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Lee Mckelvey , Martin Crozier
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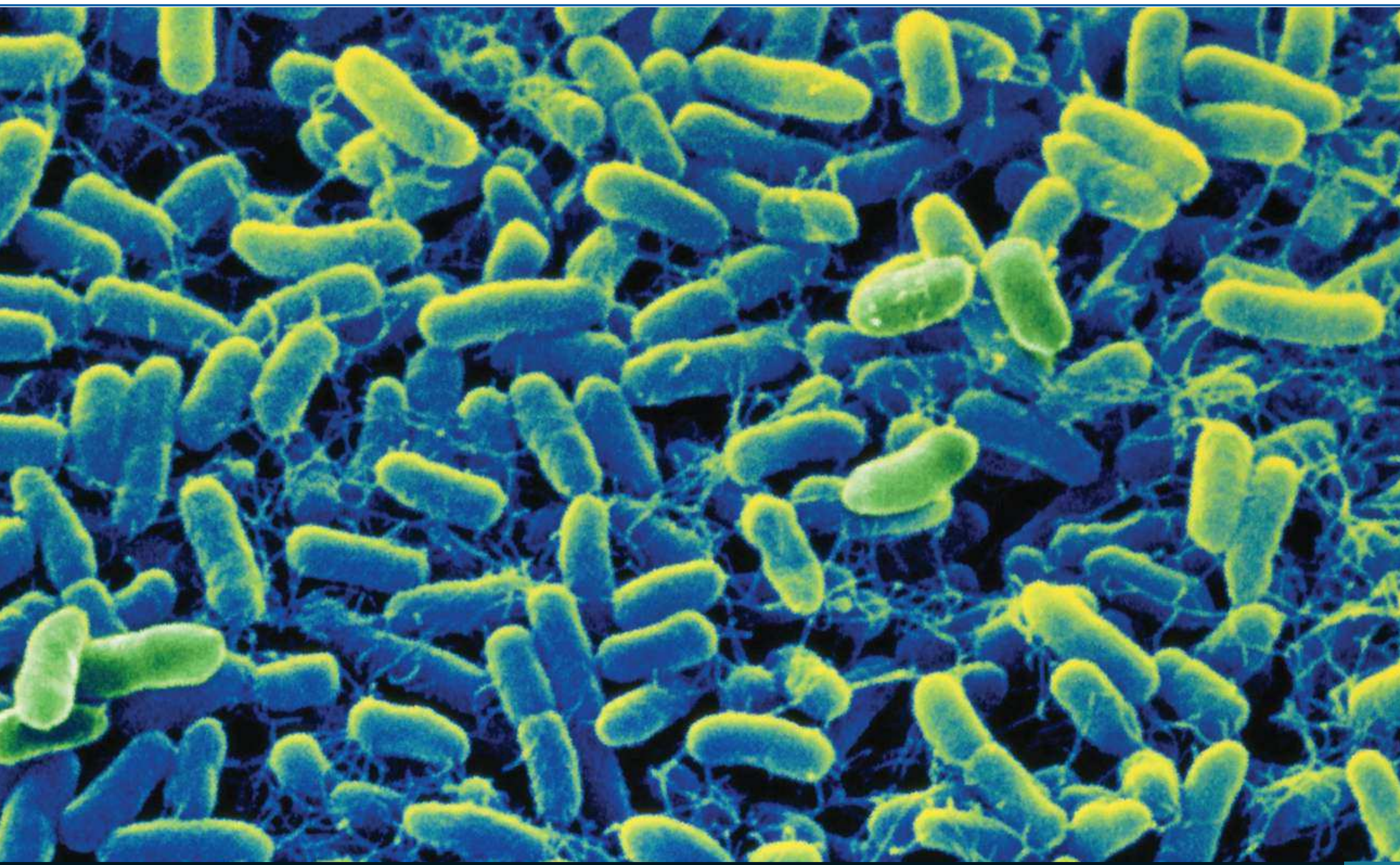
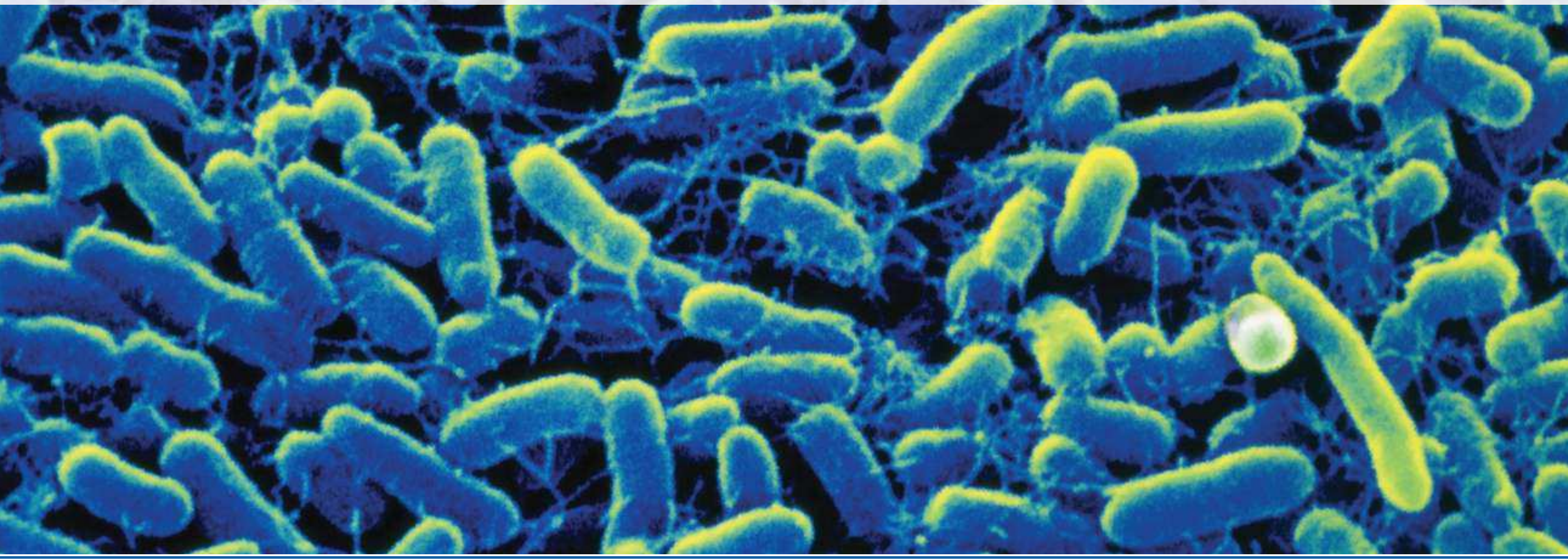


