CHAPTER 1

INTRODUCTION TO SOBOLEV SPACES

In this chapter we recall some basics on functional analysis and provide a brief introduction to Sobolev spaces. For a more detailed and comprehensive study, we refer to Adams (1975).

1.1. Banach and Hilbert spaces

Definition 1.1. Let X be a real linear space. A mapping $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if

- $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$ for all $x \in X$,
- $\|\lambda x\| = |\lambda|\,\|x\|$ for all $x \in X$, $\lambda \in \mathbb{R}$,
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a normed space.

Setting $y = -x$ in the inequality and using the other two properties we get $\|x\| \geq 0$ for all $x \in X$.

Definition 1.2. A sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence, if for all $\varepsilon > 0$ there exists an index $n_0(\varepsilon)$ such that for all $m, n > n_0(\varepsilon)$ it holds $\|x_m - x_n\| < \varepsilon$.

Definition 1.3. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ if for all $\varepsilon > 0$ there is an index $n_0(\varepsilon)$ such that for all $n > n_0(\varepsilon)$ it holds $\|x_n - x\| < \varepsilon$.

Definition 1.4. A normed space $(X, \|\cdot\|)$ is called complete if every Cauchy sequence in X converges in X.

Definition 1.5. A complete normed space is called Banach space.

Example 1.1. Let $\Omega \subset \mathbb{R}^d$, d is the dimension, be an open and bounded domain. We denote by $L^p(\Omega)$, $1 \le p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which

$$\int_\Omega |f(x)|^p \, \mathrm{d}x < \infty.$$

Similarly, the set of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\operatorname{ess\,sup}\{\, |f(x)| \,:\, x \in \Omega \,\} < \infty$$

will be denoted by $L^\infty(\Omega)$. Then, $L^p(\Omega)$, $p \in [1,\infty]$ is a real linear space. Setting for $1 \le p < \infty$

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}\{\, |f(x)| \,:\, x \in \Omega \},$$

respectively, $L^p(\Omega)$ becomes a normed space for any $p \in [1,\infty]$ if we identify functions which are equal up to a set of measure zero. This identification is necessary because we only have for a set $M \subset \Omega$ of measure zero

$$\int_\Omega |f(x)|^p \, \mathrm{d}x = 0 \qquad \Rightarrow \qquad f(x) = 0 \text{ for all } x \in \Omega \backslash M.$$

Strictly speaking, the elements in $L^p(\Omega)$, $p \in [1,\infty]$ are classes of equivalent functions (equal up to a set of measure zero). Then, the first condition in Definition 1.1 becomes true. The second condition follows directly from the definition of $\|\cdot\|_{L^p(\Omega)}$ and the third condition is just the inequality stated in Lemma 1.1.

Lemma 1.1 (Minkowski's inequality). *For $f, g \in L^p(\Omega)$, $p \in [1,\infty]$, we have*

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Theorem 1.1. *$L^p(\Omega)$, $p \in [1,\infty]$, is a Banach space.*

Lemma 1.2 (Hölders's inequality). *For $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p, q \in [1,\infty]$, $1/p + 1/q = 1$, we have $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \, \|g\|_{L^q(\Omega)}.$$

Definition 1.6. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a normed space X are called equivalent if there exist positive constants C_1, C_2 such that

$$C_1 \|x\|_1 \le \|x\|_2 \le C_2 \|x\|_1 \quad \text{for all } x \in X.$$

Note that topological properties do not change when switching to an equivalent norm, for example, a sequence is convergent in $(X, \|\cdot\|_1)$ iff it is convergent in $(X, \|\cdot\|_2)$.

Definition 1.7. Let $(X, \|\cdot\|_X)$ be a normed space. A mapping $g : X \to \mathbb{R}$ is called linear if

$$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y) \quad \text{for all } \alpha, \beta \in \mathbb{R}, \;\; x, y \in X.$$

A linear mapping $g : X \to \mathbb{R}$ is continuous if there is a constant C such that

$$|g(x)| \le C \|x\|_X \quad \text{for all } x \in X.$$

Definition 1.8. We define the sum of two continuous linear functionals, g_1 and g_2, and the multiplication of a continuous linear functional g with a real number α by

$$(g_1+g_2)(x) := g_1(x)+g_2(x) \quad \text{and} \quad (\alpha g)(x) := \alpha g(x), \quad \alpha \in \mathbb{R}, x \in X.$$

Then, it can be shown that the set of all continuous linear functionals create a real linear space, the dual space X^*. In the following, for the elements $g \in X^*$ we use the notation $\langle g, x\rangle := g(x)$.

