16

The action of $\mathsf{GL}(n)$ on flags

16.0 Introduction

This chapter begins a more systematic approach to the study of the polynomial algebra $\mathsf{P}(n)$ as a $\mathbb{F}_2\mathsf{GL}(n)$-module, based on the flag module $\mathsf{FL}(n)$. We have seen in Chapter 10 that a close approximation to the $\mathbb{F}_2\mathsf{GL}(3)$-modules $\mathsf{Q}(3)$ and $\mathsf{K}(3)$ is given by quotients of $\mathsf{FL}(3)$, the permutation module of dimension 21 given by the action of $\mathsf{GL}(3)$ on right cosets of the lower triangular subgroup $\mathsf{L}(3)$.

We identify elements of the defining module $\mathsf{V}(n)$ for $\mathbb{F}_2\mathsf{GL}(n)$ with row vectors v, so that $A \in \mathsf{GL}(n)$ acts by $v \to v \cdot A = vA$. A (complete) flag X in $\mathsf{V}(n)$ is a nested sequence of subspaces, one of each dimension. There is a natural correspondence between flags and right cosets $\mathsf{L}(n)A$ of the lower triangular subgroup $\mathsf{L}(n)$. The module $\mathsf{FL}(n)$ is defined by the permutation action $(\mathsf{L}(n)A) \cdot B = \mathsf{L}(n)AB$, where $A, B \in \mathsf{GL}(n)$, on these cosets. It can be alternatively described as the representation of $\mathsf{GL}(n)$ induced from the trivial 1-dimensional representation of $\mathsf{L}(n)$. If the base field $\mathbb{F}_2$ is replaced by $\mathbb{F}_p$ for a general prime p, then this remains true if we take $\mathsf{L}(n)$ to be the Borel subgroup $\mathsf{B}(n, \mathbb{F}_p)$ of all lower triangular matrices, rather than $\mathsf{L}(n, \mathbb{F}_p)$, the subgroup of lower triangular matrices with 1s on the diagonal. For this reason we shall use the alternative notation $\mathsf{B}(n)$ for $\mathsf{L}(n)$ in the case $p = 2$ also.

Much of the combinatorial structure of the group $\mathsf{GL}(n)$ is determined by the Weyl subgroup $\mathsf{W}(n)$ of permutation matrices. We distinguish $\mathsf{W}(n)$ from the symmetric group $\Sigma(n)$ of permutations of the set $Z[n] = \{1, 2, \ldots, n\}$. Because we work with the right action of $\mathsf{GL}(n)$ on $\mathsf{V}(n)$, we associate to a permutation $\rho \in \Sigma(n)$ the matrix W obtained by applying ρ to the *columns* of the identity matrix I_n. This fixes an anti-isomorphism between $\Sigma(n)$ and $\mathsf{W}(n)$. In Section 16.1 we treat the length and descent set of a permutation ρ, and the Bruhat order on $\Sigma(n)$, from the point of view of permutation matrices.

In Section 16.2 we discuss the Bruhat decomposition $A = BWB'$ of $A \in \mathsf{GL}(n)$, where $B, B' \in \mathsf{B}(n)$ and $W \in \mathsf{W}(n)$. Since W is uniquely determined by A, we obtain a decomposition of $\mathsf{GL}(n)$ as the disjoint union of the double cosets $\mathsf{B}(n)W\mathsf{B}(n)$, or 'Bruhat cells'. Although the matrices B and B' are not unique in general, we obtain a unique decomposition by restricting B' to be in a subgroup $\mathsf{B}(W) \subseteq \mathsf{B}(n)$ determined by W, which we call a 'Bruhat subgroup'. In Section 16.3 we define a natural correspondence between cosets $\mathsf{B}(n)A$ and complete flags in $\mathsf{V}(n)$, and introduce the flag module $\mathsf{FL}(n)$. The Bruhat decomposition of $\mathsf{GL}(n)$ corresponds to a decomposition of $\mathsf{FL}(n)$ into $\mathbb{F}_2$-subspaces $\mathsf{Sch}(W)$, called Schubert cells.