Chapter 1
Roots of polynomial equations

In this chapter you will learn how to:

- recall and use the relations between the roots and coefficients of polynomial equations
- use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation.



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PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Pure Mathematics 1, Chapter 2	Use simple substitutions to make another variable the subject.	**1** Rewrite the following equations in terms of the new variable. **a** $x^2 - 3x + 5 = 0, y = x - 2$ **b** $x^3 + 2x^2 - 4 = 0, y = \frac{2}{x}$ **c** $x^3 - 3x + 7 = 0, y = \frac{1}{x+1}$
AS & A Level Mathematics Probability & Statistics 1, Chapter 2	Work with basic sigma notation, such as Σx and Σx^2.	**2** Evaluate the following. **a** $\sum_{r=1}^{10} r$ **b** $\sum_{r=1}^{10} 3$ **c** $\sum_{r=1}^{10} (r+2)$
AS & A Level Mathematics Pure Mathematics 1, Chapter 6	Work with basic recurrence relations.	**3** Write the first six terms for the following relations. **a** $u_{n+1} = 3u_n + 2, u_1 = 1$ **b** $u_{n+2} = 2u_{n+1} - u_n + 5, u_1 = 1, u_2 = 1$

What are polynomials?

Polynomials are algebraic expressions made up of one or more variables and a sum of terms involving non-negative integer powers of variables. For example, $2x^2 - 3xy + 5x$ is a polynomial, but neither $3x^{\frac{1}{2}}$ nor $\frac{5}{y}$ are polynomials. Engineers use polynomials to ensure that a new building can withstand the force of an earthquake. Medical researchers use them to model the behaviour of bacterial colonies.

We already know how to divide a polynomial by a linear term and identify the quotient and any remainder. We have worked with simpler polynomials when completing the square of a quadratic or finding the **discriminant**. Now we will extend this knowledge to work with higher powers. We will also use algebraic manipulation to understand the conditions for complex solutions and to combine polynomials with summation notation and recurrence relations.

