MATRIX ANALYSIS AND APPLICATIONS

This balanced and comprehensive study presents the theory, methods and applications of matrix analysis in a new theoretical framework, allowing readers to understand second-order and higher-order matrix analysis in a completely new light.

Alongside the core subjects in matrix analysis, such as singular value analysis, the solution of matrix equations and eigenanalysis, the author introduces new applications and perspectives that are unique to this book. The very topical subjects of gradient analysis and optimization play a central role here. Also included are subspace analysis, projection analysis and tensor analysis, subjects which are often neglected in other books. Having provided a solid foundation to the subject, the author goes on to place particular emphasis on the many applications matrix analysis has in science and engineering, making this book suitable for scientists, engineers and graduate students alike.

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To John Zhang, Ellen Zhang and Andrew Wei

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Preface

Linear algebra is a vast field of fundamental importance in most areas of pure (and applied) mathematics, while matrices are a key tool for the researchers, scientists, engineers and graduate students majoring in the science and engineering disciplines.

From the viewpoint of applications, matrix analysis provides a powerful mathematical modeling and computational framework for posing and solving important scientific and engineering problems. It is no exaggeration to say that matrix analysis is one of the most creative and flexible mathematical tools and that it plays an irreplaceable role in physics, mechanics, signal and information processing, wireless communications, machine learning, computer vision, automatic control, system engineering, aerospace, bioinformatics, medical image processing and many other disciplines, and it effectively supports research in them all. At the same time, novel applications in these disciplines have spawned a number of new results and methods of matrix analysis, such as quadratic eigenvalue problems, joint diagonalization, sparse representation and compressed sensing, matrix completion, nonnegative matrix factorization, tensor analysis and so on.

Goal of the Book

The main goal of this book is to help the reader develop the skills and background needed to recognize, formulate and solve linear algebraic problems by presenting systematically the theory, methods and applications of matrix analysis.

A secondary goal is to help the reader understand some recent applications, perspectives and developments in matrix analysis.

Structure of the Book

In order to provide a balanced and comprehensive account of the subject, this book covers the core theory and methods in matrix analysis, and places particular emphasis on its typical applications in various science and engineering disciplines. The book consists of ten chapters, spread over three parts.

Part I is on matrix algebra: it contains Chapters 1 through 3 and focuses on the necessary background material. Chapter 1 is an introduction to matrix algebra that is devoted to basic matrix operations. This is followed by a description of the vectorization of matrices, the representation of vectors as matrices, i.e. matricization, and the application of sparse matrices to face recognition. Chapter 2 presents some special matrices used commonly in matrix analysis. Chapter 3 presents the matrix differential, which is an important tool in optimization.

Part II is on matrix analysis: this is the heart of the book, and deals with the topics that are most frequently needed. It covers both theoretical and practical aspects and consists of six chapters, as follows.

Chapter 4 is devoted to the gradient analysis of matrices, with applications in smooth and nonsmooth convex optimization, constrained convex optimization, Newton's algorithm and the original–dual interior-point method.

In Chapter 5 we describe the singular value analysis of matrices, including singular value decomposition, generalized singular value decomposition, low-rank sparse matrix decomposition and matrix completion.

Researchers, scientists, engineers and graduate students from a wide variety of disciplines often have to use matrices for modeling purposes and to solve the resulting matrix equations. Chapter 6 focuses on ways to solve such equations and includes the Tikhonov regularization method, the total least squares method, the constrained total least squares method, nonnegative matrix factorization and the solution of sparse matrix equations.

Chapter 7 deals with eigenvalue decomposition, matrix reduction, generalized eigenvalue decomposition, the Rayleigh quotient, the generalized Rayleigh quotient, quadratic eigenvalue problems and joint diagonalization.

Chapter 8 is devoted to subspace analysis methods and subspace tracking algorithms in adaptive signal processing.

Chapter 9 focuses on orthogonal and oblique projections with their applications.

Part III is on higher-order matrix analysis and consists simply of Chapter 10. In it, matrix analysis is extended from the second-order case to higher orders via a presentation of the basic algebraic operations, representation as matrices, Tuckey decomposition, parallel factor decomposition, eigenvalue decomposition of tensors, nonnegative tensor decomposition and tensor completion, together with applications.

Features of the Book

The book introduces a novel theoretical framework for matrix analysis by dividing it into second-order matrix analysis (including gradient analysis, singular value analysis, eigenanalysis, subspace analysis and projection analysis) and higher-order matrix analysis (tensor analysis).

Gradient analysis and optimization play an important role in the book. This is a very topical subject and is central to many modern applications (such as communications, signal processing, pattern recognition, machine learning, radar, big data analysis, multimodal brain image fusion etc.) though quite classical in origin.