In the case $n = 2$, the module $\mathsf{FL}(2)$ gives the representation of $\mathsf{GL}(2) \cong \Sigma(3)$ which permutes the three nonzero elements $u = v_1, v = v_2$ and $w = v_1 + v_2$ of $\mathsf{V}(2)$. The elements of $\mathsf{FL}(2)$ correspond to formal sums of u, v, w, and $\mathsf{FL}(2)$ is isomorphic to the direct sum of the 1-dimensional module generated by $u + v + w$ and the 2-dimensional module generated by $u + v$. In the general case, $\mathsf{FL}(n)$ is the direct sum of 2^{n-1} indecomposable submodules indexed by subsets $I \subseteq Z[n-1] = \{1, \ldots, n-1\}$, which correspond to 'partial' flags of type I, i.e. nested sequences of subspaces with dimensions in I.

The partial flag module $\mathsf{FL}^I(n)$ is introduced in Section 16.4. It is given by the permutation action of $\mathsf{GL}(n)$ on cosets $\mathsf{P}^I(n)A$, where $\mathsf{P}^I(n)$ is the parabolic subgroup of type I containing $\mathsf{B}(n)$. The elements of $\mathsf{P}^I(n)$ are block lower triangular matrices. For $J \subseteq I \subseteq Z[n-1]$, $\mathsf{FL}^J(n)$ can be embedded as a direct summand in $\mathsf{FL}^I(n)$ by associating to a partial flag of type J the sum of all partial flags of type I which contain it, and so we obtain a quotient module $\mathsf{FL}_I(n)$ of $\mathsf{FL}^I(n)$ by factoring out all such summands for $J \subset I$. In Section 16.5 we show that $\mathsf{FL}(n)$ is the direct sum of the modules $\mathsf{FL}_I(n)$. We shall prove in Chapter 17 that the modules $\mathsf{FL}_I(n)$ are indecomposable, so that this decomposition of $\mathsf{FL}(n)$ is maximal. In Section 16.6, we give a more detailed treatment of the module $\mathsf{FL}(3)$.

16.1 Permutation matrices

Let $\mathsf{W}(n) \subset \mathsf{GL}(n)$ be the subgroup of **permutation matrices**, i.e. matrices with one entry 1 in each row and column and all other entries 0. The defining module $\mathsf{V}(n)$ is the right $\mathbb{F}_2\mathsf{GL}(n)$-module obtained by identifying elements of $\mathbb{F}_2^n$ with row vectors, so that $\mathsf{GL}(n)$ acts by matrix multiplication. In particular, a permutation matrix $W = (w_{i,j})$ acts on $\mathsf{V}(n)$ by $x_i \cdot W = x_j$ if $w_{i,j} = 1$. We denote the symmetric group of permutations of $Z[n] = \{1, 2, \ldots, n\}$ by $\Sigma(n)$, and we associate to W the permutation ρ defined by $w_{i,j} = 1$ if and only if $\rho(i) = j$.

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We use one-row notation $(\rho(1),\ldots,\rho(n))$ for permutations. For example,

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow (2,4,3,1) = \rho. \tag{16.1}$$

Thus the matrix W is obtained by applying the permutation ρ to the columns, or the inverse permutation ρ^{-1} to the rows, of the identity matrix I_n.

This choice of notation is awkward in one respect. Since $(\rho_1 \circ \rho_2)(i) = \rho_1(\rho_2(i))$, our notation for permutations gives a left action of the symmetric group $\Sigma(n)$ on $Z[n]$, whereas the action of $\mathsf{GL}(n)$ on $\mathsf{V}(n)$ is on the right. Hence the bijection $\rho \leftrightarrow W$ is an anti-isomorphism: if $\rho_1 \leftrightarrow W_1$ and $\rho_2 \leftrightarrow W_2$, then $\rho_1 \circ \rho_2 \leftrightarrow W_2 W_1$. The definitions which follow are usually introduced for $\Sigma(n)$, but we approach them from the point of view of $\mathsf{W}(n)$.