In this chapter, we will look at ways to find characteristics of polynomials, finding the sum and product of roots as well as other properties linked to their roots.

1.1 Quadratics

To begin with, let us look back at the **quadratic** equation $ax^2 + bx + c = 0$. If we write this in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then we can compare it to the form $(x - \alpha)(x - \beta) = 0$. This shows that the sum of the **roots** is $\alpha + \beta = -\frac{b}{a}$, and the product of the roots is $\alpha\beta = \frac{c}{a}$, as shown in Key point 1.1. Hence, we can say that $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

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KEY POINT 1.1

If we write a quadratic equation in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, the sum of the roots is $\alpha + \beta = -\frac{b}{a}$.

The product of the roots of the quadratic equation is $\alpha\beta = \frac{c}{a}$.

WORKED EXAMPLE 1.1

The quadratic equation $x^2 - 2px + p = 0$ is such that one root is three times the value of the other root. Find p.

Answer

$\alpha + 3\alpha = 2p$ Using $\alpha + \beta = -\frac{b}{a}$.

$p = 2\alpha$

$\alpha \times 3\alpha = p$ Using $\alpha\beta = \frac{c}{a}$.

$p = 3\alpha^2$

$\frac{p}{3} = \left(\frac{p}{2}\right)^2$ Equate the two results.

$4p - 3p^2 = 0$ Cross multiply.

$p = \frac{4}{3}$ Factorise and omit the case when $p = 0$.

Using $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$, we can begin to define many other results, but first we must introduce some new notation. The sum of the roots can be written as $\Sigma\alpha = \alpha + \beta$ and the product can be written as $\Sigma\alpha\beta = \alpha\beta$.

Let us consider how to determine the value of $\alpha^2 + \beta^2$. The natural first step is to expand $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$. Hence, we can say that $\alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$. We denote $\alpha^2 + \beta^2$ as $\Sigma\alpha^2$.

TIP

Note the difference between $(\Sigma\alpha)^2$ and $\Sigma\alpha^2$.

Next, look at $(\alpha - \beta)^2$. Again, expanding the brackets is a good start. So $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$. Hence, we can see that $(\alpha - \beta)^2 = \Sigma\alpha^2 - 2\Sigma\alpha\beta$.

We can write $\frac{1}{\alpha} + \frac{1}{\beta}$ as $\Sigma\frac{1}{\alpha}$. How do we find the sum of $\frac{1}{\alpha} + \frac{1}{\beta}$? First, combine the two fractions to get $\frac{\alpha + \beta}{\alpha\beta}$. We can see that this is $\frac{\Sigma\alpha}{\Sigma\alpha\beta}$. Similarly, we can write $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ as $\Sigma\frac{1}{\alpha^2}$ and we can show $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\Sigma\alpha^2}{(\Sigma\alpha\beta)^2}$.

WORKED EXAMPLE 1.2

Find $\alpha^3 + \beta^3$ in summation notation.

Answer

$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ — Use the binomial expansion for $(x+y)^n$.

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$ — Rearrange and factorise.

$\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha$ — Sum for each possible root.

Alternatively, use $-3\Sigma\alpha^2\beta$ in place of $-3\Sigma\alpha\beta\Sigma\alpha$. However, it is not as easy to calculate with this form.

Some of the results found can be written in alternative forms, using a recurrence relation such as $S_n = \alpha^n + \beta^n$. If we consider the quadratic equation $x^2 + 5x + 7 = 0$, we can see that $\alpha + \beta = -5$. This result can also be viewed as $S_1 = \alpha + \beta = -5$. To determine the value of $\alpha^2 + \beta^2$, we can approach this from another angle.

Given that α and β are roots of the original equation, we can state that $\alpha^2 + 5\alpha + 7 = 0$ and $\beta^2 + 5\beta + 7 = 0$. Adding these together gives the result $(\alpha^2 + \beta^2) + 5(\alpha + \beta) + 14 = 0$ or $S_2 + 5S_1 + 14 = 0$. Now we can work out the value of S_2 or $\alpha^2 + \beta^2$. From $S_2 + 5S_1 + 14 = 0$ and $S_1 = -5$ we have $S_2 = \alpha^2 + \beta^2 = 11$. Note this could also have been found from $\alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (-5)^2 - 2(7) = 11$.

WORKED EXAMPLE 1.3

Given that $2x^2 + 3x - 2 = 0$ has roots α, β, find the values of $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$.