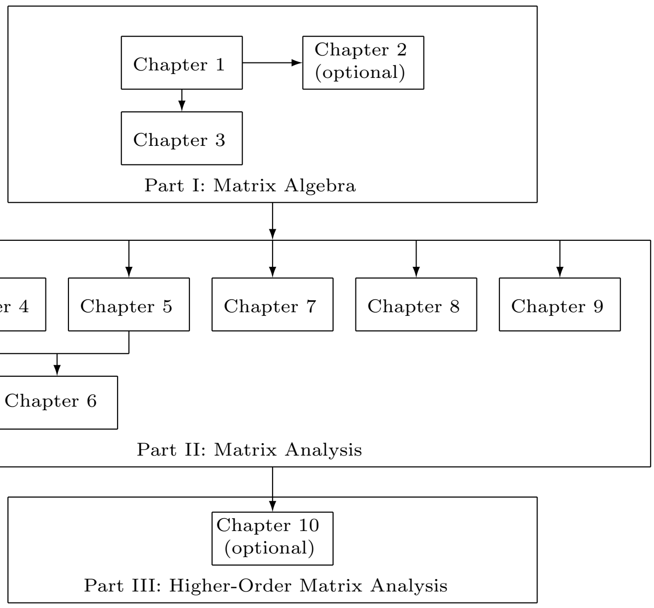
Some more contemporary topics of matrix analysis such as subspace analysis, projection analysis and tensor analysis, and which are often missing from other books, are included in our text.

Particular emphasis is placed on typical applications of matrix methods in science and engineering. The 80 algorithms for which summaries are given should help readers learn how to conduct computer experiments using related matrix analysis in their studies and research.

In order to make these methods easy to understand and master, this book adheres to the principle of both interpreting physics problems in terms of mathematics, and mathematical results in terms of physical ideas. Thus some typical or important matrix analysis problems are introduced by modeling a problem from physics, while some important mathematical results are explained and understood by revealing their physical meaning.

Reading the Book

The following diagram gives a schematic organization of this book to illustrate the chapter dependences.



Chapters 2 and 10 are optional. In particular, Chapter 10 is specifically devoted to readers involved in multi-channel or multi-way data analysis and processing.

Intended Readership

Linear algebra and matrix analysis are used in a very wide range of subjects including physics, statistics, computer science, economics, information science and technology (including signal and image processing, communications, automation control, system engineering and pattern recognition), artificial intelligence, bioinformatics, biomedical engineering, to name just a selection. This book is dedicated to providing individuals in those disciplines with a solid foundation of the fundamental skills needed to develop and apply linear algebra and matrix analysis methods in their work.

The only background required of the reader is a good knowledge of advanced calculus, so the book will be suitable for graduate students in science and engineering.

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Since 2004 we have taught graduate courses on matrix analysis and applications at Tsinghua University. Over the years I have benefited from keen interest, feedback and suggestions from many people, including my own graduate students, and students in our courses. I wish to thank Dr. Fang-Ming Han for his contribution to co-teaching and then teaching these courses, and Xi-Lin Li, Lu Wu, Dong-Xia Chang, Kun Wang, Zhong Chen, Xi-Yuan Wang and Yan-Yi Rao for their assistance with the teaching.

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Notation

Sets

$\mathbb{R}$	real numbers
$\mathbb{R}^n$	real n-vectors ($n \times 1$ real matrices)
$\mathbb{R}^{m\times n}$	real $m \times n$ matrices
$\mathbb{R}[x]$	real polynomials
$\mathbb{R}[x]^{m\times n}$	real $m \times n$ polynomial matrices
$\mathbb{R}^{I\times J\times K}$	real third-order tensors
$\mathbb{R}^{I_1\times\cdots\times I_N}$	real Nth-order tensor
$\mathbb{R}_+$	nonnegative real numbers, nonnegative orthant
$\mathbb{R}_{++}$	positive real numbers
$\mathbb{C}$	complex numbers
$\mathbb{C}^n$	complex n-vectors
$\mathbb{C}^{m\times n}$	complex $m \times n$ matrices
$\mathbb{C}[x]$	complex polynomials
$\mathbb{C}[x]^{m\times n}$	complex $m \times n$ polynomial matrices
$\mathbb{C}^{I\times J\times K}$	complex third-order tensors
$\mathbb{C}^{I_1\times\cdots\times I_N}$	complex Nth-order tensors
$\mathbb{K}$	real or complex numbers
$\mathbb{K}^n$	real or complex n-vectors
$\mathbb{K}^{m\times n}$	real or complex $m \times n$ matrices
$\mathbb{K}^{I\times J\times K}$	real or complex third-order tensors
$\mathbb{K}^{I_1\times\cdots\times I_N}$	real or complex Nth-order tensors
$\mathbb{Z}$	integers
$\mathbb{Z}_+$	nonnegative integers

Sets (*continued*)

$\mathbb{S}^{n\times n}$	symmetric $n \times n$ matrices
$\mathbb{S}_{+}^{n\times n}$	symmetric positive semi-definite $n \times n$ matrices
$\mathbb{S}_{++}^{n\times n}$	symmetric positive definite $n \times n$ matrices
$\mathbb{S}^{[m,n]}$	symmetric mth-order n-dimensional tensors $\mathcal{A}^{I_1\times\cdots\times I_m}, I_1=\cdots=I_n$
$\mathbb{S}_{+}^{[m,n]}$	symmetric mth-order n-dimensional nonnegative tensors
$\forall$	for all
$x \in A$	x belongs to the set A, i.e. x is an element of A
$x \notin A$	x is not an element of the set A
$U \mapsto V$	U maps to V
$U \to W$	U transforms to W
$\ni$	such that
$\exists$	exists
$A \Rightarrow B$	A implies B
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$A + B$	sum set of sets A and B
$A - B$	set-theoretic difference of sets A and B
$X \setminus A$	complement of the set A in the set X
$X_1 \times \cdots \times X_n$	Cartesian product of sets $X_1, \ldots, X_n$
$\mathcal{L}$	linear manifold
$\mathrm{Gr}(n, r)$	Grassmann manifold
$\mathrm{St}(n, r)$	Stiefel manifold
O_r	orthogonal group
$S^{\perp}$	orthogonal complement of the subspace S
$\mathcal{K}^m(\mathbf{A}, \mathbf{f})$	order-m Krylov subspace generated by $\mathbf{A}$ and $\mathbf{f}$
$\mathrm{Col}(\mathbf{A})$	column space of the matrix $\mathbf{A}$
$\mathrm{Ker}(\mathbf{A})$	kernel space of the matrix $\mathbf{A}$
$\mathrm{Null}(\mathbf{A})$	null space of the matrix $\mathbf{A}$
$\mathrm{nullity}(\mathbf{A})$	nullity of the matrix $\mathbf{A}$
$\mathrm{Range}(\mathbf{A})$	range space of the matrix $\mathbf{A}$

