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On the role of totally disconnected groups in the structure of locally compact groups

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Abstract

We describe to what extent a general locally compact group decomposes into a totally disconnected part and a connected part that can be approximated by Lie groups. We also present a construction that highlights the relevance of groups acting on trees in the structure of general locally compact groups.

In the first part of this lecture we establish a fundamental theorem which says that if G is a totally disconnected locally compact group, every neighbourhood of $e \in G$ contains a compact open subgroup. Since such groups are profinite, this implies that the multiplication in a neighbourhood of e can be approximated with arbitrary accuracy by the multiplication in finite groups.

In the second part of the lecture we address the question to which extent a locally compact group decomposes into a totally disconnected part and a connected part; using the concept of amenable radical we will establish a decomposition result using the Gleason–Yamabe structure theorem.

In the third part we describe a construction which relates totally disconnected groups which are compactly generated to groups acting on trees via an appropriate Cayley–Abels graph ("Nebengruppenbild", see [3], §4).

Parts 2 and 3 of this lecture are taken from [1], 3.3–3.5, with some modifications. The heuristic principle is that certain questions, concerning for instance amenable actions of general compactly generated locally compact groups, can be analyzed by treating separately the case of semisimple Lie groups and closed cocompact subgroups of the automorphism group of a regular tree. For van Dantzig's theorem, one may consult [2], Chapter II, §7, and a good reference for Hilbert's 5th problem is [4].

1.1 Van Dantzig's theorem

Let us first fix some terminology: a topological space X is locally compact if every point admits a compact neighbourhood; if in addition X is T_2 (that is Hausdorff), then every point admits a fundamental system of compact neighbourhoods.

Let now G be a topological group; then the connected component G° of $e \in G$ is a normal subgroup and closed in addition; thus G/G° with quotient topology is a T_2 topological group and:

Lemma 1.1 *The topological group G/G° is totally disconnected.*

Recall

Definition 1.2 A non-empty topological space is totally disconnected if all its connected components are reduced to points.

Observe then that a topological group L is totally disconnected if and only if $L^\circ = (e)$.

Proof Let $\pi : G \to L := G/G^\circ$ be the canonical quotient homomorphism and assume that $\pi^{-1}(L^\circ) = F_1 \cup F_2$ where $F_i \subset G$ are closed disjoint subsets. Since $G^\circ \subset \pi^{-1}(L^\circ)$, we have for every $g \in \pi^{-1}(L^\circ)$ either $gG^\circ \subset F_1$ or $gG^\circ \subset F_2$. Thus, F_1 and F_2 are union of G°-cosets and the decomposition descends to the quotient. Since L° is connected, we conclude that $F_1 = \varnothing$ or $F_2 = \varnothing$. Hence $\pi^{-1}(L^\circ) \supset G^\circ$ is connected and hence $\pi^{-1}(L^\circ) = G^\circ$ which finally implies that $L^\circ = (e)$ and $L = G/G^\circ$ is totally disconnected. □

Corollary 1.3 *If G is locally compact T_2, then G/G° is locally compact, T_2 and totally disconnected.*

Our aim is to establish the following result of D. van Dantzig [5].

Theorem 1.4 *If G is locally compact, T_2 and totally disconnected, then every compact neighbourhood of e contains a closed and open subgroup.*

In the sequel we will call “clopen” the subsets which are closed and open.

The following lemma is the essential ingredient:

Lemma 1.5 *Let X be a compact T_2 space. Then for every $x \in X$, the subset*

$$K_x := \bigcap \{U : U \ni x,\ U \text{ is clopen}\}$$

is connected.

Proof Assume that $K_x = K_1 \cup K_2$ is a disjoint union of closed subsets with $K_1 \ni x$; pick open subsets $U_i \supset K_i$ with $U_1 \cap U_2 = \varnothing$ and consider $F := \overline{U}_2 \backslash U_2$. Since $K_2 \subset U_2$, we have $F \cap K_2 = \varnothing$ and since $F \subset \overline{U}_2 \subset U_1^c$ we have $F \cap K_1 = \varnothing$. Thus, $F \cap K_x = \varnothing$ and by compactness of X, there are finitely many clopen subsets $U_i \ni x$, $1 \leq i \leq n$, with

$$\bigcap_{i=1}^{n} U_i \cap F = \varnothing.$$

Setting $V := \bigcap_{i=1}^{n} U_i$, we deduce from $V \cap (\overline{U}_2 \backslash U_2) = \varnothing$ that $V \cap U_2^c = V \cap \overline{U}_2^c$ which shows that $V \cap U_2^c$ is a clopen set containing x and avoiding K_2. Thus, $K_2 = \varnothing$ and K_x is connected. □