Answer

$2\alpha^2 + 3\alpha - 2 = 0$

$2\beta^2 + 3\beta - 2 = 0$

$\Rightarrow 2S_2 + 3S_1 - 4 = 0$ — Add the two equations to get the recurrence form.

$S_1 = -\frac{3}{2}$ — State $S_1 = -\frac{b}{a}$ from the original quadratic equation.

$S_2 = \alpha^2 + \beta^2 = \frac{17}{4}$ — Substitute the S_1 value into the equation.

$2x^2 + 3x - 2 = 0$

$\Rightarrow 2x^3 + 3x^2 - 2x = 0$ — Multiply by x.

$\Rightarrow 2S_3 + 3S_2 - 2S_1 = 0$ — Add $2\alpha^3 + 3\alpha^2 - 2\alpha = 0$ and $2\beta^3 + 3\beta^2 - 2\beta = 0$.

$S_3 = \alpha^3 + \beta^3 = -\frac{63}{8}$ — Use the values of S_1 and S_2.

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EXERCISE 1A

1 Each of the following quadratic equations has roots α, β. Find the values of $\alpha + \beta$ and $\alpha\beta$.

a $x^2 + 5x + 9 = 0$ **b** $x^2 - 4x + 8 = 0$ **c** $2x^2 + 3x - 7 = 0$

2 Given that $3x^2 + 4x + 12 = 0$ has roots α, β, find:

a $\alpha + \beta$ and $\alpha\beta$ **b** $\alpha^2 + \beta^2$

3 $x^2 - (2 + p)x + (7 + p) = 0$ has roots that differ by 1. Find the value of p given that $p > 0$.

PS **4** If $a + b = -3$ and $a^2 + b^2 = 7$, find the value of ab and, hence, write down a quadratic equation with roots a and b.

P **5** If $x^2 + bx + c = 0$ has roots α and β, prove that:

a if $\alpha = 3\beta$, then $b^2 = \frac{16}{3}c$

b if $\alpha = \beta - 2$, then $b^2 = 4(c + 1)$.

PS **6** You are given the quadratic equation $px^2 + qx - 16 = 0$, which has roots α and β. Given also that $\alpha + \beta = -\frac{1}{2}$ and $\alpha\beta = -8$, find the values of p and q.

7 The quadratic equation $x^2 + 2x - 6 = 0$ has roots α and β. Find the values of $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

PS **8** A quadratic equation has roots α and β. Given that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$ and $\alpha^2 + \beta^2 = 12$, find two possible quadratic equations that satisfy these values.

9 The quadratic equation $3x^2 + 2x - 4 = 0$ has roots α and β. Find the values of S_1, S_2 and S_{-1}.

PS **10** You are given the quadratic equation $4x^2 - x + 6 = 0$ which has roots α and β.

a Find $\alpha^2 + \beta^2$.

b Without solving the quadratic equation, state what your value for part **a** tells you about the roots.

1.2 Cubics

In this section we will be looking at **cubic equations**. We will use the same concepts as in Section 1.1, but this time the roots will be α, β and γ.

Beginning with $ax^3 + bx^2 + cx + d = 0$, the first step is to divide by the constant a to get $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$.

Next, relate this to $(x - \alpha)(x - \beta)(x - \gamma) = 0$ to establish the relation:

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$$

Then $\alpha + \beta + \gamma = -\frac{b}{a}$, which is known as $\Sigma\alpha$ or S_1.

Other results are $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, written as $\Sigma\alpha\beta$, and $\alpha\beta\gamma = -\frac{d}{a}$, written as $\Sigma\alpha\beta\gamma$.

Recall from quadratics that $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$. This is the same result for a cubic equation, where the term $(\Sigma\alpha)^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$, as shown in Key point 1.2.

TIP

Following on from the idea you saw in Worked example 1.3, if we consider the notation $S_n = \alpha^n + \beta^n + \gamma^n$ and then use it to represent our roots, just as with quadratic equations, we can use S_2 to represent $\alpha^2 + \beta^2 + \gamma^2$ and so on.

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KEY POINT 1.2

$$(\Sigma\alpha)^2 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma$$

WORKED EXAMPLE 1.4

Find the summation form for the results $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ and $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$.

Answer

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$ — Combine the fractions.

$\Rightarrow \Sigma\frac{1}{\alpha}=\frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$ — State the result.

