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Controllability of Parabolic Systems: The Moment Method

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Abstract

We give some recent controllability results of linear hyperbolic systems and we will apply them to solve some nonlinear control problems.

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Contents

1.1 Introduction

The main goal of these notes is to give a review of results relating to controllability issues for some parabolic systems obtained via the *moment method*. We will follow Fattorini and Russell who, in the 1970s, solved controllability problems for scalar parabolic equations (see [10, 11]). This method is very efficient in the one-dimensional space setting. But it has also been used to prove the boundary null-controllability of the heat equation for particular geometries of the space domain (disks, parallelipepidons, etc.).

At the beginning of the 1990s, Fursikov and Imanuvilov [12] solved the null-controllability problem for a general second-order parabolic equation. They did this by proving a global Carleman inequality for solutions of quite general parabolic equations. This Carleman inequality implies observability inequality and thus controllability of the corresponding parabolic equation when the control function acts on an arbitrary open subset of the space domain or on an arbitrary relatively open subset of its boundary. At the same time, Lebeau and Robbiano [16] also proved the null-controllability of the heat equation with constant coefficients. Their method of proof is less general than that of Fursikov–Imanuvilov when dealing with parabolic equations but it generalizes to abstract diagonal systems.

Since then, a huge literature has been devoted to solving control problems by a systematic use of Carleman estimates: Stokes and Navier–Stokes equations, Burger's equations, etc. But as usual in mathematics, any powerful tool or method has its limitations. These appeared in particular when dealing with parabolic systems. It is one of the goals of these notes to explain these limits.

1.2 Parabolic Systems and Controllability Concepts

Consider the following system:

$$\begin{cases} (\partial_t - D\Delta - A)y = Bu1_\omega, & Q_T := (0,T)\times\Omega, \\ y = Cv1_{\Gamma_0}, & \Sigma_T := (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0, & \Omega, \end{cases} \tag{1.1}$$

where

- $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\omega \subset \Omega$ is an open set, $\Gamma_0 \subset \partial\Omega$ is a relatively open subset;

- $D = \mathrm{diag}(d_1, \ldots, d_n)$, $A = (a_{ij})_{1 \le i,j \le N} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n))$,
- $B = (b_{ij})$, $C = (c_{ij}) \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$: control matrices.

Definition 1.1 System (1.1) is approximately controllable at time $T > 0$ if for all $\varepsilon > 0$, for all $(y^0, y^1) \in \mathbb{X} \times \mathbb{X}$, there exists $(u, v) \in L^2(Q_T) \times L^2(\Sigma_T)$ such that $\|y(T) - y^1\|_{\mathbb{X}} \le \varepsilon$.

System (1.1) is null-controllable at time $T > 0$ if for all $y^0 \in \mathbb{X}$, there exists $(u, v) \in L^2(Q_T) \times L^2(\Sigma_T)$ such that $y(T) = 0$ in Ω.

Here $\mathbb{X}$ is a space where the system (1.1) is well-posed. For example, when $C = 0$ (*distributed control*), it is enough to work with $\mathbb{X} = L^2(\Omega; \mathbb{R}^n)$. In this case, variational methods should prove that for $(y^0, u) \in \mathbb{X} \times L^2(Q_T; \mathbb{R}^m)$, system (1.1) admits a unique solution

$$y \in C([0, T]; \mathbb{X}) \cap L^2\left(0, T; H_0^1(\Omega, \mathbb{R}^n)\right).$$

When $B = 0$ and $C \neq 0$ (*boundary control*), a suitable space is $\mathbb{X} = H^{-1}(\Omega; \mathbb{R}^n)$. The transposition method proves that for $(y^0, u) \in \mathbb{X} \times L^2(\Sigma_T; \mathbb{R}^m)$, system (1.1) admits a unique solution

$$y \in C([0, T]; \mathbb{X}) \cap L^2(Q_T; \mathbb{R}^n).$$

The previous two controllability concepts have dual equivalent concepts. Introduce the backward adjoint system:

$$\begin{cases} (\partial_t + D\Delta + A^*)\varphi = 0, & \text{in} \quad Q_T, \\ \varphi = 0, & \text{on} \quad \Sigma_T, \\ \varphi(T) = \varphi^0, & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

If $\varphi^0 \in L^2(\Omega, \mathbb{R}^n)$ (resp. $\varphi^0 \in H_0^1(\Omega, \mathbb{R}^n)$) then there exists a unique solution φ to (1.2) such that:

$$\varphi \in C\left(0, T; L^2(\Omega, \mathbb{R}^n)\right) \cap L^2\left(0, T; H_0^1(\Omega, \mathbb{R}^n)\right),$$

$$\left(\text{resp. } \varphi \in C\left(0, T; H_0^1(\Omega, \mathbb{R}^n)\right) \cap L^2\left(0, T; H^2 \cap H_0^1(\Omega, \mathbb{R}^n)\right)\right).$$

The following characterizations have been known for a long time and their proof can be found in [9] for instance.