Sets (*continued*)

$\text{Row}(\mathbf{A})$	row space of the matrix $\mathbf{A}$
$\text{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$	span of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$

Vectors

$\mathbf{x}^*$	conjugate of the vector $\mathbf{x}$
$\mathbf{x}^T$	transpose of the vector $\mathbf{x}$
$\mathbf{x}^H$	conjugate transpose (Hermitian conjugate) of the vector $\mathbf{x}$
$\mathcal{L}(\mathbf{u})$	linear transform of the vector $\mathbf{u}$
$\Vert\mathbf{x}\Vert_0$	ℓ_0-norm: the number of nonzero entries in the vector $\mathbf{x}$
$\Vert\mathbf{x}\Vert_1$	ℓ_1-norm of the vector $\mathbf{x}$
$\Vert\mathbf{x}\Vert_2$	Euclidean norm of the vector $\mathbf{x}$
$\Vert\mathbf{x}\Vert_p$	ℓ_p-norm or Hölder norm of the vector $\mathbf{x}$
$\Vert\mathbf{x}\Vert_*$	nuclear norm of the vector $\mathbf{x}$
$\Vert\mathbf{x}\Vert_\infty$	ℓ_∞-norm of the vector $\mathbf{x}$
$\langle\mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^H\mathbf{y}$	inner product of vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathbf{x} \circ \mathbf{y} = \mathbf{x}\mathbf{y}^H$	outer product of vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathbf{x} \perp \mathbf{y}$	orthogonality of vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathbf{x} > \mathbf{0}$	positive vector, with components $x_i > 0, \forall\, i$
$\mathbf{x} \geq \mathbf{0}$	nonnegative vector, with components $x_i \geq 0, \forall\, i$
$\mathbf{x} \geq \mathbf{y}$	vector elementwise inequality $x_i \geq y_i, \forall\, i$
$\text{unvec}(\mathbf{x})$	matricization of the column vector $\mathbf{x}$
$\text{unrvec}(\mathbf{x})$	row matricization of the column vector $\mathbf{x}$
$\theta_i^{(m)}, \mathbf{y}_i^{(m)}$	Rayleigh–Ritz (RR) values, RR vectors
$(\theta_i^{(m)}, \mathbf{y}_i^{(m)})$	Ritz pair

Matrices

$\mathbf{A} \in \mathbb{R}^{m\times n}$	real $m \times n$ matrix $\mathbf{A}$
$\mathbf{A} \in \mathbb{C}^{m\times n}$	complex $m \times n$ matrix $\mathbf{A}$
$\mathbf{A}[x] \in \mathbb{R}[x]^{m\times n}$	real $m \times n$ polynomial matrix $\mathbf{A}$
$\mathbf{A}[x] \in \mathbb{C}[x]^{m\times n}$	complex $m \times n$ polynomial matrix $\mathbf{A}$
$\mathbf{A}^*$	conjugate of $\mathbf{A}$
$\mathbf{A}^T$	transpose of $\mathbf{A}$

Matrices (*continued*)