With this at hand we obtain the following important information concerning the topology of totally disconnected spaces:

Lemma 1.6 *Let X be locally compact, T_2 and totally disconnected. The family of compact open subsets is a basis for the topology of X.*

Proof Let $x \in X$ and $C \ni x$ a compact neighbourhood of x. Since X is totally disconnected T_2, Lemma 1.5 implies

$$\{x\} = \bigcap \{U : U \subset C,\ \ U \text{ clopen in } C,\ \ U \ni x\}.$$

If $\mathring{C}$ denotes the interior of C, we have $(C \backslash \mathring{C}) \cap \{x\} = \varnothing$, hence there are $U_1, \ldots, U_n$ clopen in C with $x \in \bigcap_{i=1}^{n} U_i$ and $(\bigcap_{i=1}^{n} U_i) \cap (C \backslash \mathring{C}) = \varnothing$. Thus, $\bigcap_{i=1}^{n} U_i$ is open in X; since at any rate it is compact, this finishes the proof of the lemma. □

Lemma 1.7 *Let G be a topological group and $C \subset U$ with C compact and U open. Then there is $V \ni e$ open, with $V = V^{-1}$ and $C \cdot V \subset U$.*

Proof By continuity of the multiplication and the inverse, there is for every $x \in C$ and open $V_x \ni e$ with $V_x = V_x^{-1}$ and $xV_x^2 \subset U$. In particular, $C \subset \bigcup_{x \in C} xV_x$

and hence there exist $x_1, \dots, x_n$ in C with $C \subset \bigcap_{i=1}^{n} x_i V_{x_i}$. Now set $V := \bigcap_{i=1}^{n} V_{x_i}$; then $V = V^{-1} \ni e$ is open and

$$C \cdot V \subset \bigcup_{i=1}^{n} x_i V_{x_i} V \subset \bigcup_{i=1}^{n} x_i V_{x_i}^2 \subset U.$$

□

Proof of Theorem 1.4. Let $U \ni e$ be a neighbourhood of e. By Lemma 1.6 there exists $C \subset G$ compact open subset with $e \in C \subset U$. By Lemma 1.7, since C is both compact and open, there exists $V \ni e$, $V = V^{-1}$ open with $CV \subset C$ and hence $CV = C$. Thus, the subgroup

$$L := \{g \in G\colon\ C \cdot g = C\}$$

contains $V \ni e$ and hence is open in G; since $C \ni e$, we have $L \subset C$, and since L is closed it is therefore compact. □

Thus, it follows from Theorem 1.4 that for such locally compact totally disconnected groups, the local multiplicative structure is completely encoded in compact groups. Now one can go one step further and apply Theorem 1.4 to compact groups:

Corollary 1.8 *A compact, T_2 totally disconnected group is a projective limit of finite groups.*

Proof By Theorem 1.4, the set $\mathscr{V}$ of all compact open subgroups of G form a fundamental system of neighbourhoods of e. Since G is compact, for every $H \in \mathscr{O}$, G/H is finite and thus $\bigcap_{x \in G} xHx^{-1}$ is still open and normal in G. Thus, $\mathscr{V} = \{H \in \mathscr{O}\colon H \lhd G\}$ also form a fundamental system of neighbourhoods of e. For every H, the group G/H is finite and we deduce from the above that the continuous homomorphism

$$\begin{array}{ccc} G & \longrightarrow & \prod_{H \in \mathscr{V}} G/H \\ g & \longmapsto & (g \bmod H) \end{array}$$

is injective, and hence provides a topological isomorphism from G to a closed subgroup of the above product of finite groups. □

1.2 An application of the Gleason–Yamabe theorem

An interesting application of Theorem 1.4 is the following: let G be locally compact totally disconnected; then any continuous homomorphism $\pi\colon G \to$

$GL(n,\mathbf{R})$ has open kernel. Indeed, let $U \ni Id$ be an open neighbourhood of Id in $GL(n,\mathbf{R})$ which does not contain any non-trivial subgroup; then $\pi^{-1}(U)$ is an open neighbourhood of e and hence contains a compact open subgroup L. But then $\pi(L) \subset U$ must be the trivial group. Thus, one of the possible obstructions to be a Lie group is to be non-discrete, totally disconnected.