Proposition 1.2

- **Assume that $C = 0$ (distributed control)**

  System (1.1) is approximately controllable if, and only if, for any $\varphi^0 \in L^2(\Omega, \mathbb{R}^n)$ the associated solution to (1.2) satisfies the property:

$$B^*\varphi = 0 \quad \text{in} \quad (0, T) \times \omega \Rightarrow \varphi = 0 \quad \text{in} \quad Q_T. \tag{1.3}$$

System (1.1) is null-controllable if, and only if, there exists $C = C_T > 0$ such that for any solution to (1.2)

$$\|\varphi(0)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq C \int_0^T \int_\omega |B^*\varphi|^2 \, dxdt. \tag{1.4}$$

- **Assume $B = 0$ (boundary control)**

  System (1.1) is approximately controllable if, and only if, for any $\varphi^0 \in H^1_0(\Omega, \mathbb{R}^n)$ the associated solution to (1.2) satisfies the property:

$$C^* \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{in} \quad (0, T) \times \Gamma_0 \Rightarrow \varphi = 0 \quad \text{in} \quad Q_T. \tag{1.5}$$

  System (1.1) is null-controllable if, and only if, there exists $C = C_T > 0$ such that for any solution to (1.2)

$$\|\varphi(0)\|^2_{H^1_0(\Omega,\mathbb{R}^n)} \leq C \int_0^T \int_{\Gamma_0} \left| C^* \frac{\partial \varphi}{\partial \nu} \right|^2 dxdt.$$

1.3 Controllability Results for the Scalar Case: The Carleman Inequality

We describe in this section known controllability results for the scalar parabolic equation and give (without proof) the general form of the Carleman inequality proved in [12].

Theorem 1.3 *The problem*

$$\begin{cases} (\partial_t - \Delta - a)\, y = u 1_\omega, & Q_T := (0, T) \times \Omega, \\ y = 0, & \Sigma_T := (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, & \Omega, \end{cases} \tag{1.6}$$

is null and approximately controllable in $\mathbb{X} = L^2(\Omega)$ for any open set $\omega \subset \Omega$, provided that $a \in L^\infty(Q_T)$.

As a consequence, the problem

$$\begin{cases} (\partial_t - \Delta - a) y = 0, & Q_T := (0, T) \times \Omega, \\ y = v 1_{\Gamma_0}, & \Sigma_T := (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, & \Omega, \end{cases} \tag{1.7}$$

is null and approximately controllable in $\mathbb{X} = H^{-1}(\Omega)$ for any relatively open set $\Gamma_0 \subset \partial\Omega$.

To prove this result, let $\beta_0 \in C^2\left(\overline{\Omega}\right)$ and $s \in \mathbb{R}$ a parameter. Introduce the functions

$$\eta(t,x) := s\,\frac{\beta_0(x)}{t(T-t)}, \qquad (t,x) \in Q_T,$$
$$\rho(t) := \frac{s}{t(T-t)}, \qquad (t,x) \in Q_T$$

and the functional

$$I(\tau,\varphi) = \int_{Q_T} \rho^{\tau-1} e^{-2\eta}\left(|\varphi_t|^2 + |\Delta\varphi|^2 + \rho^2|\nabla\varphi|^2 + \rho^4|\varphi|^2\right).$$

Theorem 1.4 (Carleman inequality) *There exist a positive function $\beta_0 \in C^2(\overline{\Omega})$, $s_0 > 0$ and $C > 0$ such that $\forall s \geq s_0$ and $\forall \tau \in \mathbb{R}$:*

$$I(\tau,\varphi) \leq C\left(\int_{Q_T} \rho^{\tau} e^{-2\eta}|\varphi_t \pm c\Delta\varphi|^2 + \int_0^T \int_{\omega} \rho^{\tau+3} e^{-2\eta}|\varphi|^2\right), \tag{1.8}$$

for any function φ satisfying $\varphi = 0$ on Σ_T and for which the right-hand side is defined.

More detailed information about the function β_0 can be found in [12].

