How to Integrate It

A Practical Guide to Finding Elementary Integrals

While differentiating elementary functions is merely a skill, finding their integrals is an *art*. This practical introduction to the art of integration gives readers the tools and confidence to tackle common and uncommon integrals.

After a review of the basic properties of the Riemann integral, each chapter is devoted to a particular technique of elementary integration. Thorough explanations and plentiful worked examples prepare the reader for the extensive exercises at the end of each chapter. These exercises increase in difficulty from warm-up problems, through drill examples, to challenging extensions that illustrate such advanced topics as the irrationality of π and e, the solution of the Basel problem, Leibniz's series, and Wallis's product.

The author's accessible and engaging manner will appeal to a wide audience, including students, teachers, and self-learners. It can serve as a complete introduction to finding elementary integrals, or as a supplementary text for any beginning course in calculus.

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Preface

Calculus occupies an important place in modern mathematics. At its heart it is the study of continuous change. It forms the foundation of mathematical analysis while the immense wealth of its ideas and usefulness of the tools to have emerged from its development make it capable of handling a wide variety of problems both within and outside of mathematics. Indeed, the sheer number of applications that the calculus finds means it continues to remain a central component for any serious study of mathematics for future mathematicians, scientists, and engineers alike.

The material presented in this volume deals with one of the major branches of calculus known as the *integral* calculus – the other being the *differential*, with the two being intimately bound. The integral calculus deals with the notion of an integral, its properties, and method of calculation. Our word for 'integrate' is derived from the Latin *integratus* meaning 'to make whole'. As calculus deals with continuous change, integration, then, is a general method for finding the whole change when you know all the intermediate (infinitesimal) changes.

A precursor to the concept of an integral dates back to the ancient Greeks, to Eudoxus in the fourth century BCE and Archimedes in the third century BCE, and their work related to the method of exhaustion. The method of exhaustion was used to calculate areas of plane figures and volumes of solids based on approximating the object under consideration by exhaustively partitioning it into ever smaller pieces using the simplest possible planar figures or bodies, such as rectangles or cylinders. Summing its constituent parts together then gave the area or volume of the whole. Integration thus renders something whole by bringing together all its parts. Its modern development came much later. Starting in the late seventeenth century with the seminal work of Newton and Leibniz, it was carried forward in the eighteenth century by Euler and the Bernoulli brothers, Jacob and Johann, and in the nineteenth century most notably by Cauchy before the first rigorous treatment of the integral was given by Riemann during the middle part of that century. Since this time many other notions for the integral have emerged. In this text we focus exclusively on the first and perhaps simplest of these notions to emerge, that of the *Riemann* integral.

Unlike differentiation, which once learnt is merely a skill where a set of rules are applied, as there is no systematic procedure for finding an integral, even for functions that behave ‘nicely’, many look upon integration as an ‘art’. Finding an integral tends to be a complicated affair involving a search for patterns and is hard to do. The unavailability of a mechanical approach to integration means many different techniques for finding integrals of well-behaved functions in terms of familiar functions have been developed. This makes integration hard and is exactly how it is perceived by most beginning students. When encountering integration for the first time one is often bewildered by the number of different methods that need to be known before the problem of integration can be successfully tackled.

This book is an attempt at taking some of the mystery out of the art of integration. The text provides a self-contained presentation of the properties of the definite integral together with many of the familiar, and some not so familiar, techniques that are available for finding elementary integrals. Prerequisites needed for the proper study of the material presented in the text are minimal. The reader is expected to be familiar with the differential calculus including the concept of a limit, continuity, and differentiability, together with a working knowledge of the rules of differentiation.

The book takes the reader through the various elementary methods that can be used to find (Riemann) integrals together with introducing and developing the various properties associated with the definite integral. The focus is primarily on ideas and techniques used for integrating functions and on the properties associated with the definite integral rather than on applications. By doing so the aim is to develop in the student the skills and confidence needed to approach the general problem of how to find an integral in terms of familiar functions. Once these have been developed and thoroughly mastered, the student should be in a far better position to move onto the multitude of applications the integral finds for itself. Of course this is not to say applications for the integral are not to be found in the text. They are. While the text makes no attempt to use integration to calculate areas or volumes in any schematic way, applications developed through the process of integration that lead to important results in other areas of mathematics are given. As examples, the proof of the irrationality of the numbers π and e is presented, as are the solutions to the Basel problem, Leibniz’s series, and Wallis’s product.

Most chapters of the book are quite short and succinct. Each chapter is self-contained and is structured such that after the necessary theory is introduced and developed, a range of examples of increasing level of difficulty are presented showing how the technique is used and works in practice. Along the way, various strategies and sound advice are given. At the conclusion of each chapter, an extensive set of exercises appear. The text can serve as a complete introduction and guide to finding elementary integrals. Alternatively, it can serve as a resource or supplemental text for any beginning course in calculus by allowing students to focus on particular problem areas they might be having by working through one or more relevant chapters at a time.