$\mathbf{A}^H$	conjugate transpose (Hermitian conjugate) of $\mathbf{A}$
$(\mathbf{A}, \mathbf{B})$	matrix pencil
$\det(\mathbf{A}), \lvert\mathbf{A}\rvert$	determinant of $\mathbf{A}$
$\mathrm{tr}(\mathbf{A})$	trace of $\mathbf{A}$
$\mathrm{rank}(\mathbf{A})$	rank of $\mathbf{A}$
$\mathrm{eig}(\mathbf{A})$	eigenvalues of the Hermitian matrix $\mathbf{A}$
$\lambda_i(\mathbf{A})$	ith eigenvalue of the Hermitian matrix $\mathbf{A}$
$\lambda_{\max}(\mathbf{A})$	maximum eigenvalue(s) of the Hermitian matrix $\mathbf{A}$
$\lambda_{\min}(\mathbf{A})$	minimum eigenvalue(s) of the Hermitian matrix $\mathbf{A}$
$\lambda(\mathbf{A}, \mathbf{B})$	generalized eigenvalue of the matrix pencil $(\mathbf{A}, \mathbf{B})$
$\sigma_i(\mathbf{A})$	ith singular value of $\mathbf{A}$
$\sigma_{\max}(\mathbf{A})$	maximum singular value(s) of $\mathbf{A}$
$\sigma_{\min}(\mathbf{A})$	minimum singular value(s) of $\mathbf{A}$
$\rho(\mathbf{A})$	spectral radius of $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of the nonsingular matrix $\mathbf{A}$
$\mathbf{A}^{\dagger}$	Moore–Penrose inverse of $\mathbf{A}$
$\mathbf{A} \succ 0$	positive definite matrix $\mathbf{A}$
$\mathbf{A} \succeq 0$	positive semi-definite matrix $\mathbf{A}$
$\mathbf{A} \prec 0$	negative definite matrix $\mathbf{A}$
$\mathbf{A} \preceq 0$	negative semi-definite matrix $\mathbf{A}$
$\mathbf{A} > \mathbf{O}$	positive (or elementwise positive) matrix $\mathbf{A}$
$\mathbf{A} \geq \mathbf{O}$	nonnegative (or elementwise nonnegative) matrix $\mathbf{A}$
$\mathbf{A} \geq \mathbf{B}$	matrix elementwise inequality $a_{ij} \geq b_{ij}, \forall\, i, j$
$\lVert\mathbf{A}\rVert_1$	maximum absolute column-sum norm of $\mathbf{A}$
$\lVert\mathbf{A}\rVert_\infty$	maximum absolute row-sum norm of $\mathbf{A}$
$\lVert\mathbf{A}\rVert_{\mathrm{spec}}$	spectrum norm of $\mathbf{A}: \sigma_{\max}(\mathbf{A})$
$\lVert\mathbf{A}\rVert_F$	Frobenius norm of $\mathbf{A}$
$\lVert\mathbf{A}\rVert_\infty$	max norm of $\mathbf{A}$: the absolute maximum of all entries of $\mathbf{A}$
$\lVert\mathbf{A}\rVert_{\mathbf{G}}$	Mahalanobis norm of $\mathbf{A}$
$\mathrm{vec}\,\mathbf{A}$	column vectorization of $\mathbf{A}$
$\mathrm{rvec}\,\mathbf{A}$	row vectorization of $\mathbf{A}$
$\mathrm{off}(\mathbf{A})$	off function of $\mathbf{A} = [a_{ij}]: \sum_{i=1, i\neq j}^{m} \sum_{j=1}^{n} \lvert a_{ij}\rvert^2$
$\mathrm{diag}(\mathbf{A})$	diagonal function of $\mathbf{A} = [a_{ij}]: \sum_{i=1}^{n} \lvert a_{ii}\rvert^2$

Matrices (*continued*)

$\mathbf{diag}(\mathbf{A})$	diagonal vector of $\mathbf{A} = [a_{ij}] : [a_{11}, \ldots, a_{nn}]^T$
$\mathbf{Diag}(\mathbf{A})$	diagonal matrix of $\mathbf{A} = [a_{ij}] : \mathbf{Diag}(a_{11}, \ldots, a_{nn})$
$\langle \mathbf{A}, \mathbf{B} \rangle$	inner product of $\mathbf{A}$ and $\mathbf{B}$: $(\mathrm{vec}\, \mathbf{A})^H \mathrm{vec}\, \mathbf{B}$
$\mathbf{A} \otimes \mathbf{B}$	Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{A} \odot \mathbf{B}$	Khatri–Rao product of matrices $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{A} * \mathbf{B}$	Hadamard product of matrices $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{A} \oplus \mathbf{B}$	direct sum of matrices $\mathbf{A}$ and $\mathbf{B}$
$\{\mathbf{A}\}_N$	matrix group consisting of matrices $\mathbf{A}_1, \ldots, \mathbf{A}_N$
$\{\mathbf{A}\}_N \otimes \mathbf{B}$	generalized Kronecker product of $\{\mathbf{A}\}_N$ and $\mathbf{B}$
$\delta \mathbf{x}, \delta \mathbf{X}$	perturbations of the vector $\mathbf{x}$ and the matrix $\mathbf{X}$
$\mathrm{cond}(\mathbf{A})$	condition number of the matrix $\mathbf{A}$
$\mathrm{In}(\mathbf{A})$	inertia of a symmetric matrix $\mathbf{A}$
$i_+(\mathbf{A})$	number of positive eigenvalues of $\mathbf{A}$
$i_-(\mathbf{A})$	number of negative eigenvalues of $\mathbf{A}$
$i_0(\mathbf{A})$	number of zero eigenvalues of $\mathbf{A}$
$\mathbf{A} \sim \mathbf{B}$	similarity transformation
$\mathbf{A}(\lambda) \cong \mathbf{B}(\lambda)$	balanced transformation
$\mathbf{A} \doteq \mathbf{B}$	essentially equal matrices
$\mathbf{J} = \mathbf{PAP}^{-1}$	Jordan canonical form of the matrix $\mathbf{A}$
$d_k(x)$	kth determinant divisor of a polynomial matrix $\mathbf{A}(x)$
$\sigma_k(x)$	kth invariant factor of a polynomial matrix $\mathbf{A}(x)$
$\mathbf{A}(\lambda)$	λ-matrix of the matrix $\mathbf{A}$
$\mathbf{S}(\lambda)$	Smith normal form of the λ-matrix $\mathbf{A}(\lambda)$