Lemma 1.3. *The set X^* of continuous linear functionals $x \mapsto \langle g, x\rangle$ on X is a Banach space with respect to the norm*

$$\|g\|_{X^*} := \sup_{0\neq x\in X} \frac{|\langle g, x\rangle|}{\|x\|_X}.$$

Proof. Using the definition of the dual norm $\|\cdot\|_{X^*}$, we see

$$\|g\|_{X^*} = 0 \quad \Leftrightarrow \quad \langle g, x\rangle = 0 \text{ for all } x \in X \quad \Leftrightarrow \quad g = 0.$$

Further, we have

$$\|\lambda g\|_{X^*} = \sup_{0\neq x\in X} \frac{|\langle \lambda g, x\rangle|}{\|x\|_X} = \sup_{0\neq x\in X} \frac{|\lambda\langle g, x\rangle|}{\|x\|_X} = |\lambda| \sup_{0\neq x\in X} \frac{|\langle g, x\rangle|}{\|x\|_X} = |\lambda|\,\|g\|_{X^*}$$

and for the sum of two functionals it holds

$$\begin{aligned}\|g_1+g_2\|_{X^*} &= \sup_{0\neq x\in X} \frac{|\langle g_1+g_2, x\rangle|}{\|x\|_X} = \sup_{0\neq x\in X} \frac{|\langle g_1, x\rangle + \langle g_2, x\rangle|}{\|x\|_X} \\ &\leq \sup_{0\neq x\in X} \frac{|\langle g_1, x\rangle|}{\|x\|_X} + \sup_{0\neq x\in X} \frac{|\langle g_2, x\rangle|}{\|x\|_X} = \|g_1\|_{X^*} + \|g_2\|_{X^*}\end{aligned}$$

It remains to show that $(X^*, \|\cdot\|_{X^*})$ is complete. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X^*, that is, for all $\varepsilon > 0$ there is an index $n_0(\varepsilon)$ such that for all $m, n > n_0(\varepsilon)$ it holds $\|g_m - g_n\|_{X^*} < \varepsilon$. Then, for a fixed $x \in X$, the sequence of real numbers $(\langle g_n, x\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ due to

$$|\langle g_m, x\rangle - \langle g_n, x\rangle| = |\langle g_m - g_n, x\rangle| \leq \|g_m - g_n\|_{X^*}\,\|x\|_X < \varepsilon\|x\|_X$$

for all $m, n > n_0(\varepsilon)$. Any Cauchy sequence of real numbers is convergent in $\mathbb{R}$, that means $\lim_{n\to\infty}\langle g_n, x\rangle = g(x)$ for all $x \in X$. The linearity of g follows from

$$\begin{aligned}g(\alpha x+\beta y) &= \lim_{n\to\infty}\langle g_n, \alpha x+\beta y\rangle = \lim_{n\to\infty}\big(\alpha\langle g_n, x\rangle + \beta\langle g_n, y\rangle\big) \\ &= \alpha \lim_{n\to\infty}\langle g_n, x\rangle + \beta \lim_{n\to\infty}\langle g_n, y\rangle = \alpha g(x) + \beta g(y).\end{aligned}$$

Concerning the continuity of g we recall that Cauchy sequences are bounded, thus

$$|\langle g_n, x\rangle| \leq \|g_n\|_{X^*}\,\|x\|_X \leq C\|x\|_X \quad \text{for all } x \in X.$$

The limit $n \to \infty$ shows that the linear mapping $g : X \to \mathbb{R}$ is continuous. ■

Definition 1.9. Let V be a linear space. A mapping $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called an inner product on V if

- $(u,v) = (v,u)$ for all $u,v \in V$,
- $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$ for all $u,v,w \in V$, $\alpha,\beta \in \mathbb{R}$,
- $(u,u) > 0$ for $u \neq 0$.

A norm on V is induced by setting $\|u\| := \sqrt{(u,u)}$. If $(V, \|\cdot\|_V)$ is complete we call V a Hilbert space.