$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$ — Combine the fractions, as before.

$\Sigma\frac{1}{\alpha^2}=\frac{\Sigma(\alpha\beta)^2}{(\Sigma\alpha\beta\gamma)^2}$ — State the result.

All of the results derived for quadratic equations can also be written for cubics, but the algebra is more complicated. Try to convince yourself that for a cubic it is true that $\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma$.

WORKED EXAMPLE 1.5

Given that $x^3+2x^2+5=0$, find, using summation form, the values of S_1, S_2, S_3 and S_{-1}.

Answer

$S_1=-2$ — Recall this is $-\frac{b}{a}=-\frac{2}{1}$.

$S_2=(\Sigma\alpha)^2-2\Sigma\alpha\beta$

$\Rightarrow S_2=(-2)^2-2\times 0=4$ — Recall the value of $\Sigma\alpha\beta$ is given by $\frac{c}{a}=\frac{0}{1}$ as the linear term coefficient is 0.

$S_3=(\Sigma\alpha)^3-3\Sigma\alpha\beta\Sigma\alpha+3\Sigma\alpha\beta\gamma$ — The last term is $3\alpha\beta\gamma$.

$S_3=(-2)^3-3\times(0)\times(-2)+3\times(-5)$ — Substitute the values into the equation.

$S_3=-23$

$S_{-1}=\frac{0}{-5}=0$ — Recall that this result is equivalent to $\frac{c}{-d}$, which is obtained by taking the negative of the coefficient of the linear term and dividing by the constant term.

Worked example 1.5 uses the summation form, but there is a more efficient way of finding S_3 and higher powers.

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In Worked example 1.6 we will use the recurrence form to evaluate results such as S_3 and S_4.

Consider the equation $x^3 + 3x^2 + 6 = 0$. Since α, β, γ all satisfy our cubics, we can see that $\alpha^3 + 3\alpha^2 + 6 = 0, \beta^3 + 3\beta^2 + 6 = 0$ and $\gamma^3 + 3\gamma^2 + 6 = 0$.

Adding the three equations gives $\alpha^3 + \beta^3 + \gamma^3 + 3(\alpha^2 + \beta^2 + \gamma^2) + 18 = 0$ or $S_3 + 3S_2 + 18 = 0$.

WORKED EXAMPLE 1.6

For the cubic equation $3x^3 + 2x^2 - 4x + 1 = 0$, find the value of S_3.

Answer

$S_1 = -\frac{2}{3}$ — From $-\frac{b}{a}$.

$S_2 = \left(-\frac{2}{3}\right)^2 - 2 \times \left(-\frac{4}{3}\right) = \frac{28}{9}$ — Using $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

$3S_3 + 2S_2 - 4S_1 + 3 = 0$ — We know that $3\alpha^3 + 2\alpha^2 - 4\alpha + 1 = 0$ and similar equations can be made for β and γ. We add the three equations.

$S_3 = -\frac{107}{27}$ — Substitute for the final result.

We have already seen how to manipulate a polynomial to get a higher power result, such as obtaining S_3 from a quadratic equation. Imagine we want to obtain a value such as S_{-2} from a cubic equation, using only recurrence methods.

The first step would be to multiply our cubic by x^{-2} to give $ax + b + \frac{c}{x} + \frac{d}{x^2} = 0$. The recurrence formula would then be $aS_1 + 3b + cS_{-1} + dS_{-2} = 0$. Note the constant term, b, is multiplied by 3. Now we need to find only S_1 and S_{-1}, and from the original equation this is straightforward.

WORKED EXAMPLE 1.7

For the cubic equation $x^3 - 3x^2 + 4 = 0$, find the value of S_{-3}.

Answer

$S_{-1} = \frac{0}{-4} = 0$ — Recall that $S_{-1} = \frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$.

$x^3 - 3x^2 + 4 = 0$

$\Rightarrow 1 - \frac{3}{x} + \frac{4}{x^3} = 0$ — Divide by x^3.

$3 - 3S_{-1} + 4S_{-3} = 0$ — Remember that the 1 is counted three times when adding the three equations for α, β and γ.

$S_{-3} = -\frac{3}{4}$ — Substitute values into the equation.

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We can generalise from Worked example 1.7. For a general cubic of the form $ax^3 + bx^2 + cx + d = 0$, if we multiply by x^n then our recurrence formula is $aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$. Note that only constant terms get counted multiple times.