Special Vectors and Special Matrices

$\mathbf{P}_S$	projector onto the subspace S
$\mathbf{P}_S^\perp$	orthogonal projector onto the subspace S
$\mathbf{E}_{H\|S}$	oblique projector onto the subspace H along the subspace S
$\mathbf{E}_{S\|H}$	oblique projector onto the subspace S along the subspace H
$\mathbf{1}$	summing vector with all entries 1
$\mathbf{0}$	null or zero vector with all components 0
$\mathbf{e}_i$	basic vector with $e_i = 1$ and all other entries 0
$\boldsymbol{\pi}$	extracting vector with the last nonzero entry 1

Special Vectors and Special Matrices (*continued*)

$\mathbf{i}_N$	index vector : $[\langle 0\rangle, \langle 1\rangle, \ldots, \langle N-1\rangle]^T$
$\mathbf{i}_{N,\mathrm{rev}}$	bit-reversed index vector of $\mathbf{i}_N$
$\mathbf{O}$	null or zero matrix, with all components zero
$\mathbf{I}$	identity matrix
$\mathbf{K}_{mn}$	$mn \times mn$ commutation matrix
$\mathbf{J}_n$	$n \times n$ exchange matrix : $\mathbf{J}_n = [\mathbf{e}_n, \ldots, \mathbf{e}_1]$
$\mathbf{P}$	$n \times n$ permutation matrix : $\mathbf{P} = [\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}], i_1, \ldots, i_n \in \{1, \ldots, n\}$
$\mathbf{G}$	generalized permutation matrix or g-matrix : $\mathbf{G} = \mathbf{PD}$
$\mathbf{C}_n$	$n \times n$ centering matrix $= \mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n^T$
$\mathbf{F}_N$	$N \times N$ Fourier matrix with entry $F(i,k) = (\mathrm{e}^{-\mathrm{j}2\pi/N})^{(i-1)(k-1)}$
$\mathbf{F}_{N,\mathrm{rev}}$	$N \times N$ bit-reversed Fourier matrix
$\mathbf{H}_n$	Hadamard matrix : $\mathbf{H}_n\mathbf{H}_n^T = \mathbf{H}_n^T\mathbf{H}_n = n\mathbf{I}_n$
$\mathbf{A}$	symmetric Toeplitz matrix : $[a_{\lvert i-j\rvert}]_{i,j=1}^n$
$\mathbf{A}$	complex Toeplitz matrix: Toep$[a_0, a_1, \ldots, a_n]$ with $a_{-i} = a_i^*$
$\mathbf{Q}_n$	$n \times n$ real orthogonal matrix : $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$
$\mathbf{U}_n$	$n \times n$ unitary matrix : $\mathbf{UU}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}$
$\mathbf{Q}_{m\times n}$	$m \times n$ semi-orthogonal matrix : $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_n$ or $\mathbf{QQ}^T = \mathbf{I}_m$
$\mathbf{U}_{m\times n}$	$m \times n$ para-unitary matrix : $\mathbf{U}^H\mathbf{U} = \mathbf{I}_n,\ m > n,$ or $\mathbf{UU}^H = \mathbf{I}_m,\ m < n$
$\mathbf{S}_b$	between-class scatter matrix
$\mathbf{S}_w$	within-class scatter matrix

Tensors

$\mathcal{A} \in \mathbb{K}^{I_1\times\cdots\times I_N}$	Nth-order real or complex tensor
$\mathcal{I}, \mathcal{E}$	identity tensor
$\mathbf{A}_{i::}$	horizontal slice matrix of $\mathcal{A} \in \mathbb{K}^{I\times J\times K}$
$\mathbf{A}_{:j:}$	lateral slice matrix of $\mathcal{A} \in \mathbb{K}^{I\times J\times K}$
$\mathbf{A}_{::k}$	frontal slice matrix of $\mathcal{A} \in \mathbb{K}^{I\times J\times K}$
$\mathbf{a}_{:jk}, \mathbf{a}_{i:k}, \mathbf{a}_{ij:}$	mode-1, model-2, model-3 vectors of $\mathcal{A} \in \mathbb{K}^{I\times J\times K}$
vec $\mathcal{A}$	vectorization of tensor $\mathcal{A}$
unvec $\mathcal{A}$	matricization of tensor $\mathcal{A}$
$\mathbf{A}^{(JK\times I)}, \mathbf{A}^{(KI\times J)}, \mathbf{A}^{(IJ\times K)}$	matricization of tensor $\mathcal{A} \in \mathbb{K}^{I\times J\times K}$

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Tensors (*continued*)

$\langle \mathcal{A}, \mathcal{B} \rangle$	inner product of tensors: $(\mathrm{vec}\,\mathcal{A})^H \mathrm{vec}\,\mathcal{B}$
$\Vert\mathcal{A}\Vert_F$	Frobenius norm of tensor $\mathcal{A}$
$\mathcal{A} = \mathbf{u} \circ \mathbf{v} \circ \mathbf{w}$	outer product of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$
$\mathcal{X} \times_n \mathbf{A}$	Tucker mode-n product of $\mathcal{X}$ and $\mathbf{A}$
$\mathrm{rank}(\mathcal{A})$	rank of tensor $\mathcal{A}$
$[\![\mathcal{G}; \mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(N)}]\!]$	Tucker operator of tensor $\mathcal{G}$
$\mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$	Tucker decomposition (third-order SVD)
$\mathcal{G} \times_1 \mathbf{U}^{(1)} \cdots \times_N \mathbf{U}^{(N)}$	higher-order SVD of Nth-order tensor
$x_{ijk} = \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} a_{ip} b_{jq} c_{kr}$	CP decomposition of the third-order tensor $\mathcal{X}$
$\mathcal{A}\mathbf{x}^m, \mathcal{A}\mathbf{x}^{m-1}$	tensor–vector product of $\mathcal{A} \in \mathbb{S}^{[m,n]}, \mathbf{x} \in \mathbb{C}^{n\times 1}$
$\det(\mathcal{A})$	determinant of tensor $\mathcal{A}$
$\lambda_i(\mathcal{A})$	ith eigenvalue of tensor $\mathcal{A}$
$\sigma(\mathcal{A})$	spectrum of tensor $\mathcal{A}$