Lemma 1.4 (Schwarz inequality)**.** *Let V be a Hilbert space. Then,*

$$|(u,v)| \leq \|u\|_V \, \|v\|_V \quad \textit{for all } u,v \in V.$$

Example 1.2. $V = L^2(\Omega)$ equipped with the inner product

$$(f,g) := \int_\Omega f(x)g(x)\,\mathrm{d}x$$

is a Hilbert space.

Theorem 1.2 (Riesz representation theorem)**.** *Let V be a Hilbert space. There is a unique linear mapping $R : V^* \to V$ from the dual space V^* into the Hilbert space V such that for all $g \in V^*$ and $v \in V$*

$$(Rg, v) = \langle g, v\rangle \quad \textit{and} \quad \|Rg\|_V = \|g\|_{V^*}.$$

Fixed point theorems are often used to establish the unique solvability of an abstract operator equation. Let us consider the following problem in a Banach space V with an operator $P : V \to V$:

$$\text{Find } u \in V \text{ such that } u = Pu. \tag{1.1}$$

Solutions $u \in V$ of Problem (1.1) are called fixed points of the mapping $P : V \to V$.

Definition 1.10. The mapping $P : V \to V$ is called contractive or to be a contraction if there is a positive constant $\rho < 1$ such that

$$\|Pv_1 - Pv_2\|_V \leq \rho \, \|v_1 - v_2\|_V \quad \text{for all } v_1, v_2 \in V.$$

Theorem 1.3 (Banach's fixed point theorem)**.** *Let V be a Banach space and $P : V \to V$ be a contraction. Then, we have the following statements:*

(1) *There is a unique fixed point $u^* \in V$ solving Problem* (1.1).
(2) *The sequence $u_n = Pu_{n-1}$, $n \geq 1$, converges to the fixed point $u^* \in V$ for any initial guess $u_0 \in V$.*
(3) *It holds the error estimate*

$$\|u_n - u^*\|_V \leq \frac{\rho^n}{1-\rho} \|Pu_0 - u_0\|_V \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We start by estimating the distance of $u_i \in V$ to $u_{i-1} \in V$

$$\begin{aligned} \|u_i - u_{i-1}\|_V &= \|Pu_{i-1} - Pu_{i-2}\|_V \le \rho \|u_{i-1} - u_{i-2}\|_V \le \rho^2 \|u_{i-2} - u_{i-3}\|_V \\ \vdots \quad &\le \quad \vdots \\ \|u_i - u_{i-1}\|_V &\le \rho^{i-1} \|Pu_0 - u_0\|_V \qquad \text{for all } i \in \mathbb{N}. \end{aligned}$$

From that we get for $m > n \ge 1$

$$\begin{aligned} \|u_m - u_n\|_V &= \left\| \sum_{i=n+1}^{m} (u_i - u_{i-1}) \right\|_V \le \sum_{i=n+1}^{m} \|u_i - u_{i-1}\|_V \\ &\le \sum_{i=n+1}^{m} \rho^{i-1} \|Pu_0 - u_0\|_V \le \rho^n \frac{1-\rho^{m-n}}{1-\rho} \|Pu_0 - u_0\|_V \\ &\le \frac{\rho^n}{1-\rho} \|Pu_0 - u_0\|_V \to 0 \quad \text{for } n \to \infty. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists an index bound $n_0(\varepsilon)$ such that for all $m > n \ge n_0(\varepsilon)$

$$\|u_m - u_n\|_V \le \frac{\rho^n}{1-\rho} \|Pu_0 - u_0\|_V < \varepsilon,$$

consequently, (u_n) is a Cauchy sequence and converges to some $u^* \in V$. We show that u^* is a fixed point of P. Indeed,

$$\begin{aligned} \|u^* - Pu^*\|_V &\le \|u^* - u_n\|_V + \|Pu_{n-1} - Pu^*\|_V \\ &\le \|u^* - u_n\|_V + \rho \|u_{n-1} - u^*\|_V \to 0 \quad \text{for } n \to \infty. \end{aligned}$$

Next we prove that there is at most one fixed point. Assume that there are two fixed points $u_1^* \ne u_2^*$. Then, we conclude

$$\|u_1^* - u_2^*\|_V = \|Pu_1^* - Pu_2^*\|_V \le \rho \|u_1^* - u_2^*\|_V \quad \Rightarrow \quad (1-\rho) \|u_1^* - u_2^*\|_V \le 0.$$