Definition 16.1.1 The **length** $\text{len}(W)$ of $W \in \mathsf{W}(n)$ is the number of submatrices of W of the form

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The submatrices are given by the entries in any two rows and any two columns: for example, in (16.1) $\text{len}(W) = 4$. The identity matrix I_n is the unique element of length 0 in $\mathsf{W}(n)$, and the anti-diagonal matrix W_0 corresponding to the reversal $\rho_0 = (n, n-1, \ldots, 1)$ is the unique element of maximal length $n(n-1)/2$. The elements of length 1 are the switch matrices S_i obtained by exchanging rows i and $i+1$ of I_n for $1 \le i \le n-1$. More generally, the switch matrix $S_{i,j}$ obtained by exchanging rows i and j of I_n has length $2|i-j|-1$.

Definition 16.1.2 Let $W \in \mathsf{W}(n)$. For $1 \le i \le n-1$, i is a **descent** of W if rows i and $i+1$ of W contain a submatrix J. More generally, for $1 \le i < j \le n$, (i,j) is an **inversion** of W if rows i and j of W contain a submatrix J. We write $\text{des}(W)$ for the set of descents of W, and $\text{inv}(W)$ for the set of inversions.

Thus the length of W is the number of its inversions. For the matrix W in (16.1), $\text{des}(W) = \{2,3\}$ and $\text{inv}(W) = \{(1,4),(2,3),(2,4),(3,4)\}$. The identity matrix I_n is the unique matrix with no descents, and the reversal matrix W_0 is the unique matrix with descent set $Z[n-1]$. The switch matrix S_i has descent set $\{i\}$ and inversion set $\{(i,i+1)\}$. For $j > i+1$, the switch matrix $S_{i,j}$ has descent set $\{i, j-1\}$ and inversion set consisting of (i,j) and all (i,k) and (k,j)

for $i < k < j$. The set $\mathrm{inv}(W)$ is transitive, in the sense that $(i,j) \in \mathrm{inv}(W)$ if $(i,k),(k,j) \in \mathrm{inv}(W)$ for some k such that $i < k < j$.

If $\rho \in \Sigma(n)$ is the permutation associated to W, we write the length and the descent and inversion sets of W alternatively as $\mathrm{len}(\rho)$, $\mathrm{des}(\rho)$ and $\mathrm{inv}(\rho)$. Thus $i \in \mathrm{des}(\rho)$ if $\rho(i) > \rho(i+1)$, and $(i,j) \in \mathrm{inv}(\rho)$ if $i < j$ and $\rho(i) > \rho(j)$.

Proposition 16.1.3 *For $W \in \mathsf{W}(n)$ and the reversal matrix W_0*

(i) $len(W^{-1}) = len(W)$,
(ii) $i \in des(WW_0)$ *if and only if* $i \notin des(W)$,
(iii) $i \in des(W_0W)$ *if and only if* $n - i \notin des(W)$,
(iv) $len(WW_0) = len(W_0W) = n(n-1)/2 - len(W)$.

Proof For (i) we observe that $W^{-1} = W^{\mathrm{tr}}$ and transposition of a matrix preserves submatrices of the form J and I_2. For (ii) and (iii) we observe that reversal of the rows or columns exchanges these submatrices, and, in the case of column reversal, inversions and non-inversions are exchanged. Hence $(i,j) \in \mathrm{inv}(WW_0)$ if and only if $(i,j) \notin \mathrm{inv}(W)$, and $(i,j) \in \mathrm{inv}(W_0W)$ if and only if $(n+1-j, n+1-i) \notin \mathrm{inv}(W)$. In particular (iv) follows. □

Proposition 16.1.4 *For $W \in \mathsf{W}(n)$*

(i) $$len(S_iW) = \begin{cases} len(W)+1, & \text{if } i \notin des(W),\\ len(W)-1, & \text{if } i \in des(W);\end{cases}$$

