0

Introduction

0.1 On the Subject

Derived categories were introduced by A. Grothendieck and J.-L. Verdier around 1960 and were first published in the book [62] by R. Hartshorne. The basic idea was as follows. They had realized that the derived functors of classical homological algebra, namely the functors $\mathrm{R}^q F, \mathrm{L}_q F : \mathsf{M} \to \mathsf{N}$ derived from an additive functor $F : \mathsf{M} \to \mathsf{N}$ between abelian categories, are too limited to allow several rather natural manipulations. Perhaps the most important operation that was lacking was the composition of derived functors; the best approximation of it was a spectral sequence.

The solution to the problem was to invent a new category, starting from a given abelian category M. The objects of this new category are the complexes of objects of M. These are the same complexes that play an auxiliary role in classical homological algebra, as resolutions of objects of M. The complexes form a category $\mathbf{C}(\mathsf{M})$, but this category is not sufficiently intricate to carry in it the information of derived functors. So it must be modified.

A morphism $\phi : M \to N$ in $\mathbf{C}(\mathsf{M})$ is called a *quasi-isomorphism* if in each degree q the cohomology morphism $\mathrm{H}^q(\phi) : \mathrm{H}^q(M) \to \mathrm{H}^q(N)$ in M is an isomorphism. The modification that is needed is to make the quasi-isomorphisms invertible. This is done by a formal localization procedure, and the resulting category (with the same objects as $\mathbf{C}(\mathsf{M})$) is the derived category $\mathbf{D}(\mathsf{M})$. There is a functor $\mathrm{Q} : \mathbf{C}(\mathsf{M}) \to \mathbf{D}(\mathsf{M})$, which is the identity on objects, and it has a universal property (it is initial among the functors that send the quasi-isomorphisms to isomorphisms). A theorem (analogous to Ore localization in noncommutative ring theory) says that every morphism θ in $\mathbf{D}(\mathsf{M})$ can be written as a simple left or right fraction:

$$\theta = \mathrm{Q}(\psi_0)^{-1} \circ \mathrm{Q}(\phi_0) = \mathrm{Q}(\phi_1) \circ \mathrm{Q}(\psi_1)^{-1}, \tag{0.1.1}$$

where ϕ_i and ψ_i are morphisms in $\mathbf{C}(\mathsf{M})$ and ψ_i are quasi-isomorphisms.

The cohomology functors $\mathrm{H}^q : \mathbf{D}(\mathsf{M}) \to \mathsf{M}$, for all $q \in \mathbb{Z}$, are still defined. It turns out that the functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})$, which sends an object M to the complex M concentrated in degree 0, is fully faithful.

The next step is to say what is a left or a right derived functor of an additive functor $F : \mathsf{M} \to \mathsf{N}$. The functor F can be extended in an obvious manner to a functor on complexes $F : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{N})$. A *right derived functor* of F is a functor

$$\mathrm{R}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{N}), \tag{0.1.2}$$

together with a morphism of functors $\eta^{\mathrm{R}} : \mathrm{Q}_{\mathsf{N}} \circ F \to \mathrm{R}F \circ \mathrm{Q}_{\mathsf{M}}$. The pair $(\mathrm{R}F, \eta^{\mathrm{R}})$ has to be *initial among all such pairs*. The uniqueness of such a functor $\mathrm{R}F$, up to a unique isomorphism, is relatively easy to prove (using the language of 2-categories). As for existence of $\mathrm{R}F$, it relies on the existence of suitable resolutions (similar to the injective resolutions in the classical situation). If these resolutions exist, and if the original functor F is left exact, then there is a canonical isomorphism of functors

$$\mathrm{R}^q F \cong \mathrm{H}^q \circ \mathrm{R}F : \mathsf{M} \to \mathsf{N} \tag{0.1.3}$$

for every $q \geq 0$.

The left derived functor

$$\mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{N}) \tag{0.1.4}$$

is defined similarly. When suitable resolutions exist, and when F is right exact, there is a canonical isomorphism of functors

$$\mathrm{L}_q F \cong \mathrm{H}^{-q} \circ \mathrm{L}F : \mathsf{M} \to \mathsf{N} \tag{0.1.5}$$

for every $q \geq 0$.

There are several variations: F could be a contravariant additive functor, or it could be an additive bifunctor, contravariant in one or two of its arguments. In all these situations the derived (bi)functors $\mathrm{R}F$ and $\mathrm{L}F$ can be defined.