Functions and Derivatives

$\stackrel{\mathrm{def}}{=}$	defined to be equal
$\sim$	asymptotically equal (in scaling sense)
$\approx$	approximately equal (in numerical value)
$f : \mathbb{R}^m \to \mathbb{R}$	real function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m, f \in \mathbb{R}$
$f : \mathbb{R}^{m\times n} \to \mathbb{R}$	real function $f(\mathbf{X}), \mathbf{X} \in \mathbb{R}^{m\times n}, f \in \mathbb{R}$
$f : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{R}$	real function $f(\mathbf{z}, \mathbf{z}^*), \mathbf{z} \in \mathbb{C}^m, f \in \mathbb{R}$
$f : \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n} \to \mathbb{R}$	real function $f(\mathbf{Z}, \mathbf{Z}^*), \mathbf{Z} \in \mathbb{C}^{m\times n}, f \in \mathbb{R}$
$\mathbf{dom}\, f, \mathcal{D}$	definition domain of function f
$\mathcal{E}$	domain of equality constraint function
$\mathcal{I}$	domain of inequality constraint function
$\mathcal{F}$	feasible set
$B_c(\mathbf{c}; r), B_o(\mathbf{c}; r)$	closed, open neighborhoods of $\mathbf{c}$ with radius r
$B_c(\mathbf{C}; r), B_o(\mathbf{C}; r)$	closed, open neighborhoods of $\mathbf{C}$ with radius r
$f(\mathbf{z}, \mathbf{z}^*), f(\mathbf{Z}, \mathbf{Z}^*)$	function of complex variables $\mathbf{z}$, or $\mathbf{Z}$
$\mathrm{d}f(\mathbf{z}, \mathbf{z}^*), \mathrm{d}f(\mathbf{Z}, \mathbf{Z}^*)$	complex differentials
$D(\mathbf{p}\Vert\mathbf{g})$	distance between vectors $\mathbf{p}$ and $\mathbf{g}$

Functions and Derivatives (*continued*)

$D(\mathbf{x},\mathbf{y})$	dissimilarity between vectors $\mathbf{x}$ and $\mathbf{y}$
$D_E(\mathbf{x},\mathbf{y})$	Euclidean distance between vectors $\mathbf{x}$ and $\mathbf{y}$
$D_M(\mathbf{x},\mathbf{y})$	Mahalanobis distance between vectors $\mathbf{x}$ and $\mathbf{y}$
$D_J^g(\mathbf{x},\mathbf{y})$	Bregman distance between vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathrm{D}_{\mathbf{x}}, \mathrm{D}_{\mathrm{vec}\,\mathbf{X}}$	row partial derivative operators
$\mathrm{D}_{\mathbf{x}} f(\mathbf{x})$	row partial derivative vectors of $f(\mathbf{x})$
$\mathrm{D}_{\mathrm{vec}\,\mathbf{X}} f(\mathbf{X})$	row partial derivative vectors of $f(\mathbf{X})$
$\mathrm{D}_{\mathbf{X}}$	Jacobian operator
$\mathrm{D}_{\mathbf{X}} f(\mathbf{X})$	Jacobian matrix of the function $f(\mathbf{X})$
$\mathrm{D}_{\mathbf{z}}, \mathrm{D}_{\mathrm{vec}\,\mathbf{Z}}$	complex cogradient operator
$\mathrm{D}_{\mathbf{z}^*}, \mathrm{D}_{\mathrm{vec}\,\mathbf{Z}^*}$	complex conjugate cogradient operator
$\mathrm{D}_{\mathbf{z}} f(\mathbf{z},\mathbf{z}^*)$	cogradient vector of complex function $f(\mathbf{z},\mathbf{z}^*)$
$\mathrm{D}_{\mathrm{vec}\,\mathbf{Z}} f(\mathbf{Z},\mathbf{Z}^*)$	cogradient vector of complex function $f(\mathbf{Z},\mathbf{Z}^*)$
$\mathrm{D}_{\mathbf{z}^*} f(\mathbf{z},\mathbf{z}^*)$	conjugate cogradient vector of $f(\mathbf{z},\mathbf{z}^*)$
$\mathrm{D}_{\mathrm{vec}\,\mathbf{Z}^*} f(\mathbf{Z},\mathbf{Z}^*)$	conjugate cogradient vector of $f(\mathbf{Z},\mathbf{Z}^*)$
$\mathrm{D}_{\mathbf{Z}}, \nabla_{\mathbf{Z}^*}$	Jacobian, gradient matrix operator
$\mathrm{D}_{\mathbf{Z}} f(\mathbf{Z},\mathbf{Z}^*)$	Jacobian matrices of $f(\mathbf{Z},\mathbf{Z}^*)$
$\nabla_{\mathbf{Z}} f(\mathbf{Z},\mathbf{Z}^*)$	gradient matrices of $f(\mathbf{Z},\mathbf{Z}^*)$
$\mathrm{D}_{\mathbf{Z}^*} f(\mathbf{Z},\mathbf{Z}^*)$	conjugate Jacobian matrices of $f(\mathbf{Z},\mathbf{Z}^*)$
$\nabla_{\mathbf{Z}^*} f(\mathbf{Z},\mathbf{Z}^*)$	conjugate gradient matrices of $f(\mathbf{Z},\mathbf{Z}^*)$
$\nabla_{\mathbf{x}}, \nabla_{\mathrm{vec}\,\mathbf{X}}$	gradient vector operator
$\nabla_{\mathbf{x}} f(\mathbf{x})$	gradient vector of function $f(\mathbf{x})$
$\nabla_{\mathrm{vec}\,\mathbf{X}} f(\mathbf{X})$	gradient vector of function $f(\mathbf{X})$
$\nabla f(\mathbf{X})$	gradient matrix of function f
$\nabla^2 f$	Hessian matrix of function f
$\mathbf{H}_{\mathbf{x}} f(\mathbf{x})$	Hessian matrix of function f
$\mathbf{H} f(\mathbf{z},\mathbf{z}^*)$	full Hessian matrix of $f(\mathbf{z},\mathbf{z}^*)$
$\mathbf{H}_{\mathbf{z},\mathbf{z}}, \mathbf{H}_{\mathbf{z},\mathbf{z}^*}, \mathbf{H}_{\mathbf{z}^*,\mathbf{z}^*}$	part Hessian matrices of function $f(\mathbf{z},\mathbf{z}^*)$
$\mathrm{d}f, \partial f$	differential or subdifferential of function f
$\mathbf{g} \in \partial f$	subgradient of function f
$\Delta\mathbf{x}$	descent direction of function $f(\mathbf{x})$
$\Delta\mathbf{x}_{\mathrm{nt}}$	Newton step of function $f(\mathbf{x})$
$\max\ f, \min\ f$	maximize, minimize function f