Since $\rho < 1$ we get $u_1^* = u_2^*$. Finally, we recall the estimate for $m > n \ge 1$

$$\|u_n - u^*\|_V \le \|u_n - u_m\|_V + \|u_m - u^*\|_V \le \frac{\rho^n}{1-\rho} \|Pu_0 - u_0\|_V + \|u_m - u^*\|_V$$

from which the last statement follows by $m \to \infty$. ■

1.2. Weak derivatives

First we introduce a generalization of the integration by parts formula known for smooth functions $u, v : [a, b] \to \mathbb{R}$:

$$\int_a^b u' v \, \mathrm{d}x = uv|_a^b - \int_a^b uv' \, \mathrm{d}x.$$

We start with a characterization of geometric properties of a domain $\Omega \subset \mathbb{R}^d$.

Definition 1.11. The domain $\Omega \subset \mathbb{R}^d$ is said to have a Lipschitz continuous boundary, if for every point of the boundary $\Gamma = \partial\Omega$ there exists a local coordinate system in which the boundary corresponds to some hypersurface with the domain Ω lying on one side of that surface and $\partial\Omega$ can locally be represented by the graph of a Lipschitz continuous mapping.

Theorem 1.4 (Gauss theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary Γ and outer normal $\mathbf{n}$. Then, for $\mathbf{w} \in C^1(\overline{\Omega})^d$ we have*

$$\int_\Omega \operatorname{div} \mathbf{w} \, \mathrm{d}x = \int_\Omega \sum_{i=1}^d \frac{\partial w_i}{\partial x_i} \, \mathrm{d}x = \int_\Gamma \mathbf{w} \cdot \mathbf{n} \, \mathrm{d}\gamma \, .$$

Let $j \in \{1, 2, \ldots, d\}$ be a fixed index. The formula of integration by parts follows by setting $w_i = uv$ for $i = j$ and $w_i = 0$ for $i \neq j$, and using the product rule.

Lemma 1.5 (Integration by parts)**.** *For $u, v \in C^1(\overline{\Omega})$ we have*

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, \mathrm{d}x = \int_\Gamma u v n_j \, \mathrm{d}\gamma - \int_\Omega u \frac{\partial v}{\partial x_j} \, \mathrm{d}x, \quad j = 1, \ldots, d.$$

Now we have the tools to generalize the concept of differentiability.

Definition 1.12. The support of a function $f : \Omega \to \mathbb{R}$ is defined by

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The set of all functions that are differentiable of any order with compact support in Ω will be denoted by $C_0^\infty(\Omega)$.

For $\Omega = (0, 1)$, the function $x \mapsto x(1 - x)$ does not belong to $C_0^\infty(\Omega)$ because its support is $[0, 1] \not\subset (0, 1)$. Indeed, functions in $C_0^\infty(\Omega)$ are zero in the neighbourhood of the boundary $\partial\Omega$. As a consequence, a function $f \in C_0^\infty(\Omega)$ can be extended to a smooth function defined on $\mathbb{R}^d$ by setting $f = 0$ outside of Ω.

Definition 1.13. We define

$$L^1_{\mathrm{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} : f \in L^1(A) \text{ for all compact } A \subset \Omega\}.$$

For bounded domains Ω, we have $L^2(\Omega) \subset L^1(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$.

Example 1.3. Consider $\Omega = (0, 1)$. The mapping $x \mapsto 1/x$ belongs to $L^1_{\mathrm{loc}}(\Omega)$ but not to $L^1(\Omega)$. The mapping $x \mapsto 1/\sqrt{x}$ belongs to $L^1(\Omega)$ but not to $L^2(\Omega)$.

Definition 1.14. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index, that is, α_i are non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. A function, $f \in L^1_{\mathrm{loc}}(\Omega)$ has a weak derivative $v = D^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ if for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega v \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \, \mathrm{d}x.$$

The notion of a weak derivative generalizes the classical partial derivatives. Indeed, a partial derivative in the classical sense satisfies the above condition by partial integration.

