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QUANTUM FIELDS AND PROCESSES

Wick ordering of creation and annihilation operators is of fundamental importance for computing averages and correlations in quantum field theory and, by extension, in the Hudson–Parthasarathy theory of quantum stochastic processes, quantum mechanics, stochastic processes, and probability. This book develops the unified combinatorial framework behind these examples, starting with the simplest mathematically, and working up to the Fock space setting for quantum fields. Emphasizing ideas from combinatorics such as the role of the lattice of partitions for multiple stochastic integrals by Wallstrom–Rota and combinatorial species by Joyal, it presents insights coming from quantum probability. It also introduces a "field calculus" that acts as a succinct alternative to standard Feynman diagrams and formulates quantum field theory (cumulant moments, Dyson–Schwinger equation, tree expansions, 1-particle irreducibility) in this language. Featuring many worked examples, the book is aimed at mathematical physicists, quantum field theorists, and probabilists, including graduate and advanced undergraduate students.

John Gough is professor of mathematical and theoretical physics at Aberystwyth University, Wales. He works in the field of quantum probability and open systems, especially quantum Markovian models that can be described in terms of the Hudson–Parthasarathy quantum stochastic calculus. His more recent work has been on the general theory of networks of quantum Markovian input-output and their applications to quantum feedback control.

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Quantum Fields and Processes

A Combinatorial Approach

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To Margarita, Sigrid, and John Junior.

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Preface

It is probably safe to say that very few people have developed an interest in physical sciences due to combinatorics. Yet combinatorics makes its presence felt in modern mathematics and physics in a fundamental and elegant manner, and goes far beyond standard school problems such as to determine how many ways to rearrange the letters of the word MISSISSIPPI.

It has been argued in many places that modern physics owes much to the work of Ludwig Boltzmann, who in many ways was the first scientist to think like a modern physicist. Certainly, he was the first to use probability in an essential way, and his ideas on the microscopic foundations of thermodynamics directly influenced Gibbs and primed both Planck and Einstein at the start of the twentieth century. The machinery needed for modern mathematical physics was assembled in the twentieth century, and concentrated on the areas of functional analysis and probability theory, moving inexorably toward the description of stochastic processes and quantized fields. Behind this, however, was the realization that combinatorics played an important role.

In writing this book, we have been influenced by several recurring ideas in mathematical physics that all have an underlying combinatorial core. Wallstrom and Rota were the first to notice that several disparate strands, such as multiple Itō integrals, Wick products, and normal ordering, could be conveniently expressed in terms of the combinatorics of the lattice of partitions. They, in turn, were influenced by Meyer's book, *Quantum Probability for Probabilists* (1995). In many respects, our focal point for this book has been (Bose and Fermi) Fock space: this is the framework for quantum field theory and the quantum stochastic calculus of Hudson and Parthasarathy, and is well known to probabilists. The creation and annihilation operators, satisfying canonical (anti-)commutation relations, are then the objects of much of our attention, and we give a good deal of attention to both their mathematical and physical aspects. The elements of combinatorics that we cover here are those arising from quantum fields and stochastic processes. However, this gives ample room to bring in modern approaches, especially the combinatorial species of Joyal, to give a more algebraic feel than the traditional Feynman diagram approach. We combine elements of species with the Guichardet notation for symmetric Fock spaces in order to construct a field calculus that leads to explicit combinatorial formulas in place of the diagrammatic expansions. We have tried to resist the temptation to show off the often surprising and striking combinatorial expressions that arise. In truth, combinatorics appears as a tool in many branches of mathematics – and very frequently in the proofs of mathematical physics propositions – but doing justice to these varied and multifaceted techniques would be beyond the scope of this book.

We are grateful for the input of many colleagues over the years. We would especially like to thank Robin Hudson, Luigi Accardi, Yun Gang Lu, Igor Volovich, Wilhelm von Waldenfels, Hans Maassen, Madalin Guta, Hendra Nurdin, Matthew James, Joe Pulé, and Aubrey Truman, and to acknowledge the enormous influence of the extended quantum probability community at large. The completion of the project is in no small part due to Professor Oleg Smolyanov, whose constant refrain, “you need to write a book,” propelled us forward. We are also very grateful to Diana Gillooly and the staff at Cambridge University Press for their help and advice while writing. Finally, we would like to thank our families – Margarita and John Junior, and Sigrid – for their support and patience in waiting for a book dedicated to them.