Functions and Derivatives (*continued*)

$\max\{x, y\}$	maximum of x and y
$\min\{x, y\}$	minimum of x and y
inf	infimum
sup	supremum
Re, Im	real part, imaginary part of complex number
arg	argument of objective function or complex number
$\mathcal{P}_C(\mathbf{y}), \mathbf{P}_C\mathbf{y}$	projection operator of the vector $\mathbf{y}$ onto the subspace C
$\mathbf{x}^+$	nonnegative vector with entry $[\mathbf{x}^+]_i = \max\{x_i, 0\}$
$\mathbf{prox}_h(\mathbf{u})$	proximal operator of function $h(\mathbf{x})$ to point $\mathbf{u}$
$\mathbf{prox}_h(\mathbf{U})$	proximal operator of function $h(\mathbf{X})$ to point $\mathbf{U}$
$\mathrm{soft}(x, \tau), \mathcal{S}_\tau[x]$	soft thresholding operator of real variable x

Probability

$\mathrm{soft}(\mathbf{x}, \tau), \mathrm{soft}(\mathbf{X}, \tau)$	soft thresholding operator of real variables $\mathbf{x}, \mathbf{X}$
$\mathcal{D}_\mu(\mathbf{\Sigma})$	singular value (matrix) thresholding (operation)
$I_C(\mathbf{x})$	indicator function
$\mathbf{x}_{\mathrm{LS}}, \mathbf{X}_{\mathrm{LS}}$	least squares solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{DLS}}, \mathbf{X}_{\mathrm{DLS}}$	data least squares solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{WLS}}, \mathbf{X}_{\mathrm{WLS}}$	weighted least squares solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{opt}}, \mathbf{X}_{\mathrm{opt}}$	optimal solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{Tik}}, \mathbf{X}_{\mathrm{Tik}}$	Tikhonov solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{TLS}}, \mathbf{X}_{\mathrm{TLS}}$	total least squares (TLS) solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{GTLS}}, \mathbf{X}_{\mathrm{GTLS}}$	generalized TLS solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$\mathbf{x}_{\mathrm{ML}}, \mathbf{X}_{\mathrm{ML}}$	maximum likelihood solutions to $\mathbf{Ax} = \mathbf{b}, \mathbf{AX} = \mathbf{B}$
$D_{AB}^{(\alpha,\beta)}(\mathbf{P}\|\mathbf{G})$	alpha–beta (AB) divergence of matrices $\mathbf{P}$ and $\mathbf{G}$
$D_\alpha(\mathbf{P}\|\mathbf{G})$	alpha-divergence of matrices $\mathbf{P}$ and $\mathbf{G}$
$D_\beta(\mathbf{P}\|\mathbf{G})$	beta-divergence of matrices $\mathbf{P}$ and $\mathbf{G}$
$D_{\mathrm{KL}}(\mathbf{P}\|\mathbf{G})$	Kullback–Leibler divergence of matrices $\mathbf{P}$ and $\mathbf{G}$
$\ln_q(x)$	Tsallis logarithm
$\exp_q(x)$	q-exponential
$\ln_{1-\alpha}(x)$	deformed logarithm
$\exp_{1-\alpha}(x)$	deformed exponential