Example 1.4. Let $f : (-1,+1) \to \mathbb{R}$ with $f(x) = 2 - \sqrt{|x|}$. Then, the generalized derivative $Df \in L^1_{\text{loc}}(-1,+1)$ is given by

$$Df(x) = -\frac{\operatorname{sgn} x}{2\sqrt{|x|}} \quad x \in (-1,0) \cup (0,+1).$$

Proof. Applying integration by parts separately in the two subintervals, we get for $\varphi \in C_0^\infty(-1,+1)$

$$\begin{aligned}
-\int_{-1}^{+1} \frac{\operatorname{sgn} x}{2\sqrt{|x|}} \varphi \, \mathrm{d}x &= \int_{-1}^{0} \frac{1}{2\sqrt{|x|}} \varphi \, \mathrm{d}x - \int_{0}^{+1} \frac{1}{2\sqrt{|x|}} \varphi \, \mathrm{d}x \\
&= \left(2 - \sqrt{|x|}\right) \varphi|_{x=-1}^{0} - \int_{-1}^{0} \left(2 - \sqrt{|x|}\right) \varphi' \, \mathrm{d}x \\
&\quad + \left(2 - \sqrt{|x|}\right) \varphi|_{x=0}^{+1} - \int_{0}^{+1} \left(2 - \sqrt{|x|}\right) \varphi' \, \mathrm{d}x \\
&= -\int_{-1}^{+1} \left(2 - \sqrt{|x|}\right) \varphi' \, \mathrm{d}x,
\end{aligned}$$

where we used the continuity of φ at $x = 0$ and $\varphi(\pm 1) = 0$. ■

1.3. Sobolev spaces

Definition 1.15. We denote by $W^{m,p}(\Omega)$, $1 \le p \le \infty$, $m \ge 0$, the set of all functions, $f \in L^p(\Omega)$ with weak derivatives $D^\alpha f \in L^p(\Omega)$ up to the order $|\alpha| \le m$. The sets $W^{m,p}(\Omega)$ are called Sobolev spaces.

Lemma 1.6. *The Sobolev space $W^{m,p}(\Omega)$ equipped with the norm*

$$\|f\|_{W^{m,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha f(x)|^p \, \mathrm{d}x \right)^{1/p} \qquad \text{if } 1 \le p < \infty$$

and

$$\|f\|_{W^{m,p}(\Omega)} := \max_{|\alpha| \le m} \left(\operatorname{ess\,sup}_{x \in \Omega} |D^\alpha f(x)| \right) \qquad \text{if } p = \infty$$

is a Banach space. In particular, the Sobolev space $H^m(\Omega) := W^{m,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_{H^m(\Omega)} := \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v) = \sum_{|\alpha| \le m} \int_\Omega D^\alpha u \, D^\alpha v \, \mathrm{d}x \quad \text{for all } u, v \in H^m(\Omega).$$

Proof. Here, we give the arguments for $1 \le p < \infty$. For the case $p = \infty$, we refer to Adams (1975). The sets $W^{m,p}(\Omega)$ are real linear spaces (below we show that the sum $(f+g)$ of two elements $f,g \in W^{m,p}(\Omega)$ is an element of $W^{m,p}(\Omega)$). Next we have to show that the mapping $f \mapsto \|f\|_{W^{m,p}(\Omega)}$ is a norm, that is, the three conditions in Definition 1.1 hold. If $f = 0$, then $\|f\|_{W^{m,p}(\Omega)} = 0$ by definition of $\|\cdot\|_{W^{m,p}(\Omega)}$. Further, since $0 = \|f\|_{W^{m,p}(\Omega)} \ge \|f\|_{L^p(\Omega)}$ we conclude $f = 0$ in the sense of $L^p(\Omega)$, that is, functions which are equal up to a set of measure zero are identified. Thus, the first condition in Definition 1.1 holds. The second condition follows directly from the definition

$$\|\lambda f\|_{W^{m,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha \lambda f(x)|^p \,\mathrm{d}x\right)^{1/p} = \left(\int_\Omega |\lambda|^p \sum_{|\alpha|\le m} |D^\alpha f(x)|^p \,\mathrm{d}x\right)^{1/p}$$
$$= |\lambda| \left(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha f(x)|^p \,\mathrm{d}x\right)^{1/p} = |\lambda|\, \|f\|_{W^{m,p}(\Omega)}.$$

Let $f,g \in W^{m,p}(\Omega)$ and $|\alpha| \le m$. The Minkowski's inequality, Lemma 1.1, implies that $D^\alpha(f+g) = D^\alpha f + D^\alpha g \in L^p(\Omega)$ and

$$\|D^\alpha(f+g)\|_{L^p(\Omega)} \le \|D^\alpha f\|_{L^p(\Omega)} + \|D^\alpha g\|_{L^p(\Omega)}.$$