The derived category $\mathbf{D}(\mathsf{M})$ is additive, but it is not abelian. The notion of short exact sequence (in M and in $\mathbf{C}(\mathsf{M})$) is replaced by that of *distinguished triangle*, and thus $\mathbf{D}(\mathsf{M})$ is a *triangulated category*. The derived functors $\mathrm{R}F$ and $\mathrm{L}F$ are *triangulated functors*, which means that they send distinguished triangles in $\mathbf{D}(\mathsf{M})$ to distinguished triangles in $\mathbf{D}(\mathsf{N})$.

Already in classical homological algebra, we are interested in the *bifunctors* $\mathrm{Hom}(-,-)$ and $(- \otimes -)$. These bifunctors can also be derived. To simplify matters, let's assume that A is a commutative ring and $\mathsf{M} = \mathsf{N} = \mathsf{Mod}\, A$, the category of A-modules. We then have bifunctors

$$\mathrm{Hom}_A(-,-) : (\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

and

$$(- \otimes_A -) : \mathsf{Mod}\, A \times \mathsf{Mod}\, A \to \mathsf{Mod}\, A,$$

where the superscript "op" denotes the opposite category, which encodes the contravariance in the first argument of Hom. In this situation, all resolutions exist, and we have the right derived bifunctor

$$\mathrm{RHom}_A(-,-) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathbf{D}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, A) \tag{0.1.6}$$

and the left derived bifunctor

$$(- \otimes^{\mathrm{L}}_A -) : \mathbf{D}(\mathsf{Mod}\, A) \times \mathbf{D}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, A)\,. \tag{0.1.7}$$

The compatibility with the classical derived bifunctors is this: there are canonical isomorphisms

$$\mathrm{Ext}^q_A(M,N) \cong \mathrm{H}^q(\mathrm{RHom}_A(M,N)) \tag{0.1.8}$$

and

$$\mathrm{Tor}^A_q(M,N) \cong \mathrm{H}^{-q}(M \otimes^{\mathrm{L}}_A N) \tag{0.1.9}$$

for all $M, N \in \mathsf{Mod}\, A$ and $q \geq 0$.

This is what derived categories and derived functors are. As to what can be done with them, here are some of the things we will explore in this book:

- *Dualizing complexes* and *residue complexes* over noetherian commutative rings. Besides the original treatment from [62], which we present in detail here, we also include *Van den Bergh rigidity* in the commutative setting, which gives rise to *rigid residue complexes*.
- *Perfect DG modules* and *tilting DG bimodules* over noncommutative DG rings, and a few variants of *derived Morita Theory*, including the *Rickard–Keller Theorem*.
- *Derived torsion* and *balanced dualizing complexes* over connected graded NC rings, and *rigid dualizing complexes* over NC rings, including a full proof of the *Van den Bergh Existence Theorem* for NC rigid dualizing complexes.

A topic that is beyond the scope of this book, but of which we provide an outline here, is

- The *rigid approach to Grothendieck Duality* on noetherian schemes and Deligne–Mumford stacks.

Derived categories have important roles in several areas of mathematics; below is a partial list. We will not be able to talk about any of these topics in this book, so instead we give some references alongside each topic.

- $\mathcal{D}$-modules, perverse sheaves and representations of algebraic groups and Lie algebras. See [16] and [27]. More recently, the focus in this area is on the *Geometric Langlands Correspondence*, which can only be stated in terms of derived categories (see the survey [50]).
- Algebraic analysis, including differential, microdifferential and DQ modules (see [74], [134], [77]) and microlocal sheaf theory (see [75]), with its application to symplectic topology (see [149], [110]).
- Representations of finite groups and quivers, including *cluster algebras* and the *Broué Conjecture*. See [59], [84], [43].
- Birational algebraic geometry. This includes *Fourier–Mukai transforms* and *semi-orthogonal decompositions*. See the surveys [64] and [103], and the book [70].
- Homological mirror symmetry. It relates the derived category of coherent sheaves on a complex algebraic variety X to the *derived Fukaya category* of the mirror partner Y, which is a symplectic manifold. See Remark 3.8.22 and the online reference [88].
- Derived algebraic geometry. Here not only is the category of sheaves derived but also the underlying geometric objects (schemes or stacks). See Example 6.2.35, Remark 6.2.38 and the references [99] and [152].