Probability (*continued*)

$\mathrm{sign}(x)$	signum function of real valued variable x
$\mathrm{SGN}(x)$	signum multifunction of real valued variable x
$\mathrm{shrink}(y, \alpha)$	shrink operator
$R(\mathbf{x})$	Rayleigh quotient, generalized Rayleigh quotient
$\mathbf{M}_{\mathrm{off}}$	off-diagonal matrix corresponding to matrix $\mathbf{M}$
$z^{-j}\mathbf{x}(n)$	time-shifting operation on vector $\mathbf{x}(n)$
$E\{\mathbf{x}\} = \bar{\mathbf{x}}$	expectation (mean) of random vector $\mathbf{x}$
$E\{\mathbf{x}\mathbf{x}^H\}$	autocorrelation matrix of random vector $\mathbf{x}$
$E\{\mathbf{x}\mathbf{y}^H\}$	cross-correlation matrix of random vectors $\mathbf{x}$ and $\mathbf{y}$
ρ_{xy}	correlation coefficient of random vectors $\mathbf{x}$ and $\mathbf{y}$
$N(\mathbf{c}, \boldsymbol{\Sigma})$	Gaussian random vector with mean (vector) $\mathbf{c}$ and covariance (matrix) $\boldsymbol{\Sigma}$
$CN(\mathbf{c}, \boldsymbol{\Sigma})$	complex Gaussian random vector with mean (vector) $\mathbf{c}$ and covariance (matrix) $\boldsymbol{\Sigma}$
$f(x_1, \ldots, x_m)$	joint probability density function of random vector $\mathbf{x} = [x_1, \ldots, x_m]^T$
$\Phi_{\mathbf{x}}(\boldsymbol{\omega})$	characteristic function of random vector $\mathbf{x}$

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More Information

Abbreviations

AB	alpha–beta
ADMM	alternating direction method of multipliers
ALS	alternating least squares
ANLS	alternating nonnegative least squares
APGL	accelerated proximal gradient line
ARNLS	alternating regularization nonnegative least squares
BBGP	Barzilai–Borwein gradient projection
BCQP	bound-constrained quadratic program
BFGS	Broyden–Fletcher–Goldfarb–Shanno
BP	basis pursuit
BPDN	basis pursuit denoising
BSS	blind source separation
CANDECOMP	canonical factor decomposition
CNMF	constrained nonnegative matrix factorization
CoSaMP	compression sampling matching pursuit
CP	CANDECOMP/PARAFAC
DCT	discrete cosine transform
DFT	discrete Fourier transform
DLS	data least squares
DOA	direction of arrival
EEG	electroencephalography
EM	expectation-maximization
EMML	expectation-maximization maximum likelihood
ESPRIT	estimating signal parameters via rotational invariance technique

EVD	eigenvalue decomposition
FAJD	fast approximate joint diagonalization
FAL	first-order augmented Lagrangian
FFT	fast Fourier transform
FISTA	fast iterative soft thresholding algorithm
GEAP	generalized eigenproblem adaptive power
GEVD	generalized eigenvalue decomposition
GSVD	generalized singular value decomposition
GTLS	generalized total least squares
HBM	heavy ball method
HOOI	higher-order orthogonal iteration
HOSVD	higher-order singular value decomposition
ICA	independent component analysis
IDFT	inverse discrete Fourier transform
iid	independent and identically distributed
inf	infimum
KKT	Karush–Kuhn–Tucker
KL	Kullback–Leibler
LARS	least angle regressive
Lasso	least absolute shrinkage and selection operator
LDA	linear discriminant analysis
LMV	Lathauwer–Moor–Vanderwalle
LP	linear programming
LS	least squares
LSI	latent semantic indexing
max	maximize, maximum
MCA	minor component analysis
MIMO	multiple-input–multiple-output
min	minimize, minimum
ML	maximum likelihood
MP	matching pursuit
MPCA	multilinear principal component analysis
MUSIC	multiple signal classification

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More Information

NeNMF	Nesterov nonnegative matrix factorization
NMF	nonnegative matrix factorization
NTD	nonnegative tensor decomposition
OGM	optimal gradient method
OMP	orthogonal matching pursuit
PARAFAC	parallel factor decomposition
PAST	projection approximation subspace tracking
PASTd	projection approximation subspace tracking via deflation
PCA	principal component analysis
PCG	preconditioned conjugate gradient
PCP	principal component pursuit
pdf	positive definite
PMF	positive matrix factorization
psdf	positive semi-definite
PSF	point-spread function
PSVD	product singular value decomposition
RIC	restricted isometry constant
RIP	restricted isometry property
ROMP	regularization orthogonal matching pursuit
RR	Rayleigh–Ritz
QCLP	quadratically constrained linear programming
QEP	quadratic eigenvalue problem
QP	quadratic programming
QV	quadratic variation
sign	signum
SPC	smooth PARAFAC tensor completion
StOMP	stagewise orthogonal matching pursuit
sup	supremum
SVD	singular value decomposition
SVT	singular value thresholding
TLS	total least squares
TV	total variation
UPCA	unfold principal component analysis
VQ	vector quantization