Let us see how this inequality is applied to prove null and approximate controllability of a system (1.6). Consider the associated backward adjoint system:

$$\begin{cases} (\partial_t + \Delta + a)\,\varphi = 0, & Q_T := (0,T) \times \Omega, \\ \varphi = 0, & \Sigma_T := (0,T) \times \partial\Omega, \\ \varphi(T,\cdot) = \varphi^0, & \Omega. \end{cases} \tag{1.9}$$

From Theorem 1.4, for any $\varphi^0 \in L^2(\Omega)$, the solution of (1.9) satisfies (1.8) which, in particular gives the estimate:

$$\int_{Q_T} \rho^{\tau+3} e^{-2\eta}|\varphi|^2 \leq C\left(\int_{Q_T} \rho^{\tau} e^{-2\eta}|a\varphi|^2 + \int_0^T \int_{\omega} \rho^{\tau+3} e^{-2\eta}|\varphi|^2\right).$$

Since

$$\int_{Q_T} \rho^{\tau} e^{-2\eta}|a\varphi|^2 \leq \|a\|_\infty^2 \int_{Q_T} \rho^{\tau} e^{-2\eta}\,|\varphi|^2$$

it appears that

$$\int_{Q_T} \rho^{\tau}\left(\rho^3 - \|a\|_\infty^2\right) e^{-2\eta}|\varphi|^2 \leq C\int_0^T \int_{\omega} \rho^{\tau+3} e^{-2\eta}|\varphi|^2\,. \tag{1.10}$$

But, for $s > 0$, we have $\rho^3 \geq \frac{4^3 s^3}{T^6}$ and taking $s \geq \frac{T^2}{2^{5/3}}\|a\|_\infty^{2/3}$, we see that $\rho^3 - \|a\|_\infty^2 > 0$ on $(0,T)$. With this choice of the parameter s, the approximate controllability property is readily implied by (1.10).

To prove the null-controllability property, something more has to be done. According to (1.4), we have to deduce from (1.10) that

$$\int_\Omega |\varphi(0,x)|^2\,dx \le C_T \int_0^T \int_\omega |\varphi|^2,$$

for any solution of (1.9). After noting that $e^{-2\eta} \ge e^{-2s\overline{\beta_0}\rho}$ (here $\overline{\beta_0} = \max_{\overline{\Omega}} \beta_0$), the other argument is that there exists $\alpha = \alpha(\|a\|_\infty)$ such that the function $t \mapsto E(t) := e^{\alpha t}\int_\Omega \varphi^2$ is increasing on $(0,T)$ (this is quite easy: it suffices to compute $E'(t)$, to use the equation satisfied by φ and to choose α in such a way that $E'(t) \le 0$ for $t \in (0,T)$). Using this, we get

$$\begin{aligned}\int_{Q_T} \rho^\tau\left(\rho^3 - \|a\|_\infty^2\right) e^{-2\eta}|\varphi|^2 &\ge \int_0^T \rho^\tau\left(\rho^3 - \|a\|_\infty^2\right) e^{-2s\overline{\beta_0}\rho-\alpha t}\left(e^{\alpha t}\int_\Omega |\varphi|^2\,dx\right)dt\\ &\ge \int_0^T \rho^\tau\left(\rho^3 - \|a\|_\infty^2\right) e^{-2s\overline{\beta_0}\rho-\alpha t}dt \int_\Omega |\varphi(0,x)|^2\,dx\\ &\ge m_T \int_\Omega |\varphi(0,x)|^2\,dx.\end{aligned}$$

On the other hand, there exists $c_T > 0$ such that

$$\int_0^T \int_\omega \rho^{\tau+3} e^{-2\eta}|\varphi|^2 \le c_T \int_0^T \int_\omega |\varphi|^2.$$

We arrive to: Ω

$$\int_\Omega |\varphi(0,x)|^2\,dx \le C_T \int_0^T \int_\omega |\varphi|^2,$$

which is exactly the observability inequality (1.4). This proves the distributed null-controllability.

Due to this distributed null-controllability property holding true for any open subset $\omega \subset \Omega$, it allows to deduce the boundary controllability result for an arbitrary relatively open subset $\Gamma_0 \subset \partial\Omega$. Here is the (heuristic) proof. Let $\Omega' \supset \Omega$ another smooth bounded domain such that $\Omega' = \Omega \cup \Omega_0$ with $\Omega \cap \Omega_0 = \varnothing$ and $\overline{\Omega} \cap \overline{\Omega_0} \subset \Gamma_0$. By the previous result, the problem (1.6) is null-controllable on $Q'_T = (0,T) \times \Omega'$ with any $\omega \subset \Omega_0$. The restriction to $Q_T = (0,T) \times \Omega$ of a controlled solution on Q'_T is a controlled solution of system (1.7) (and the control function is just the Dirichlet trace of this controlled solution to $(0,T) \times \Gamma_0$.