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More Information

Notation

General Mathematical

ran(f)	range of a function f
$A^\top$	transpose
A^*	adjoint/Hermitian conjugate/complex conjugate
spec(A)	spectrum of a matrix/operator A
$\Delta_n(t,s)$	the simplex $\{(t_n,\dots,t_1) : t \geq t_n > t_{n-1} > \cdots t_1 \geq s\}$
$\Delta_n^\sigma(t,s)$	the simplex $\{(t_n,\dots,t_1) : t \geq t_{\sigma(n)} > t_{\sigma(n-1)} > \cdots t_{\sigma(1)} \geq s\}$
$\Theta(x)$	Heaviside function

Spaces, Norms, Etc.

$\mathbb{C}$	complex numbers
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\mathbb{N}$	the natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{N}_+$	nonnegative integers $\{0, 1, 2, 3, \dots\}$
$\mathbb{N}_-$	the set $\{0, 1\}$
$\mathbb{M}$	Minkowski spacetime
$\hat{\mathbb{M}}$	dual Minkowski spacetime
$\mathfrak{h}, \mathfrak{H}$	Hilbert space (always assumed separable!)
$L^2(\mathbb{R}, \mathbb{C})$	complex-valued square integrable functions on $\mathbb{R}$
$\ell^2(\mathbb{N}, \mathbb{C})$	complex-valued square summable sequences
$\mathfrak{B}(\mathfrak{h})$	bounded operators on a Hilbert space $\mathfrak{h}$
$\mathfrak{T}(\mathfrak{h})$	bounded operators on a Hilbert space $\mathfrak{h}$
$\Gamma(\mathfrak{h})$	Fock space (bosonic) over a Hilbert space $\mathfrak{h}$
$\mathfrak{S}_n$	Permutations on $\{1, 2, \dots, n\}$
$\Gamma(\mathfrak{h})$	Fock space over a Hilbert space $\mathfrak{h}$

$\Gamma^{+}(\mathfrak{h})$	Boson Fock space over a Hilbert space $\mathfrak{h}$
$\Gamma^{-}(\mathfrak{h})$	Fermion Fock space over a Hilbert space $\mathfrak{h}$
$\otimes$	tensor product
$\vee$	symmetric tensor product
$\wedge$	antisymmetric tensor product

Combinatorics

$x^{\underline{n}}$	falling factorial powers, i.e., $x(x-1)(x-2)\cdots(x-n+1)$
$\left[{n \atop m} \right]$	Stirling number of the first kind
$\left\{ {n \atop m} \right\}$	Stirling number of the second kind
Perm(X)	permutations on a set X
Part(X)	partitions of a set X
Pair(X)	pair partitions of a set X
Power(X)	power set (set of finite subsets) of a set X
$H_n(x)$	Hermite polynomial, $= \sum_{k=1}^{[n/2]} \frac{(-1)^k n!}{2^k k!(n-2k)!} x^{n-2k}$
$\diamond$	Wick product

Quantum Mechanics

$\psi(x)$	wavefunction
$\vert\psi\rangle$	state vector/ket
ρ	density operator
$\langle\psi\vert\phi\rangle$	Dirac notation for inner product
P	projection
$D(\beta)$	Weyl displacement operator
$\vert\exp\beta\rangle$	Exponential vector (Bargmann state)
$\circledast$	group product for the Heisenberg group
$:X:$	Wick (normal) ordering of an operator X

Classical Probability

$\mathbf{P}$	probability measure
$\mathbf{E}$	expectation
$\mathcal{A}$	σ-algebra
Ω	sample space
χ_E	indicator function for event E
$[[X,Y]]_t$	the quadratic covariance, that is, $\int_0^t dX_s.dY_s$
$X_t^{[n]}$,	the process $\int_0^t (dX_s)^n$

$⨍_{[0,t]^n} dX^{(n)}_{t_n} \dots dX^{(1)}_{t_1}$	the diagonal-free stochastic integral
$\not{e}^{X_t}$	the diagonal-free stochastic exponential
$X\,\delta Y$	Stratonovich differential (i.e., $X\,dY + \frac{1}{2}dX\,dY$).
$\vec{T}_S e^{Z_t}$	Stratonovich time-ordered exponential

Quantum Field Theory

Φ	space of all field configurations
$\mathfrak{J}$	space of all source fields (= test functions in the dual of Φ)
$\phi_X = \prod_{x\in X} \phi(x)$	
G_X	Green's functions (moments = expectation of ϕ_X)
K_X	connected Green's functions (= cumulants)
$A(f)$	annihilation operator with test function f
$D(f)$	Weyl unitary with test function f
$W(f) \equiv D(if)$	
$\Gamma(U)$	second quantization of a unitary U
$d\Gamma(H)$	differential second quantization
$\hat{\Phi}(f)$	Segal's field operator, $= A(f) + A^*(f)$
$\mathcal{P}$, $\mathcal{P}^{\uparrow}$	Poincaré group, the orthochronous Poincaré group