0.2 A Motivating Discussion: Duality

Let us now approach derived categories from another perspective, very different from the one taken in the previous section, by considering the idea of *duality in algebra*.

We begin with something elementary: linear algebra. Take a field $\mathbb{K}$. Given a $\mathbb{K}$-module M (i.e. a vector space), let $D(M) := \operatorname{Hom}_{\mathbb{K}}(M, \mathbb{K})$, the dual module. There is a canonical homomorphism

$$\operatorname{ev}_M : M \to D(D(M)), \tag{0.2.1}$$

called *Hom-evaluation*, whose formula is $\operatorname{ev}_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated, then ev_M is an isomorphism (actually this is “if and only if”).

To formalize this situation, let $\mathsf{Mod}\,\mathbb{K}$ denote the category of $\mathbb{K}$-modules. Then $D : \mathsf{Mod}\,\mathbb{K} \to \mathsf{Mod}\,\mathbb{K}$ is a contravariant functor, and $\operatorname{ev} : \operatorname{Id} \to D \circ D$ is

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a morphism of functors (i.e. a natural transformation). Here Id is the identity functor of $\mathsf{Mod}\,\mathbb{K}$.

Now let us replace $\mathbb{K}$ by some nonzero commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\,A \to \mathsf{Mod}\,A, \quad D(M) := \mathrm{Hom}_A(M, A), \tag{0.2.2}$$

and a morphism of functors $\mathrm{ev} : \mathrm{Id} \to D \circ D$. It is easy to see that $\mathrm{ev}_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free A-module. Of course, we can't expect reflexivity (i.e. ev_M being an isomorphism) if M is not finitely generated, but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since $\mathbb{Z}$-modules are just abelian groups, the category $\mathsf{Mod}\,\mathbb{Z}$ is often denoted by Ab. Let Ab_{f} be the full subcategory of finitely generated abelian groups. Every finitely generated abelian group is of the form $M \cong T \oplus F$, with T finite and F free. (The letters "T" and "F" stand for "torsion" and "free," respectively.) It is important to note that this is *not a canonical isomorphism*. There is a canonical short exact sequence

$$0 \to T \xrightarrow{\phi} M \xrightarrow{\psi} F \to 0 \tag{0.2.3}$$

in Ab_{f}, but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \to M$ of this sequence.

Exercise 0.2.4 Prove that the exact sequence (0.2.3) is functorial, namely that there are functors $T, F : \mathsf{Ab}_{\mathrm{f}} \to \mathsf{Ab}_{\mathrm{f}}$, and natural transformations $\phi : T \to \mathrm{Id}$ and $\psi : \mathrm{Id} \to F$, such that for each $M \in \mathsf{Ab}_{\mathrm{f}}$ the group $T(M)$ is finite, the group $F(M)$ is free and the sequence of homomorphisms

$$0 \to T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \to 0 \tag{0.2.5}$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \to \mathrm{Id}$, such that for every M the homomorphism $\sigma_M : F(M) \to M$ splits the sequence (0.2.5). (Hint: find a counterexample.)

We know that for a free finitely generated abelian group F, there is reflexivity, i.e. $\mathrm{ev}_F : F \to D(D(F))$ is an isomorphism. But for a finite abelian group T we have $D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0$. Thus, for a group $M \in \mathsf{Ab}_{\mathrm{f}}$ with a nonzero torsion subgroup T, reflexivity fails: $\mathrm{ev}_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual: $D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. There is a morphism of functors $\mathrm{ev}' : \mathrm{Id} \to$

$D' \circ D'$. For a finite abelian group T the homomorphism $\mathrm{ev}'_T : T \to D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential function; then D' becomes *Pontryagin Duality*.)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F. So once more, this is not a good duality for all finitely generated abelian groups.

This is where the *derived category* enters. For every commutative ring A, there is the derived category $\mathsf{D}(\mathsf{Mod}\, A)$. Here is a very quick explanation of it, in concrete terms – as opposed to the abstract point of view taken in the previous section.

Recall that a *complex* of A-modules is a diagram

$$M = (\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^0} M^1 \to \cdots) \tag{0.2.6}$$

in the category $\mathsf{Mod}\, A$. Namely the M^i are A-modules, and the d_M^i are homomorphisms. The condition is that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. We sometimes write $M = \{M^i\}_{i \in \mathbb{Z}}$. The collection $\mathrm{d}_M = \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}$ is called the *differential* of M.