Using the Minkowski's inequality for sums, we get

$$\|f+g\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha(f+g)\|^p_{L^p(\Omega)} \le \sum_{|\alpha|\le m} \left(\|D^\alpha f\|_{L^p(\Omega)} + \|D^\alpha g\|_{L^p(\Omega)}\right)^p$$
$$\le \left[\left(\sum_{|\alpha|\le m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} + \left(\sum_{|\alpha|\le m} \|D^\alpha g\|^p_{L^p(\Omega)}\right)^{1/p}\right]^p$$
$$\le \left[\|f\|_{W^{m,p}(\Omega)} + \|g\|_{W^{m,p}(\Omega)}\right]^p$$

from which the third condition follows. It remains to prove the completeness of the normed space $W^{m,p}(\Omega)$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then, $(D^\alpha f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ for any multi-index α with $|\alpha| \le m$. Since $L^p(\Omega)$ is a Banach space, there is $f^\alpha \in L^p(\Omega)$ with $D^\alpha f_n \to f^\alpha$ in $L^p(\Omega)$. It remains to show that $f^\alpha = D^\alpha f$ where $f = f^0$. We show this for $\alpha = (1,0,\ldots,0)$. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$\left|\int_\Omega (f^\alpha \varphi + f D^\alpha \varphi)\mathrm{d}x\right| \le \int_\Omega |f^\alpha - D^\alpha f_n|\,|\varphi|\mathrm{d}x + \left|\int_\Omega (D^\alpha f_n \varphi + f_n D^\alpha \varphi)\mathrm{d}x\right|$$
$$+ \int_\Omega |f - f_n|\,|D^\alpha \varphi|\mathrm{d}x.$$

The second term on the right hand side equals zero since $D^\alpha f_n$ is the weak derivative of f_n. The convergence of f_n to f and $D^\alpha f_n$ to f^α in $L^p(\Omega)$, respectively, imply the convergence in $L^1(\Omega)$, thus the first and third term on the right hand side tend to zero for $n \to \infty$. The left hand side

is non-negative and independent of n, consequently we conclude that it is zero, which means f^α is the weak derivative of f.
The properties for the inner product in $H^m(\Omega)$, see Definition 1.9, follow from its definition and the observation that $(v,v)_{H^m(\Omega)} = \|v\|^2_{W^{m,2}(\Omega)}$ for all $v \in W^{m,2}(\Omega)$. ■

Definition 1.16. We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. Analogously, $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

In the following, for $f \in W^{m,p}(\omega)$, $m \geq 0$, $1 \leq p \leq \infty$, the norm will be shortly denoted by $\|f\|_{m,p,\omega} := \|f\|_{W^{m,p}(\omega)}$ and a seminorm is introduced by

$$|f|_{m,p,\omega} := \left(\int_\omega \sum_{|\alpha|=m} |D^\alpha f(x)|^p \, \mathrm{d}x \right)^{1/p} \qquad \text{if } 1 \leq p < \infty$$

and

$$|f|_{m,p,\omega} := \max_{|\alpha|=m} \left(\operatorname{ess\,sup}_{x\in\Omega} |D^\alpha f(x)| \right) \qquad \text{if } p = \infty.$$

In case, $\omega = \Omega$ and/or $p = 2$ we omit Ω and p, respectively. Thus,

$$\|f\|_m = \|f\|_{m,2,\Omega}, \quad \|f\|_{m,p} = \|f\|_{m,p,\Omega}, \quad |f|_m = |f|_{m,2,\Omega}, \quad |f|_{m,p} = |f|_{m,p,\Omega}.$$

Lemma 1.7 (Poincaré inequality)**.** *There is a positive constant $c_P \leq 2a$ such that*

$$\|v\|_{0,p} \leq c_P |v|_{1,p} \quad \textit{for all } v \in W_0^{1,p}(\Omega).$$

Here, 'a' denotes the diameter of Ω in an arbitrary but fixed direction.