(ii) $$len(WS_i) = \begin{cases} len(W)+1, & \text{if } i \notin des(W^{-1}),\\ len(W)-1, & \text{if } i \in des(W^{-1}).\end{cases}$$

Proof For (i), the inversion $(i, i+1)$ is added or removed from $\mathrm{des}(W)$, while inversions not involving row i or $i+1$ are unchanged, and inversions involving only one of these rows are replaced by inversions involving only the other. Since $\mathrm{len}(WS_i) = \mathrm{len}((WS_i)^{-1}) = \mathrm{len}(S_iW^{-1})$, (ii) follows by applying (i) to W^{-1}. □

Proposition 16.1.5 *The switch matrices S_i, $1 \le i \le n-1$, generate the group $\mathsf{W}(n)$. For $W \in \mathsf{W}(n)$, the minimum number r of factors in a product $W = S_{i_1} \cdots S_{i_r}$ is $len(W)$, where this product is I_n in the case $r = 0$.*

Proof This follows from Proposition 16.1.4(i) by iteration. By choosing $i_1 \in \mathrm{des}(W)$, $i_2 \in \mathrm{des}(S_{i_1}W)$ and so on, we can reduce the length of the product matrix by 1 at each step. Since I_n is the only matrix with no descents, we obtain $S_{i_r} \cdots S_{i_1} W = I_n$ after $r = \mathrm{len}(W)$ steps. Hence $W = S_{i_1} \cdots S_{i_r}$. □

Definition 16.1.6 A product $W = S_{i_1} \cdots S_{i_r}$ of length $r = \text{len}(W)$ is a **reduced word** for W.

Example 16.1.7 By iterative switching of rows i and $i+1$ where i is a descent, we find that the matrix W of (16.1) has 3 reduced words, namely $S_2S_3S_2S_1$, $S_3S_2S_1S_3$ and $S_3S_2S_3S_1$.

Remark 16.1.8 The generators $S_1, \dots, S_{n-1}$ of $\mathsf{W}(n)$ satisfy the relations $S_i^2 = I_n$, $S_iS_j = S_jS_i$ if $|i-j| > 1$, and $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$. It is a standard result of group theory that these are a set of defining relations. Thus it is possible to use these relations to convert any word in $S_1, \dots, S_{n-1}$ to a reduced word, or to interchange two reduced words for the same element of $\mathsf{W}(n)$.

Proposition 16.1.9 *There is a reduced word of the form S_iW' for a permutation matrix $W \in \mathsf{W}(n)$ if and only if $i \in des(W)$, and there is a reduced word of the form $W'S_j$ for W if and only if $j \in des(W^{-1})$.*

Proof If S_iW' is a reduced word for W, then W' is a reduced word for S_iW, so the first statement follows from Proposition 16.1.4(i). The second statement follows similarly from Proposition 16.1.4(ii), or by replacing W by W^{-1}. □

Proposition 16.1.10 *Let ρ be the permutation associated to W, and let $(i,j) \in inv(W)$, so that the $(i,\rho(i))$th and $(j,\rho(j))$th entries of W are the entries 1 in a submatrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of W. Let W' be the matrix obtained by exchanging rows i and j of W, or columns $\rho(j)$ and $\rho(i)$. If the (k,ℓ)th entries of W are 0 for all other values of k and ℓ such that $i \le k \le j$ and $\rho(j) \le \ell \le \rho(i)$, then $len(W') = len(W) - 1$, and otherwise $len(W') < len(W) - 1$.