EXERCISE 1B

1 Each of the following cubic equations has roots α, β, γ. Find, for each case, $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

a $x^3 + 3x^2 - 5 = 0$ b $2x^3 + 5x^2 - 6 = 0$ c $x^3 + 7x - 9 = 0$

2 Given that $x^3 - 3x^2 + 12 = 0$ has roots α, β, γ, find the following values:

a $\alpha + \beta + \gamma$ and $\alpha\beta + \alpha\gamma + \beta\gamma$ b $\alpha^2 + \beta^2 + \gamma^2$

3 The roots of each of the following cubic equations are α, β, γ. In each case, find the values of S_2 and S_{-1}.

a $x^3 - 2x^2 + 5 = 0$ b $3x^3 + 4x - 1 = 0$ c $x^3 + 3x^2 + 5x - 7 = 0$

4 The cubic equation $x^3 - x + 7 = 0$ has roots α, β, γ. Find the values of $\Sigma\alpha$ and $\Sigma\alpha^2$.

M 5 Given that $2x^3 + 5x^2 + 1 = 0$ has roots α, β, γ, and that $S_n = \alpha^n + \beta^n + \gamma^n$, find the values of S_2 and S_3.

M PS 6 The cubic equation $x^3 + ax^2 + bx + a = 0$ has roots α, β, γ, and the constants a, b are real and positive.

a Find, in terms of a and b, the values of $\Sigma\alpha$ and $\Sigma\frac{1}{\alpha}$.

b Given that $\Sigma\alpha = \Sigma\frac{1}{\alpha}$, does this cubic equation have complex roots? Give a reason for your answer.

M PS 7 The cubic equation $x^3 - x + 3 = 0$ has roots α, β, γ.

a Using the relation $S_n = \alpha^n + \beta^n + \gamma^n$, or otherwise, find the value of S_4.

b By considering S_1 and S_4, determine the value of $\alpha^3(\beta + \gamma) + \beta^3(\alpha + \gamma) + \gamma^3(\alpha + \beta)$.

P 8 A cubic polynomial is given as $2x^3 - x^2 + x - 5 = 0$, having roots α, β, γ.

a Show that $2S_{n+3} - S_{n+2} + S_{n+1} - 5S_n = 0$.

b Find the value of S_{-2}.

M 9 The cubic equation $px^3 + qx^2 + r = 0$ has roots α, β, γ. Find, in terms of p, q, r:

a S_1 b S_2 c S_3

10 The equation $x^3 + px^2 + qx + r = 0$ is such that $S_1 = 0$, $S_2 = -2$ and $S_{-1} = \frac{1}{5}$.

Find the values of the constants p, q, r.

1.3 Quartics

Now that we are working with **quartics**, it is best to use the recurrence formula whenever we can. This is especially true for the sum of the cubes $(\alpha^3 + \beta^3 + \gamma^3 + \delta^3)$. If we want to determine the sum of the cubes of a general quartic, the best way is to first note down S_1, then determine S_2 and S_{-1}. After this, we can use the form $aS_4 + bS_3 + cS_2 + dS_1 + 4e = 0$, then divide by x to obtain S_3. This process allows us to work out other values, especially those beyond the highest power.

As we have seen with previous polynomials, there are standard results that are defined by observation from previous cases, but the algebra for some results is too complicated to be discussed here.

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So, with our roots $\alpha, \beta, \gamma, \delta$, we have $\Sigma\alpha = -\frac{b}{a}$, $\Sigma\alpha\beta = \frac{c}{a}$, $\Sigma\alpha\beta\gamma = -\frac{d}{a}$ and $\Sigma\alpha\beta\gamma\delta = \frac{e}{a}$.

We also have $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ and $S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$ and so on.

Algebraically it is much more sensible to use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

When converting a polynomial to a recurrence formula, the constant is always multiplied by n from the original equation. As an example, $x^4 - 3x^3 - 5 = 0$ would give $S_4 - 3S_3 - 20 = 0$.

WORKED EXAMPLE 1.8

A quartic polynomial is given as $x^4 + 3x^2 - x + 5 = 0$ and has roots $\alpha, \beta, \gamma, \delta$. Find the values of S_2 and S_4.

Answer

$S_1 = 0$ — Simply state the negative of the coefficient of x^3, as the coefficient of x^4 is 1.