Remark 1.5 Note that the Carleman inequality (1.8) allows to prove both null and approximate controllability.

1.4 First Application to a Parabolic System

Consider the 2×2 parabolic system:

$$\begin{cases} (\partial_t - \Delta)y_1 = a_{11}y_1 + a_{12}y_2 & \\ (\partial_t - d\Delta)\, y_2 = a_{21}y_1 + a_{22}y_2 + u1_\omega, & Q_T, \\ y = (y_1, y_2) = 0, & \Sigma_T, \\ y(0,\cdot) = y^0, & \Omega, \end{cases} \tag{1.11}$$

where $a_{ij} \in L^\infty(Q_T)$. The following result is proved in [2] and in a most general version in [13].

Theorem 1.6 *If there exists $\omega_0 \subset \omega$ such that $a_{12} \geq \sigma > 0$ on $(0,T) \times \omega_0$ then system (1.11) is null and approximately controllable for any $d > 0$.*

The proof of this result uses Carleman inequalities for scalar parabolic equations (see Theorem 1.4) applied to each equation of the backward adjoint system:

$$\begin{cases} -(\partial_t + \Delta)\varphi_1 = a_{11}\varphi_1 + a_{21}\varphi_2 & \\ -(\partial_t + \Delta)\varphi_2 = a_{12}\varphi_1 + a_{22}\varphi_2, & Q_T, \\ \varphi = (\varphi_1, \varphi_2) = 0, & \Sigma_T, \\ \varphi(0,\cdot) = \varphi^0, & \Omega. \end{cases} \tag{1.12}$$

The assumption $a_{12} \geq \sigma > 0$ on $(0,T) \times \omega_0$ is used to get an estimate of the L^2- norm of φ_1 on $(0,T) \times \omega_0$ using the second equation in (1.12). For more precise details, see [2, 13].

Natural questions arise at this level:

- What happens if
$$\operatorname{supp}(a_{12}) \cap \omega = \varnothing?$$
The technique of proof used for the previous theorem cannot be extended to this case. It seems that Carleman estimates cannot treat this kind of situation.
- What happens for the boundary control system:
$$\begin{cases} (\partial_t - \Delta)y_1 = a_{11}y_1 + a_{12}y_2 & \\ (\partial_t - d\Delta)y_2 = a_{21}y_1 + a_{22}y_2, & Q_T, \\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1_{\Gamma_0} v, & \Sigma_T, \\ y(0,\cdot) = y^0, & \Omega, \end{cases}$$
where Γ_0 is a relatively open subset of $\partial\Omega$?

There exist only partial answers to these two questions: even in the one-dimensional space case. In any space dimension, the single result is the one proved by Alabau-Boussouira and Léautaud in [1]. They considered the special system

$$\begin{cases} (\partial_t - \Delta)y_1 = ay_1 + by_2 & \\ (\partial_t - d\Delta)y_2 = \delta by_1 + ay_2 + u1_\omega, & Q_T, \\ y = (y_1, y_2) = 0, & \Sigma_T, \\ y(0,\cdot) = y^0, & \Omega, \end{cases} \tag{1.13}$$

and proved.

Theorem 1.7 [1] *Let $b \geq 0$ on Ω. Assume that there exists $b_0 > 0$ and $\omega_b := \mathrm{supp}(b) \subset \Omega$ satisfying the Geometric Control Condition (GCC)* (see [6]) *with $b \geq b_0$ in ω_b. Assume that ω also satisfies GCC. Then there exists $\delta_0 > 0$ such that if $0 < \sqrt{\delta}\|b\|_{L^\infty(\Omega)} \leq \delta_0$, System (1.13) is null controllable at any positive time T.*

Carleman's inequalities are not used in the proof of this result. It is obtained as a consequence of the controllability of the corresponding hyperbolic system of two wave equations and the transmutation method.

In the forthcoming sections, we will study the one-dimensional version of system (1.11) by means of the moment method.

1.5 The Moment Method

1.5.1 Presentation: Example 1

We present in this section the moment method through the study of the null controllability issue for the scalar one-dimensional heat equation:

$$\begin{cases} y' - y_{xx} = f(x)\,u(t), & Q_T = (0,T) \times (0,\pi) \\ y_{|x=0,\pi} = 0, & (0,T) \\ y_{|t=0} = y^0 & (0,\pi)\,. \end{cases} \tag{1.14}$$

Here the constraint is that the control has separate variables: $f \in L^2(0,\pi)$ and $u \in L^2(0,T)$.