Proof Consider W as a partitioned matrix of the form

$$W = \left(\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ \hline * & * & * \end{array}\right)$$

where the rows are divided into rows $k < i$, $i \le k \le j$ and $k > j$, and the columns into columns $\ell < \rho(j)$, $\rho(j) \le \ell \le \rho(i)$ and $\ell > \rho(i)$. By hypothesis, the only entries 1 in the central submatrix are the $(i,\rho(i))$ and $(j,\rho(j))$ entries. The result follows by case by case consideration of the relative positions of the two 1s in the central submatrix with respect to a 1 in one of the other submatrices. □

Example 16.1.11 For W as in (16.1), $\mathrm{inv}(W) = \{(1,4),(2,3),(2,4),(3,4)\}$. The condition of Proposition 16.1.10 is satisfied for $(i,j) = (1,4)$, $(2,3)$ and $(3,4)$ but not for $(i,j) = (2,4)$. Exchanging these rows gives a matrix W' of length 3 in the first three cases, but of length 1 in the fourth case.

Definition 16.1.12 Let $W, W' \in \mathsf{W}(n)$ with $\mathrm{len}(W) = \mathrm{len}(W') + 1$. Then W covers W' in the **Bruhat order** if $W = S_{i,j}W'$ where $1 \le j \le n$, or equivalently if $W = W'S_{\rho(j),\rho(i)}$, where ρ is the permutation associated to W. Given W_1 and W_2 in $\mathsf{W}(n)$, we write $W_1 \ge W_2$ if W_1 is connected to W_2 by a chain of such coverings.

Proposition 16.1.10 gives an equivalent formulation of this definition. Note that the condition of Proposition 16.1.10 is always satisfied when adjacent rows or columns of W are switched, i.e. in the cases $j = i+1$ and $\rho(i) = \rho(j)+1$.

Example 16.1.13 The permutation matrix

$$W = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow (3,4,1,2) \tag{16.2}$$

has inversion set $\{(1,3),(1,4),(2,3),(2,4)\}$, and W covers $S_{2,3}W \leftrightarrow (3,1,4,2)$ (adjacent rows), $WS_{2,3} \leftrightarrow (2,4,1,3)$ (adjacent columns), $S_{1,3}W = WS_{1,3} \leftrightarrow (1,4,3,2)$ and $S_{2,4}W = WS_{2,4} \leftrightarrow (3,2,1,4)$ in the Bruhat order.

For $W_1, W_2 \in \mathsf{W}(n)$, we can use the following criterion to determine whether $W_1 \ge W_2$ in the Bruhat order. We first define this as a second partial order $\ge_s$ on $\mathsf{W}(n)$, and then we shall prove that $\ge_s$ is the same as the Bruhat order.

Definition 16.1.14 For $W \in \mathsf{W}(n)$ and $1 \le i,j \le n$, let $s_{i,j}(W)$ be the number of entries 1 in the north-east corner submatrix of W given by elements in rows $1,\ldots,i$ and columns $j,\ldots,n$ of W. Then $W_1 \ge_s W_2$ for $W_1, W_2 \in \mathsf{W}(n)$ if and only if $s_{i,j}(W_1) \ge s_{i,j}(W_2)$ for $1 \le i,j \le n$.

In terms of the permutation $\rho \in \Sigma(n)$ associated to W, $s_{i,j}$ is the number of integers $a \in Z[n]$ such that $a \le i$ and $\rho(a) \ge j$.

Proposition 16.1.15 *Given* $W_1, W_2 \in \mathsf{W}(n)$*,* $W_1 \ge W_2$ *in Bruhat order if and only if* $W_1 \ge_s W_2$*.*

Proof To prove necessity of the sum criterion in Definition 16.1.14, it suffices to consider the case where W_1 covers W_2. Let $\rho_1, \rho_2 \in \Sigma(n)$ be the permutations associated to $W_1, W_2 \in \mathsf{W}(n)$. If W_1 covers W_2 and $S_{k,\ell}W_1 = W_2$, then it is straightforward to check that

$$s_{i,j}(W_2) = \begin{cases} s_{i,j}(W_1) - 1, & \text{if } k \le i \le \ell \text{ and } \rho_1(\ell) < j \le \rho_1(k), \\ s_{i,j}(W_1), & \text{otherwise,} \end{cases} \tag{16.3}$$

and hence $W_1 \geq_s W_2$.

