms by Example, II*
 175 G. Navarro *Character Theory and the McKay Conjecture*
 176 P. Fleig, H. P. A. Gustafsson, A. Kleinschmidt & D. Persson *Eisenstein Series and Automorphic Representations*
 177 E. Peterson *Formal Geometry and Bordism Operators*
 178 A. Ogus *Lectures on Logarithmic Algebraic Geometry*
 179 N. Nikolski *Hardy Spaces*
 180 D.-C. Cisinski *Higher Categories and Homotopical Algebra*
 181 A. Agrachev, D. Barilari & U. Boscain *A Comprehensive Introduction to Sub-Riemannian Geometry*
 182 N. Nikolski *Toeplitz Matrices and Operators*
 183 A. Yekutieli *Derived Categories*
 184 C. Demeter *Fourier Restriction, Decoupling and Applications*

Derived Categories

AMNON YEKUTIELI
Ben-Gurion University of the Negev, Israel



CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India
79 Anson Road, #06—04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781108419338

DOI: 10.1017/9781108292825

© Amnon Yekutieli 2020

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2020

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-41933-8 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Dedicated to Alexander Grothendieck, in memoriam

Cambridge University Press
978-1-108-41933-8 — Derived Categories
Amnon Yekutieli
Frontmatter
[More Information](#)

Contents

0	Introduction	1
0.1	On the Subject	1
0.2	A Motivating Discussion: Duality	4
0.3	On the Book	9
0.4	Synopsis of the Book	11
0.5	What Is Not in the Book	22
0.6	Prerequisites and Recommended Bibliography	23
0.7	Credo, Writing Style and Goals	24
0.8	Acknowledgments	25
1	Basic Facts on Categories	26
1.1	Set Theory	26
1.2	Notation and Conventions	27
1.3	Epimorphisms and Monomorphisms	28
1.4	Products and Coproducts	30
1.5	Equivalence of Categories	31
1.6	Bifunctors	32
1.7	Representable Functors	32
1.8	Inverse and Direct Limits	34
2	Abelian Categories and Additive Functors	37
2.1	Linear Categories	37
2.2	Additive Categories	38
2.3	Abelian Categories	40
2.4	A Method for Producing Proofs in Abelian Categories	44
2.5	Additive Functors	49
2.6	Projective Objects	55
2.7	Injective Objects	57

3	Differential Graded Algebra	62
3.1	Graded Algebra	62
3.2	DG \mathbf{K} -Modules	72
3.3	DG Rings and Modules	74
3.4	DG Categories	78
3.5	DG Functors	80
3.6	Complexes in Abelian Categories	82
3.7	The Long Exact Cohomology Sequence	84
3.8	The DG Category $\mathbf{C}(A, M)$	90
3.9	Contravariant DG Functors	94
4	Translations and Standard Triangles	101
4.1	The Translation Functor	101
4.2	The Standard Triangle of a Strict Morphism	105
4.3	The Gauge of a Graded Functor	107
4.4	The Translation Isomorphism of a DG Functor	108
4.5	Standard Triangles and DG Functors	109
4.6	Examples of DG Functors	113
5	Triangulated Categories and Functors	117
5.1	Triangulated Categories	117
5.2	Triangulated and Cohomological Functors	122
5.3	Some Properties of Triangulated Categories	125
5.4	The Homotopy Category Is Triangulated	132
5.5	From DG Functors to Triangulated Functors	141
5.6	The Opposite Homotopy Category Is Triangulated	143
6	Localization of Categories	146
6.1	The Formalism of Localization	146
6.2	Ore Localization	148
6.3	Localization of Linear Categories	162
7	The Derived Category $\mathbf{D}(A, M)$	165
7.1	Localization of Triangulated Categories	165
7.2	Definition of the Derived Category	172
7.3	Boundedness Conditions in $\mathbf{K}(A, M)$	176
7.4	Thick Subcategories of \mathbf{M}	180
7.5	The Embedding of \mathbf{M} in $\mathbf{D}(\mathbf{M})$	182
7.6	The Opposite Derived Category Is Triangulated	183