Proof. Suppose the Poincaré inequality is true for functions from $C_0^\infty(\Omega)$. Then, for any $v \in W_0^{1,p}(\Omega)$ there is a sequence $v_n \in C_0^\infty(\Omega)$ with $v_n \to v$ in $W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \|v\|_{0,p} &\leq \|v - v_n\|_{0,p} + \|v_n\|_{0,p} \leq \|v - v_n\|_{0,p} + c_P |v_n|_{1,p} \\ &\leq (1 + c_P)\|v - v_n\|_{1,p} + c_P |v|_{1,p}. \end{aligned}$$

Since $\|v - v_n\|_{1,p}$ tends to zero for $n \to \infty$, the Poincaré inequality is true for functions from $W_0^{1,p}(\Omega)$.

In order to prove the Poincaré inequality for functions $v \in C_0^\infty(\Omega)$ we extend $v : \Omega \to \mathbb{R}$ by setting $v(x) = 0$ for all $x \in \mathbb{R}^d \backslash \Omega$. Let $\Omega \subset [-a, +a] \times \mathbb{R}^{d-1}$, then

$$v(x) = \int_{-a}^{x_1} \frac{\partial v}{\partial x_1}(s, x_2, \ldots, x_d) \, \mathrm{d}s$$

and Hölder's inequality with $1/p + 1/q = 1$ implies

$$|v(x)| \le \left(\int_{-a}^{x_1} \left| \frac{\partial v}{\partial x_1}(s, x_2, \ldots, x_d) \right|^p \mathrm{d}s \right)^{1/p} \left(\int_{-a}^{x_1} 1^q \, \mathrm{d}s \right)^{1/q},$$

$$|v(x)|^p \le (2a)^{p/q} \int_{-a}^{+a} \left| \frac{\partial v}{\partial x_1}(s, x_2, \ldots, x_d) \right|^p \mathrm{d}s,$$

$$\int_{-a}^{+a} |v(x)|^p \, \mathrm{d}x_1 \le (2a)^{1+p/q} \int_{-a}^{+a} \left| \frac{\partial v}{\partial x_1}(s, x_2, \ldots, x_d) \right|^p \mathrm{d}s,$$

$$\int_{\mathbb{R}^d} |v(x)|^p \, \mathrm{d}x \le (2a)^p \int_{\mathbb{R}^d} \left| \frac{\partial v}{\partial x_1}(x) \right|^p \mathrm{d}x.$$

Using $v(x) = 0$ for $x \in \mathbb{R}^d \backslash \Omega$ the inequality follows. ■

The Poincaré inequality allows to show that the seminorm $|\cdot|_{m,p}$ is equivalent to the norm $\|\cdot\|_{m,p}$ on $W_0^{m,p}(\Omega)$. Let $v \in W_0^{m,p}(\Omega)$, then $D^\alpha v \in W_0^{1,p}(\Omega)$ for $|\alpha| \le m-1$. We apply successively the inequality

$$|D^\alpha v|_{0,p} \le c_{\mathrm{P}} |D^\alpha v|_{1,p} \quad \text{for } |\alpha| = 0, 1, \ldots, m-1$$

to estimate the lower derivatives by the highest derivative and obtain

$$|v|_{m,p} \le \|v\|_{m,p} = \left(\sum_{|\alpha| \le m} |D^\alpha v|_{0,p}^p \right)^{1/p} \le C |v|_{m,p},$$

that is, the equivalence of the seminorm to the norm.

An important tool will be the Sobolev embedding theorems which state that functions from $W^{m,p}(\Omega)$ belong to W^{m-1,p^*} for some $p^* > p$. Moreover, it can be shown that for suitable numbers m, p functions from $W^{m,p}(\Omega)$ belong to classical spaces of continuous and continuously differentiable functions, respectively.

Definition 1.17. We introduce the space of Hölder continuous functions with Hölder exponent $\beta \in (0, 1]$

$$C^{0,\beta}(\overline{\Omega}) := \left\{ u \in C^0(\overline{\Omega}) : \sup_{x,y \in \Omega, x \ne y} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty \right\}$$

equipped with the norm

$$\|v\|_{C^{0,\beta}(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)| + \sup_{x,y \in \Omega, x \ne y} \frac{|u(x) - u(y)|}{|x - y|^\beta}.$$

Hölder continuous functions with a Hölder exponent $\beta = 1$ are also called Lipschitz continuous.

Exercise 1.1. Consider $f : [-1, +1] \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$. Prove that $f \in C^{0,1/2}([-1, +1])$ and that there is no $\beta \in (1/2, +1]$ with $f \in C^{0,\beta}([-1, +1])$.