To prove sufficiency, assume that $W_1 \geq_s W_2$. We shall construct W' such that $W' = S_{k,\ell}W_1$ where $(k,\ell) \in \text{inv}(W_1)$, so that W_1 covers W' in the Bruhat order, and show that $W' \geq_s W_2$. The result then follows by induction on $r = \text{len}(W_1) - \text{len}(W_2)$, and iteration of the construction produces a chain of length r from W_1 to W_2.

We choose k and ℓ as follows. Let the kth row be the first where W_1 and W_2 differ. Then by the sum criterion the entry 1 in row k of W_2 precedes the entry 1 in W_1, i.e. $\rho_2(k) < \rho_1(k)$. Since W_1 and W_2 agree in rows $< k$, the entry 1 in column $\rho_2(k)$ is in a row $> k$. Let ℓ be minimal such that $\ell > k$ and W_1 has an entry 1 in a column $\rho_1(\ell)$ such that $\rho_2(k) \le \rho_1(\ell) < \rho_1(k)$. Then the submatrix of W_1 given by rows $k, \dots, \ell$ and columns $\rho_1(\ell), \dots, \rho_1(k)$ has only two entries 1, at its north-east corner $(k, \rho_1(k))$ and at its south-west corner $(\ell, \rho_1(\ell))$. Thus the matrix $W' = S_{k,\ell}W_1$ obtained by exchanging rows k and ℓ of W_1 covers W' in the Bruhat order.

It remains to prove that $W' \geq_s W_2$. By (16.3) it suffices to prove that $s_{i,j}(W') \geq s_{i,j}(W_2)$ for $k \le i < \ell$ and $\rho_1(\ell) < j \le \rho_1(k)$, or equivalently that $s_{i,j}(W_1) > s_{i,j}(W_2)$ for the corresponding submatrices A_1 and A_2 of W_1 and W_2. Consider the submatrices $B_1 \supset A_1$ and $B_2 \supset A_2$ of W_1 and W_2 given by rows $k, \dots, \ell - 1$ and columns $\rho_2(k) \le j \le \rho_1(k)$. By our choice of ℓ, all entries of the first column of B_1 are 0s, but the $(k, \rho_2(k))$ entry of W_2 is 1. Since the sum criterion holds for B_1 and B_2, strict inequality must hold when the first column is removed, and in particular for A_1 and A_2. □

Example 16.1.16 Let $W_1, W_2 \in \mathsf{W}(6)$ be the matrices associated to the permutations $(3,6,5,1,4,2)$ and $(2,4,6,1,3,5)$ respectively. The proof of Proposition 16.1.15 gives an algorithm for constructing a chain in decreasing Bruhat order showing that $W_1 > W_2$, with corresponding permutations $(3,6,5,1,4,2) \to (2,6,5,1,4,3) \to (2,5,6,1,4,3) \to (2,4,6,1,5,3) \to (2,4,6,1,3,5)$.

16.2 The Bruhat decomposition of $\mathsf{GL}(n)$

Let $\mathsf{B}(n)$ be the lower triangular subgroup of $\mathsf{GL}(n)$. For $W \in \mathsf{W}(n)$, the right coset $\mathsf{B}(n)W$ contains all matrices obtained by applying the associated permutation ρ to the columns of a lower triangular matrix, while the left coset $W\mathsf{B}(n)$ contains all matrices obtained by applying ρ^{-1} to the rows. For example, with W as in (16.1),

$$\mathsf{B}(n)W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & * & 0 & 1 \\ 0 & * & 1 & * \\ 1 & * & * & * \end{pmatrix}, \quad W\mathsf{B}(n) = \begin{pmatrix} * & 1 & 0 & 0 \\ * & * & * & 1 \\ * & * & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{16.4}$$

where the stars represent elements 0 or 1 in $\mathbb{F}_2$. Thus matrices in $\mathsf{B}(n)W$ are obtained by replacing the 0s in W below the 1s by stars, while matrices in $W\mathsf{B}(n)$ are obtained by replacing the 0s in W to the left of the 1s by stars.