8	Derived Functors	186
8.1	2-Categorical Notation	186
8.2	Functor Categories	189
8.3	Abstract Derived Functors	192
8.4	Triangulated Derived Functors	203
8.5	Contravariant Triangulated Derived Functors	211
9	DG and Triangulated Bifunctors	216
9.1	DG Bifunctors	216
9.2	Triangulated Bifunctors	220
9.3	Derived Bifunctors	224
10	Resolving Subcategories of $\mathbf{K}(A, M)$	230
10.1	K-Injective DG Modules	230
10.2	K-Projective DG Modules	236
10.3	K-Flat DG Modules	238
10.4	Opposite Resolving Subcategories	240
11	Existence of Resolutions	242
11.1	Direct and Inverse Limits of Complexes	242
11.2	Totalizations	246
11.3	K-Projective Resolutions in $\mathbf{C}^-(M)$	254
11.4	K-Projective Resolutions in $\mathbf{C}(A)$	260
11.5	K-Injective Resolutions in $\mathbf{C}^+(M)$	272
11.6	K-Injective Resolutions in $\mathbf{C}(A)$	280
12	Adjunctions, Equivalences and Cohomological Dimension	289
12.1	Boundedness Conditions Revisited	289
12.2	The Bifunctor \mathbf{RHom}	292
12.3	The Bifunctor \otimes^L	295
12.4	Cohomological Dimensions of Functors and Objects	298
12.5	Theorems on Functors Satisfying Finiteness Conditions	303
12.6	Derived Restriction, Induction and Coinduction Functors	308
12.7	DG Ring Quasi-Isomorphisms	313
12.8	Existence of DG Ring Resolutions	315
12.9	The Derived Tensor-Evaluation Morphism	322
12.10	Hom-Tensor Formulas for Weakly Commutative DG Rings	326

13 Dualizing Complexes over Commutative Rings	330
13.1 Dualizing Complexes	330
13.2 Interlude: The Matlis Classification of Injective Modules	344
13.3 Residue Complexes	347
13.4 Van den Bergh Rigidity	357
13.5 Rigid Dualizing and Residue Complexes	366
14 Perfect and Tilting DG Modules over NC DG Rings	373
14.1 Algebraically Perfect DG Modules	373
14.2 Derived Morita Theory	387
14.3 DG Bimodules over K-Flat DG Rings	397
14.4 Tilting DG Bimodules	404
14.5 Tilting Bimodule Complexes over Rings	413
15 Algebraically Graded Noncommutative Rings	424
15.1 Categories of Algebraically Graded Modules	424
15.2 Properties of Algebraically Graded Modules	435
15.3 Resolutions and Derived Functors	444
15.4 Artin–Schelter Regular Graded Rings	455
16 Derived Torsion over NC Graded Rings	462
16.1 Quasi-Compact Finite Dimensional Functors	463
16.2 Weakly Stable and Idempotent Copointed Functors	468
16.3 Graded Torsion: Weak Stability and Idempotence	473
16.4 Representability of Derived Torsion	484
16.5 Symmetry of Derived Torsion	489
16.6 NC MGM Equivalence	498
17 Balanced Dualizing Complexes over NC Graded Rings	508
17.1 Graded NC Dualizing Complexes	509
17.2 Balanced DC: Definition, Uniqueness and Local Duality	514
17.3 Balanced DC: Existence	518
17.4 Balanced Trace Morphisms	531
18 Rigid Noncommutative Dualizing Complexes	542
18.1 Noncommutative Dualizing Complexes	542
18.2 Rigid NC DC: Definition and Uniqueness	552
18.3 Interlude: Graded Rings of Laurent Type	561

	<i>Contents</i>	xi
18.4	Graded Rigid NC DC	567
18.5	Filtered Rings and Existence of Rigid NC DC	575
18.6	Twisted Calabi–Yau Rings	581
	<i>References</i>	590
	<i>Index</i>	600

Cambridge University Press
978-1-108-41933-8 — Derived Categories
Amnon Yekutieli
Frontmatter
[More Information](#)