For connected groups, examples which are not Lie groups include $(S^1)^{\mathbf{N}}$ or $\prod_{n\geq 1} U(n)$. The latter example is somehow universal, in that every separable compact T_2 group is isomorphic to a closed subgroup thereof; this is an easy consequence of the Peter–Weyl theorem and had already been observed by von Neumann.

One of the major structure theorems for locally compact groups is the following

Theorem 1.9 (Gleason–Yamabe) *Let G be locally compact T_2. Then there is an open subgroup G' which is a projective limit of Lie groups.*

When G is connected we have automatically that $G = G'$ and we obtain

Corollary 1.10 *Let G be locally compact, T_2, connected. Then there is a normal compact subgroup $K \lhd G$ such that the quotient G/K is a Lie group.*

Our aim is to obtain a variant of Corollary 1.10 where one divides by a characteristic subgroup. This will involve the concept of amenable radical. Recall first

Definition 1.11 A topological group G is amenable if it fixes a point in every non-empty convex-compact G-space.

Here a convex-compact G-space is a convex compact subset S of a T_2 locally convex topological vector space on which G acts continuously, linearly, and preserving S.

Now we sketch the proof of the existence of the amenable radical:

Theorem 1.12 *Let G be locally compact T_2. Then there exists a unique maximal closed amenable normal subgroup $A(G)$ of G. It is topologically characteristic and*

$$A\big(G/A(G)\big) = (e).$$

Proof Let $\mathscr{N}$ be the set of all closed normal amenable subgroups of G and let $A(G) = \overline{\langle N \colon N \in \mathscr{N} \rangle}$ be the closed subgroup generated by them. Clearly, $A(G)$ is topologically characteristic and we proceed to show that it is amenable. Let S be a convex-compact $A(G)$-space, and let $\{N_1, \ldots, N_r\} \subset \mathscr{N}$ be any finite collection. Observe that $N_1 \dots N_{r-1}$ is a normal subgroup of G, in particular normalised by N_r; thus, N_r acts in the set $S^{N_1 \dots N_{r-1}}$ of $N_1 \dots N_{r-1}$-fixed points in S and

$$(S^{N_1,\dots,N_{r-1}})^{N_r} = \bigcap_{i=1}^{r} S^{N_i}.$$

Take first $r = 2$: since N_1 is amenable, $S^{N_1} \neq \varnothing$; since now S^{N_1} is a non-empty convex-compact N_2-space, we have $(S^{N_1})^{N_2} \neq \varnothing$, and hence $S^{N_1} \cap S^{N_2} \neq \varnothing$. Using the amenability of every N_i, one shows by induction that $(S^{N_1,\dots,N_{r-1}})^{N_r} \neq \varnothing$, and hence $\bigcap_{i=1}^{r} S^{N_i} \neq \varnothing$. Thus, since S is compact, we conclude

$$\bigcap_{N \in \mathscr{N}} S^N \neq \varnothing,$$

and hence by continuity, $S^{A(G)} \neq \varnothing$. This shows that $A(G)$ is amenable. Finally, the fact that $A(G/A(G)) = e$ is an easy exercise. □

We can now draw the following easy consequence from the existence of the amenable radical and Corollary 1.10.

Corollary 1.13 *Assume that G is locally compact, T_2 and connected. Then the quotient*

$$G/A(G) \simeq \prod_{i=1}^{n} S_i$$

is isomorphic to a direct product of connected, simple, centre-free, non-compact Lie groups.

Proof By Corollary 1.10 there exists $K \lhd G$ compact such that G/K is a Lie group. Since K is compact, in particular amenable, and normal in G, we must have that $A(G) \supset K$. Thus, $L := G/A(G)$ is a Lie group as well and in addition $A(L) = (e)$. Thus, L has trivial solvable radical as well as trivial centre; thus it is a direct product of simple adjoint Lie groups none of which can be compact. □

We turn now to our main application of the results so far obtained. Let G be locally compact T_2 and G° as above its connected component of e; then the amenable radical $A(G^\circ)$ is a characteristic subgroup of G° and hence it is normal in G; let

$$L := G/A(G^\circ)$$

denote the quotient group.