We wish to express a matrix $A \in \mathsf{GL}(n)$ as a product BWB', where B and B' are lower triangular and W is a permutation matrix. We begin by using lower triangular transvections $T_{i,j}$ which map row i to row i + row j, where $i > j$, so as to reduce A to a matrix A' in which every entry below the last 1 in each row is 0.

Example 16.2.1 In the case $n = 4$, a typical reduction is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \mapsto \quad A' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

obtained by adding row 1 to row 2, row 1 to row 4 and row 2 to row 4.

Since these row operations are equivalent to premultiplying A by lower triangular matrices, $A' = BA$ where $B \in \mathsf{B}(n)$ is the result of applying the same sequence of transvections to the identity matrix I_n. Further, by (16.4) $A' = WB'$ where $W \in \mathsf{W}(n)$ and $B' \in \mathsf{B}(n)$. Thus $A = B^{-1}WB'$. In Example 16.2.1

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; B' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

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Excerpt
More Information

The matrix W is given by the last 1 in each row of A'. Note that for $i > j$ the (i,j)th entry of B' is 1 if columns i and j contain a submatrix J, and is 0 if columns i and j contain a submatrix I_2.

Theorem 16.2.2 (Bruhat decomposition) *Every matrix $A \in \mathsf{GL}(n)$ is a product $A = BWB'$, where W is a permutation matrix uniquely determined by A, and B and B' are lower triangular.*

Proof The existence of the decomposition follows from the procedure illustrated above. To prove that W is unique, we show that if $B_1W_1B_1' = B_2W_2B_2'$ then $W_1 = W_2$. Let $A = W_1B_1'(B_2')^{-1} = B_1^{-1}B_2W_2$, so that $A \in W_1\mathsf{B}(n) \cap \mathsf{B}(n)W_2$. Let ρ_1 and ρ_2 be the permutations associated to W_1 and W_2 respectively, let $1 \le i \le n$, and let $j_1 = \rho_1(i)$, $j_2 = \rho_2(i)$. Then by (16.4) $a_{i,j_1} = a_{i,j_2} = 1$ and $a_{i,j} = 0$ for all $j > j_1$, where $A = (a_{i,j})$. Hence $j_2 \le j_1$. Since this is true for all i, $\rho_2 = \rho_1$ and so $W_2 = W_1$. □

Theorem 16.2.2 shows that $\mathsf{GL}(n)$ is the disjoint union of the double cosets $\mathsf{B}(n)W\mathsf{B}(n)$, where $W \in \mathsf{W}(n)$.

Definition 16.2.3 Given $W \in \mathsf{W}(n)$, the corresponding **Bruhat cell** of $\mathsf{GL}(n)$ is the double coset $\mathsf{B}(n)W\mathsf{B}(n)$ consisting of all matrices $A = BWB'$ where $B, B' \in \mathsf{B}(n)$.

The Bruhat cells do not all have the same size. There is a unique largest Bruhat cell $\mathsf{B}(n)W_0\mathsf{B}(n)$ with $|\mathsf{B}(n)|^2 = 2^{n(n-1)}$ elements, and a unique smallest Bruhat cell $\mathsf{B}(n) = \mathsf{B}(n)I_n\mathsf{B}(n)$. It follows from Proposition 16.2.7 below that $|\mathsf{B}(n)W\mathsf{B}(n)| = 2^m$, where $m = n(n-1)/2 + \mathrm{len}(W)$.

To prove this, we shall refine the Bruhat decomposition so as to obtain a unique factorization of each matrix $A \in \mathsf{GL}(n)$. Consider the left coset $W\mathsf{B}(n)$ of (16.4). By using lower triangular transvections we can reduce A to a matrix A' in which every entry below the last 1 in each row is 0. As in Example 16.2.1, A' has the form

$$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (16.5)$$

and conversely every element of $W\mathsf{B}(n)$ can be recovered by applying the same set of transvections to such matrices A'. Since the stars in the reduced matrix correspond to the stars in 2×2 submatrices of the form $\left(\begin{smallmatrix} * & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the number of stars is $\mathrm{len}(W)$. In terms of the permutation ρ associated to W, the (i,j)th entry of the reduced matrix is a star if and only if $j < \rho(i)$ and $i < \rho^{-1}(j)$. The reduced