Given a second complex

$$N = (\cdots \to N^{-1} \xrightarrow{\mathrm{d}_N^{-1}} N^0 \xrightarrow{\mathrm{d}_N^0} N^1 \to \cdots),$$

a *homomorphism of complexes* $\phi : M \to N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of homomorphisms $\phi^i : M^i \to N^i$ in $\mathsf{Mod}\, A$ satisfying $\phi^{i+1} \circ \mathrm{d}_M^i = \mathrm{d}_N^i \circ \phi^i$. The resulting category is denoted by $\mathsf{C}(\mathsf{Mod}\, A)$.

The ith *cohomology* of the complex M is

$$\mathrm{H}^i(M) := \mathrm{Ker}(\mathrm{d}_M^i) \,/\, \mathrm{Im}(\mathrm{d}_M^{i-1}) \in \mathsf{Mod}\, A. \tag{0.2.7}$$

A homomorphism $\phi : M \to N$ in $\mathsf{C}(\mathsf{Mod}\, A)$ induces homomorphisms $\mathrm{H}^i(\phi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$ in $\mathsf{Mod}\, A$. We call ϕ a *quasi-isomorphism* if all the homomorphisms $\mathrm{H}^i(\phi)$ are isomorphisms.

The derived category $\mathsf{D}(\mathsf{Mod}\, A)$ is the localization of $\mathsf{C}(\mathsf{Mod}\, A)$ with respect to the quasi-isomorphisms. This means that $\mathsf{D}(\mathsf{Mod}\, A)$ has the same objects as $\mathsf{C}(\mathsf{Mod}\, A)$. There is a functor

$$\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A), \tag{0.2.8}$$

which is the identity on objects it sends quasi-isomorphisms to isomorphisms, and it is universal for this property.

A single A-module M^0 can be viewed as a complex M concentrated in degree 0:

$$M = (\cdots \to 0 \xrightarrow{0} M^0 \xrightarrow{0} 0 \to \cdots).$$

This turns out to be a fully faithful embedding

$$\mathsf{Mod}\, A \to \mathbf{D}(\mathsf{Mod}\, A). \tag{0.2.9}$$

The essential image of this embedding is the full subcategory of $\mathbf{D}(\mathsf{Mod}\, A)$ on the complexes M whose cohomology is concentrated in degree 0. In this way we have *enlarged* the category of A-modules. All this is explained in Chapters 6 and 7 of the book.

Here is a very important kind of quasi-isomorphism. Suppose M is an A-module and

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0 \tag{0.2.10}$$

is a projective resolution of it. We can view M as a complex concentrated in degree 0, by the embedding (0.2.9). Define the complex

$$P := (\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots), \tag{0.2.11}$$

concentrated in nonpositive degrees. Then ρ becomes a morphism of complexes $\rho : P \to M$. The exactness of the sequence (0.2.10) says that ρ is actually a quasi-isomorphism. Thus $\mathrm{Q}(\rho) : P \to M$ is an isomorphism in $\mathbf{D}(\mathsf{Mod}\, A)$.

Let us fix a complex $R \in \mathbf{C}(\mathsf{Mod}\, A)$. For every complex $M \in \mathbf{C}(\mathsf{Mod}\, A)$ we can form the complex

$$D(M) := \mathrm{Hom}_A(M, R) \in \mathbf{C}(\mathsf{Mod}\, A).$$

This is the usual Hom complex (we recall it in Section 3.6). As M changes, we get a contravariant functor

$$D : \mathbf{C}(\mathsf{Mod}\, A) \to \mathbf{C}(\mathsf{Mod}\, A).$$

The functor D has a *contravariant right derived functor*

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, A). \tag{0.2.12}$$

If P is a bounded above complex of projective modules (as in formula (0.2.11)), or more generally a *K-projective complex* (see Section 10.2), then there is a canonical isomorphism

$$\mathrm{R}D(P) \cong D(P) = \mathrm{Hom}_A(P, R). \tag{0.2.13}$$

Every complex M admits a K-projective resolution $\rho : P \to M$, and this allows us to calculate $\mathrm{R}D(M)$. Indeed, because the morphism $\mathrm{Q}(\rho) : P \to M$ is an isomorphism in $\mathbf{D}(\mathsf{Mod}\, A)$, it follows that $\mathrm{R}D(\mathrm{Q}(\rho)) : \mathrm{R}D(M) \to \mathrm{R}D(P)$ is an isomorphism in $\mathbf{D}(\mathsf{Mod}\, A)$. And the complex $\mathrm{R}D(P)$ is known by the canonical isomorphism (0.2.13). All this is explained in Chapters 8, 10 and 11 of the book.