Theorem 1.14 *The group L° is a direct product of adjoint connected simple non-compact Lie groups. Its centraliser $Z_L(L^\circ)$ in L is totally disconnected and the direct product $L^\circ \cdot Z_L(L^\circ)$ is open and of finite index in L.*

In particular, $G/A(G^\circ)$ is virtually the direct product of its connected component of e with a totally disconnected normal subgroup. Observe that this totally disconnected group, that is $Z_L(L^\circ)$, is locally isomorphic to G/G°, so that one can at least "lift" G/G° locally to $G/A(G^\circ)$.

Proof of Theorem 1.14. Observe (exercise!) that $L^\circ = G^\circ/A(G^\circ)$; thus $A(L^\circ) = (e)$ and the first assertion of the theorem follows from Corollary 1.13. For every $g \in L$, let $i(g)\colon L \to L$ denote the automorphism given by conjugation $\ell \longmapsto g\ell g^{-1}$. By restriction to L°, we obtain a continuous homomorphism $L \to \mathrm{Aut}(L^\circ)$, $g \longmapsto i(g)|_{L^\circ}$. It follows then from the fact that L° is connected semisimple with finite centre, that the group $\mathrm{Inn}(L^\circ)$ of inner automorphisms of L° is open of finite index in $\mathrm{Aut}(L^\circ)$. As a result, the subgroup

$$L^* := \left\{g \in L : \left. i(g)\right|_{L^\circ} \in \mathrm{Inn}(L^\circ)\right\}$$

is open and finite index in L as well. Thus, for every $g \in L^*$ there exists an $h \in L^\circ$ such that $g\ell g^{-1} = h\ell h^{-1}$ for every $\ell \in L^\circ$, that is $h^{-1}g \in Z_L(L^\circ)$, which shows that $L^* = L^\circ \cdot Z_L(L^\circ)$. Observe that this product is direct since $Z_L(L^\circ) \cap L^\circ \subset Z(L^\circ) = (e)$. Finally, the subgroup $Z_L(L^\circ)$ is totally disconnected since it is isomorphic to the open subgroup L^*/L° of L/L°. □

1.3 Totally disconnected groups and actions on trees

If G is a compactly generated locally compact group, then so is $G/A(G^\circ)$ and hence the totally disconnected group $H := Z_L(L^\circ)$ is compactly generated as well. Such groups are naturally related to groups acting cofinitely on trees in the following way.

Let H be totally disconnected generated by a compact set C and let $U < H$ be a compact open subgroup. Let $\mathscr{G} = (V, E)$ be the undirected simple graph with vertex set $V := G/U$ and edge set

$$E = \{(gU, gcU) : g \in H,\ c \in UCU\}.$$

Then $\mathscr{G}$ is regular of finite valency $d := |UCU/U|$ and H acts as a group of automorphisms on $\mathscr{G}$; the kernel of this action is $K := \bigcap_{g\in H} gUg^{-1}$ and hence compact. We call $\mathscr{G}$ a **Cayley–Abels graph**.

Let $H_1 := H/K$ and $\mathscr{T}_d \to \mathscr{G}$ the universal covering of $\mathscr{G}$. Then we obtain an exact sequence

$$(e) \to \pi_1(\mathscr{G}) \to \widehat{H} \to H_1 \to (e),$$

where $\widetilde{H} < \operatorname{Aut} \mathscr{T}_d$ is the group of all automorphisms of the d-regular tree which cover elements from H_1.

Observe that $\widetilde{H}$ is a closed subgroup of $\operatorname{Aut} \mathscr{T}_d$ acting transitively on the set of vertices of $\mathscr{T}_d$, in particular is compactly generated. When H_1 is non-compact and $\mathscr{G}$ is not a tree, $\pi_1(\mathscr{G})$ is a free group on countably many generators. In particular, if $\widetilde{H}_r := \widetilde{H}/[\pi_1, [\pi_1, \dots]]$ denotes the quotient by the r^{th}-term of the derived series of π_1, the amenable radical of the compactly generated group $\widetilde{H}_r$ contains the free solvable group of rank r on countably many generators.

References for this chapter

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