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matrices form the left coset $W\mathsf{B}(W)$, where $\mathsf{B}(W)$ is the subset of $\mathsf{B}(n)$ given by matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & * & 1 \end{pmatrix}. \tag{16.6}$$

This argument shows that the Bruhat cell $\mathsf{B}(n)W\mathsf{B}(n) = \mathsf{B}(n)W\mathsf{B}(W)$, and we note that $\mathsf{B}(W)$ is a subgroup of $\mathsf{B}(n)$. For $1 \le j < i \le n$, the (i,j)th entry of (16.6) is a star if $(j,i) \in \mathrm{inv}(W^{-1})$, and is 0 otherwise. In terms of the permutation $\rho = (2,4,3,1)$ corresponding to W, the stars correspond to pairs (i,j) such that $i > j$ but i appears before j in one-line notation.

Definition 16.2.4 For $W \in \mathsf{W}(n)$, the **Bruhat subgroup** $\mathsf{B}(W) \subseteq \mathsf{B}(n)$ is the group of all matrices $B \in \mathsf{B}(n)$ such that for $1 \le j < i \le n$ the (i,j)th entry $b_{i,j}$ of B is 0 if $(j,i) \notin \mathrm{inv}(W^{-1})$. Equivalently $b_{i,j} = 0$ if columns i and j of W contain a submatrix I_2. In terms of the permutation ρ associated to W, $b_{i,j} = 0$ if j precedes i in one-line notation for ρ. We shall also write $\mathsf{B}(\rho)$ for $\mathsf{B}(W)$.

Proposition 16.2.5 *Let* $W \in \mathsf{W}(n)$. *Then*

(i) $\mathsf{B}(W)$ *is a subgroup of* $\mathsf{B}(n)$ *of order* $2^{len(W)}$,
(ii) $\mathsf{B}(W) \cap \mathsf{B}(W_0W) = I_n$ *and* $\mathsf{B}(n) = \mathsf{B}(W)\mathsf{B}(W_0W)$,
(iii) $W\mathsf{B}(W)W^{-1} \subseteq \mathsf{U}(n)$, *the upper triangular subgroup of* $\mathsf{GL}(n)$.

Proof (i) Let $C = AB$ where $A, B \in \mathsf{B}(W)$, and let $1 \le j < i \le n$. Since $c_{i,j} = \sum_{j \le k \le i} a_{i,k}b_{k,j}$, if $c_{i,j} = 1$ then $a_{i,k} = 1$ and $b_{k,j} = 1$ for some k. Hence columns k and i of W contain a submatrix J, as do columns j and k, and so columns j and i also contain a submatrix J. Hence $C \in \mathsf{B}(W)$. Since $|\mathrm{inv}(W^{-1})| = \mathrm{len}(W^{-1}) = \mathrm{len}(W)$, $\mathsf{B}(W)$ has order $2^{\mathrm{len}(W)}$.
(ii) follows by observing that W_0W is the row-reversal of W, and this exchanges submatrices I_2 and J in a given pair of columns i and j.
(iii) Let $B' = WBW^{-1}$ where $B \in \mathsf{B}(W)$. Then $b_{i,j} = b'_{\rho^{-1}(i),\rho^{-1}(j)}$, where ρ is the permutation corresponding to W. In 'star' notation, $b_{i,j} = *$ for $1 \le j < i \le n$ if and only if columns j and i of W contain a submatrix J, so that $\rho^{-1}(j) > \rho^{-1}(i)$. Hence $B' \in \mathsf{U}(n)$.

□

Example 16.2.6 Let W correspond to $\rho = (2,4,3,1)$, as in (16.1). Then applying W^{-1} to (16.5) gives the subgroup of $\mathsf{U}(4)$ whose elements have the