It turns out that there is a canonical morphism

$$\mathrm{ev}^{\mathrm{R}} : \mathrm{Id} \to \mathrm{R}D \circ \mathrm{R}D \tag{0.2.14}$$

of functors from $\mathsf{D}(\mathsf{Mod}\, A)$ to itself, called *derived Hom-evaluation*. See Section 13.1.

Let us now return to the ring $A = \mathbb{Z}$ and the complex $R = \mathbb{Z}$. So the functor D is the same one we had in (0.2.2). Given a finitely generated abelian group M, we want to calculate the complexes $\mathrm{R}D(M)$ and $\mathrm{R}D(\mathrm{R}D(M))$ and the morphism

$$\mathrm{ev}^{\mathrm{R}}_M : M \to \mathrm{R}D(\mathrm{R}D(M)) \tag{0.2.15}$$

in $\mathsf{D}(\mathsf{Mod}\, A)$. As explained above, for this we choose a projective resolution $\rho : P \to M$, and then we calculate the complexes $\mathrm{R}D(P)$ and $\mathrm{R}D(\mathrm{R}D(P))$ and the morphism $\mathrm{ev}^{\mathrm{R}}_P$. For convenience we choose a projective resolution P of the shape

$$\begin{aligned} P &= (\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots) \\ &= (\cdots \to 0 \to \mathbb{Z}^{r_1} \xrightarrow{\boldsymbol{a}\cdot(-)} \mathbb{Z}^{r_0} \to 0 \cdots), \end{aligned}$$

where $r_0, r_1 \in \mathbb{N}$ and $\boldsymbol{a}$ is a matrix of integers. The complex $\mathrm{R}D(P)$ is this:

$$\mathrm{R}D(P) \cong D(P) = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) = (\cdots \to 0 \to \mathbb{Z}^{r_0} \xrightarrow{\boldsymbol{a}^{\mathrm{t}}\cdot(-)} \mathbb{Z}^{r_1} \to 0 \cdots),$$

a complex of free modules concentrated in degrees 0 and 1, with the transpose matrix $\boldsymbol{a}^{\mathrm{t}}$ as its differential. (We are deliberately ignoring signs here; the correct signs are shown in formulas (3.6.4) and (13.1.15).)

Because $\mathrm{R}D(P) \cong D(P)$ is itself a bounded complex of free modules, its derived dual is

$$\mathrm{R}D(\mathrm{R}D(P)) \cong D(D(P)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}), \mathbb{Z}). \tag{0.2.16}$$

Under the isomorphism (0.2.16), the derived Hom-evaluation morphism $\mathrm{ev}^{\mathrm{R}}_P$ in this case is just the naive Hom-evaluation homomorphism $\mathrm{ev}_P : P \to D(D(P))$ in $\mathsf{C}(\mathsf{Mod}\, \mathbb{Z})$ from (0.2.1); see Exercise 13.1.17. Because P^0 and P^{-1} are finite rank free modules, it follows that ev_P is an isomorphism in $\mathsf{C}(\mathsf{Mod}\, \mathbb{Z})$. Therefore the morphism $\mathrm{ev}^{\mathrm{R}}_M$ in $\mathsf{D}(\mathsf{Mod}\, \mathbb{Z})$ is an isomorphism. We see that $\mathrm{R}D$ *is a duality that holds for all finitely generated* $\mathbb{Z}$*-modules* M!

Actually, much more is true. Let us denote by $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{Z})$ the full subcategory of $\mathsf{D}(\mathsf{Mod}\, \mathbb{Z})$ on the complexes M such that $\mathrm{H}^i(M)$ is finitely generated for all i. Then, according to Theorem 13.1.18, $\mathrm{ev}^{\mathrm{R}}_M$ is an isomorphism for every $M \in \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{Z})$. It follows that

$$\mathrm{R}D : \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{Z}) \to \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{Z}) \tag{0.2.17}$$

is a duality (a contravariant equivalence). This is the celebrated *Grothendieck Duality*.