If $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$, then $\{\varphi_k\}_{k\geq 1}$ is an orthonormal basis of $L^2(0,\pi)$. We look for a solution in the form

$$y(t,x) = \sum_{k\geq 1} y_k(t)\varphi_k(x)\,.$$

Set

$$f(x) = \sum_{k\geq 1} f_k \sin(kx), \quad y^0 = \sum_{k\geq 1} y_k^0 \sin(kx).$$

Then y is a solution if, and only if,

$$\begin{cases} y_k' = -k^2 y_k + f_k u(t), & (0,T) \\ y_{k|t=0} = y_k^0, & \end{cases}, \ \forall k \geq 1,$$

i.e.

$$y_k(t) = e^{-k^2 t} y_k^0 + f_k \int_0^t e^{-k^2(t-s)} u(s)\, ds, \quad \forall k \geq 1.$$

Therefore, there exists a control function $u \in L^2(0,T)$ such that the solution satisfies $y(T,x) = 0$ for any $x \in (0,\pi)$ if, and only if, there exists $u \in L^2(0,T)$ such that:

$$f_k \int_0^T e^{-k^2(T-s)} u(s)\, ds = -e^{-k^2 T} y_k^0, \quad \forall k \geq 1.$$

After a change of variable in the integral, we arrive to the reduction of the null-controllability issue to the problem ($v(t) = u(T-t)$)

$$\begin{cases} \text{Find } v \in L^2(0,T): \\ \boxed{f_k \int_0^T e^{-k^2 t} v(t)\, dt = -e^{-k^2 T} y_k^0, \quad k \geq 1.} \end{cases} \tag{1.15}$$

This is a **moment problem** in $L^2(0,T)$ with respect to the family $\{e^{-k^2 t}\}_{k\geq 1}$.

A necessary condition for the existence of a solution *for any* $y^0 \in L^2(0,\pi)$ is:

$$f_k \neq 0, \quad k \geq 1.$$

If $\{e^{-k^2 t}\}_{k\geq 1}$ admits a **biorthogonal family** $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$, i.e. a family $\{q_k\}_{k\geq 1}$ such that

$$\int_0^T e^{-k^2 t} q_\ell(t)\, dt = \delta_{k\ell}, \quad k, \ell \geq 1,$$

then a formal solution is

$$v(t) = -\sum_{k\geq 1} \frac{e^{-k^2 T}}{f_k} y_k^0 q_k.$$

The question is then: $v \in L^2(0,T)$?

The next subsection is devoted to proving the existence of this biorthogonal family $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ and to the estimate of $\|q_k\|_{L^2(0,T)}$ as k tends to ∞ (in order to prove that $v \in L^2(0,T)$).

1.5.2 Generalization of the Moment Problem

Let $\{\lambda_k\} \subset \mathbb{R}$ such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Let $\{m_k\}_{k\geq 1} \in \ell^2$ and consider the moment problem:

$$\left\{ \begin{array}{c} \text{Find } v \in L^2(0,T): \\ \boxed{\int_0^T e^{-\lambda_k t} v(t)\, dt = m_k, \quad k \geq 1.} \end{array} \right.$$

To solve this problem, we need to answer the following two questions:

1. Does the family $\{e^{-\lambda_k t}\}_{k\geq 1}$ admit a biorthogonal family $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$?
2. If a biorthogonal family $\{q_k\}_{k\geq 1}$ exists, is it possible to estimate $\|q_k\|_{L^2(0,T)}$ as $k \to \infty$?

As a first step, consider $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,\infty)$. Then following Schwartz [18], we have:

Theorem 1.8 *The family $\{e^{-\lambda_k t}\}_{k\geq 1}$ is*

1. ***complete*** *in $L^2(0,\infty)$ if $\sum_{k\geq 1} 1/\lambda_k = \infty$ and in this case, it is not minimal;*
2. ***minimal*** *in $L^2(0,\infty)$ if $\sum_{k\geq 1} 1/\lambda_k < \infty$ and in this case, it is not complete.*

Recall that a family $\{x_k\}_{k\geq 1}$ is complete in a Hilbert space H if $\overline{\operatorname{span}\{x_k, k \geq 1\}} = H$; it is minimal if for any $n \geq 1$, $x_n \notin \overline{\operatorname{span}\{x_k, k \geq 1, k \neq n\}}$.

The proof is based on classical properties of the Laplace transform and zeros of holomorphic functions.

Let $f \in L^2(0,\infty)$ and its Laplace transform F given by:

$$F(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \Re(\lambda) > 0.$$

The main properties we will use are the following (see for instance [18]):

1. $F \in \mathcal{H}(\mathbb{C}_+)$, the space of holomorphic functions on $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$.