In contemplating the material presented it cannot be overstated how important it is for one to attempt the exercises located at the end of each chapter. For those hoping to become fluent in the art of integration, this proficiency is best gained through perseverance and hard work, and in the practice of answering as many different and varied questions as possible. Indeed, a large portion of the text is devoted to such exercises and problems; they are a very important component of the book. To help aid students in their endeavours an attempt has been made to divide the exercises that appear at the end of the chapters into three types: (i) warm-ups, (ii) practice questions, and (iii) extension questions and challenge problems. The warm-ups are relatively simple questions designed to gently ease the student into the material just considered. The practice questions consolidate knowledge of the material just presented, allowing the student to gain familiarity and confidence in the workings of the technique under consideration. Finally, the extension questions and challenge problems contain a mix of questions that are either simply challenging in nature or that extend, in often quite unexpected ways, the material just considered. It is hoped many of the questions found in this last group will not only challenge the reader but pique their interest as more advanced results are gradually revealed. Of course, judging the perceived level of difficulty is often in the eye of the beholder so one may expect some overlap between the various categories. For problems considered more difficult, hints are provided along the way in the form of interrelated parts that it is hoped will help guide the student towards the final solution. In all, well over 1,000 problems relating to finding or evaluating integrals or problems associated with properties for the definite integral can be found dispersed throughout the end-of-chapter exercise sets.

Chapter 1 introduces formally what we mean by an integral in the Riemann sense. The approach taken is one via Darboux sums. The fundamental theorem of calculus, which we divide in two parts, is also given. Properties for the definite integral are given in three chapters (Chapters 2, 4, and 16). Sixteen chapters (Chapters 3 and 5–19) are then devoted to either a particular method that can be used to find a given integral or a particular class of integrals. Here methods including standard forms, integration by substitution, integration by parts, trigonometric and hyperbolic substitutions, a tangent half-angle substitution, trigonometric and hyperbolic integrals, integrating rational functions using partial fractions, integrating inverse functions, and reduction formulae can be found. The penultimate chapter, Chapter 20, introduces the improper integral. The field of improper integrals is immense and all we can do here is touch upon this important area. The final chapter, Chapter 21, is devoted to considering two very important improper integrals that arise in applications known as the Gaussian integral and the Dirichlet integral.

While all the familiar techniques of integration one would normally expect to find in any standard introductory calculus text are to be found here, the approach we take is somewhat different. Other treatments tend to be brief and hurried while the questions asked are often repetitive and uninteresting. In the present volume, as each chapter is devoted to a particular technique our focus is more concentrated and allows one to methodically work through each of these techniques. At the same time an abundance of detailed worked examples are given, and different and varied question types are asked. We also offer other useful methods not typically found elsewhere. These include integrating rational functions using the Heaviside cover-up method and Ostrogradsky's method (Chapter 11), tabular integration by parts (Chapter 7), and the rules of Bioche (Chapter 15). Finally, we provide two appendices. The first, Appendix A, on partial fractions is given for anyone who has either not encountered this topic before or is in need of a brief review. The second, Appendix B, contains answers to selected questions asked.

The genesis of this book grew from the large number of requests the author received over the years from students he taught introductory calculus to. Many students wanted additional material and questions to consolidate and test their growing skills in finding integrals and asked if a short text could be suggested to meet such a need. These many requests drove the author to seek out and create ever more varied and interesting problems, the result of which you now hold before you.

Inspiration for many of the exercises found in the text has been drawn from a wide variety of sources. Articles and problems relating to integration found in the journals *The American Mathematical Monthly*, *Mathematics Magazine*, *The College Mathematics Journal*, and *The Mathematical Gazette* have proved useful, as have online question-and-answer sites devoted to mathematics such as *Mathematics Stack Exchange* and *The Art of Problem Solving*. Joseph Edwards's *A Treatise on the Integral Calculus* (Volume 1), G. H. Hardy's *A Course of Pure Mathematics*, Michael Spivak's *Calculus*, and the Soviet text *Problems in Mathematical Analysis* edited by Boris Demidovich have also proved useful sources for questions. Answers to almost all exercises appearing in the text are given in Appendix B. While every care has been taken to ensure their accuracy, errors are regrettably unavoidable and the author would be most grateful if any errors found, could be brought to his attention.

In closing perhaps something needs to be said about why one should bother to learn any of the techniques of integration at all. After all, powerful computer algebra systems now exist that can find almost all of the integrals appearing in this text. Such a question is of course a bit like asking why bother to learn to add when all of arithmetic can be handled by a calculator. Understanding why things are the way they are is important. If nothing else, integration is incredibly useful. It is a standard topic in any introductory course on calculus and an important gateway to many areas of more advanced applied mathematics. Many of the techniques of integration are important theorems in themselves about integrable functions, providing a foundation for higher mathematics. While having the ability and the insight to see into an integral, and turn it from the inside out, may not be a very convincing reason for many, intellectually it is the most compelling reason of them all.