Here is the connection between the derived duality $\mathrm{R}D$ and the classical dualities D and D'. Take a finitely generated abelian group M, with short exact sequence (0.2.3). There are canonical isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) = D(F)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(T, \mathbb{Z}) \cong D'(T).$$

The cohomologies $\mathrm{H}^i(\mathrm{R}D(M))$ vanish for $i \neq 0, 1$. We see that if M is neither free nor finite, then $\mathrm{H}^0(\mathrm{R}D(M))$ and $\mathrm{H}^1(\mathrm{R}D(M))$ are both nonzero, so that the complex $\mathrm{R}D(M)$ is not isomorphic in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$ to an object of $\mathsf{Mod}\,\mathbb{Z}$, under the embedding (0.2.9).

Grothendieck Duality holds for many noetherian commutative rings A. A sufficient condition is that A is a finitely generated ring over a regular noetherian ring $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{Z}$ or a field). A complex $R \in \mathsf{D}(\mathsf{Mod}\,A)$ for which the contravariant functor

$$\mathrm{R}D = \mathrm{RHom}_A(-, R) : \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A) \to \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A) \tag{0.2.18}$$

is an equivalence is called a *dualizing complex*. (This is not quite accurate – see Definition 13.1.9 for the precise technical conditions on R.) A dualizing complex R over A is unique (up to a degree translation and tensoring with an invertible module). See Theorems 13.1.18, 13.1.34 and 13.1.35.

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the affine scheme $\mathrm{Spec}(A)$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is a Cohen–Macaulay ring (and $\mathrm{Spec}(A)$ is connected), then R is a single A-module (up to a shift in degrees). But if A is a more singular ring, then R must live in several degrees, as the next example shows.

Example 0.2.19 Consider the affine algebraic variety $X \subseteq \mathbf{A}^3_{\mathbb{R}}$, which is the union of a plane and a line that meet at a point, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\ t_3 \cdot t_2).$$

See Figure 1. A dualizing complex R over A must live in two adjacent degrees; namely there is some i such that both $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero. This calculation is worked out in full in Example 13.3.12.

One can also talk about dualizing complexes over *noncommutative rings*. We will do this in Chapters 17 and 18.

0.3 On the Book

This book develops the theory of derived categories, starting from the foundations, and going all the way to applications in commutative and noncommutative

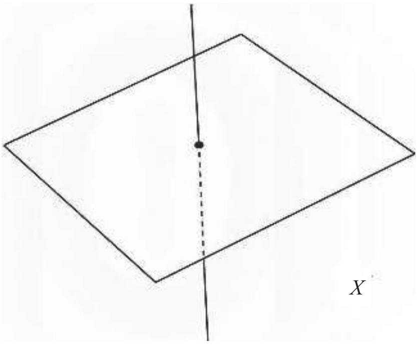

Figure 1. An algebraic variety X that is connected but not equidimensional, and hence not Cohen–Macaulay.

algebra. The emphasis is on explicit constructions (with examples), as opposed to axiomatics. The most abstract concept we use is probably that of an abelian category (which seems indispensable).

A special feature of this book is that most of the theory deals with the category $\mathbf{C}(A, \mathsf{M})$ of *DG A-modules in* M, where A is a DG ring and M is an abelian category. Here "DG" is short for "differential graded," and our DG rings are more commonly known as unital associative DG algebras. See Sections 3.3 and 3.8 for the definitions. The notion $\mathbf{C}(A, \mathsf{M})$ covers most important examples that arise in algebra and geometry:

- The category $\mathbf{C}(A)$ of DG A-modules, for any DG ring A. This includes unbounded complexes of modules over an ordinary ring A.
- The category $\mathbf{C}(\mathsf{M})$ of unbounded complexes in any abelian category M. This includes $\mathsf{M} = \mathsf{Mod}\,\mathcal{A}$, the category of sheaves of $\mathcal{A}$-modules on a ringed space $(X, \mathcal{A})$.

The category $\mathbf{C}(A, \mathsf{M})$ is a *DG category*, and its DG structure determines the *homotopy category* $\mathbf{K}(A, \mathsf{M})$ with its *triangulated structure*. We prove that every *DG functor* $F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$ induces a *triangulated functor*

$$F : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N}) \tag{0.3.1}$$

between the homotopy categories.

We can now reveal that in the previous sections we were a bit imprecise (for the sake of simplifying the exposition): what we referred to there as $\mathbf{C}(\mathsf{M})$ was actually the *strict subcategory* $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, whose morphisms are the degree 0 cocycles in the DG category $\mathbf{C}(\mathsf{M})$. For the same reason the homotopy category $\mathbf{K}(\mathsf{M})$ was suppressed there.