$S_2 = 0^2 - 2 \times 3 = -6$ — Use $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

$S_4 + 3S_2 - S_1 + 20 = 0$ — Use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

$S_4 = -2$ — Final answer.

TIP

Remember that for any polynomial, $\Sigma\frac{1}{\alpha}$ is always obtained using the negative of the coefficient of the linear term over the constant term.

WORKED EXAMPLE 1.9

For the quartic $x^4 - x^3 + 2x^2 - 2x - 5 = 0$, state the values of S_1 and S_{-1}, and determine the value of S_2. State whether or not there are any complex solutions.

Answer

$S_1 = 1$ — $-1 \times (-1)$

$S_{-1} = \frac{-(-2)}{-5} = -\frac{2}{5}$ — Use $S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$.

$S_2 = (1)^2 - 2 \times 2 = -3$ — Use $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

$S_2 < 0$ so there are complex solutions.

TIP

Don't try to use an algebraic approach for quartics, especially for S_3 and higher. Use the recurrence method.

KEY POINT 1.3

For quartics, use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$ as a recurrence model to determine results.

EXERCISE 1C

1 For each of the following quartic equations, find the values of $\Sigma\alpha$ and $\Sigma\alpha\beta$.

a $x^4 - 2x^3 + 5x^2 + 7 = 0$

b $2x^4 + 5x^3 - 3x + 4 = 0$

c $3x^4 - 2x^2 + 9x - 11 = 0$

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2 The quartic equation $5x^4 - 3x^3 + x - 13 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find:

a $\Sigma\alpha$ and $\Sigma\alpha^2$ b $\Sigma\frac{1}{\alpha}$

3 A quartic equation is given as $x^4 + x + 2 = 0$. It has roots $\alpha, \beta, \gamma, \delta$. State the values of S_1 and S_{-1}, and find the value of S_2.

PS 4 The quartic equation $2x^4 + x^3 - x + 7 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $S_3 = \frac{11}{8}$, and using S_n, find the value of S_4.

5 You are given that $x^4 - x^3 + x + 2 = 0$, where the roots are $\alpha, \beta, \gamma, \delta$. Find the values of $\Sigma\alpha$, $\Sigma\alpha^2$ and $\Sigma\frac{1}{\alpha}$. Hence, determine the value of $\Sigma\alpha^3$.

PS 6 The quartic polynomial $x^4 + ax^2 + bx + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $S_2 = S_{-1}$, find S_3 in terms of a.

PS 7 The polynomial $3x^4 + 2x^3 + 7x^2 + 4 = 0$ has roots $\alpha, \beta, \gamma, \delta$, where $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

a Find the values of S_1 and S_2.

b Find the values of S_3 and S_4.

c Are there any complex roots? Give a reason for your answer.

PS 8 For the polynomial $x^4 + ax^3 + bx^2 + c = 0$, with roots α, β, γ and δ, it is given that $\alpha + \beta + \gamma + \delta = 2$, $\alpha\beta\gamma\delta = 1$ and $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$. Find the values of the coefficients a, b and c.

P 9 The roots of the quartic $x^4 - 2x^3 + x^2 - 4 = 0$ are $\alpha, \beta, \gamma, \delta$. Show that $S_4 = 9S_3$.

1.4 Substitutions

Imagine that we are given the quadratic equation $x^2 + 3x + 5 = 0$ with roots α, β and we are asked to find a quadratic that has roots $2\alpha, 2\beta$. There are two approaches that we can take.

First, consider the quadratic $(y - 2\alpha)(y - 2\beta) = 0$, then $y^2 - (2\alpha + 2\beta)y + 4\alpha\beta = 0$. If we compare this with the original, which is $\alpha + \beta = -3, \alpha\beta = 5$, then $y^2 + 6y + 20 = 0$ is the new quadratic. This method requires us to know some results, or at least spend time working them out.

A second method is to start with $y = 2x$, since each root of y is twice that of x. Then, substituting $x = \frac{y}{2}$ into the original gives $\left(\frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right) + 5 = 0$. Alternatively, multiplying by 4, $y^2 + 6y + 20 = 0$. This second approach does not need the values of roots. It just needs the relationship between the roots of each polynomial.

TIP

You learned in AS & A Level Mathematics Pure Mathematics 1 Coursebook how to find inverse functions by interchanging x and y. The same process is helpful here.

WORKED EXAMPLE 1.10

Given that $x^2 - 2x + 12 = 0$ has roots α, β, find the quadratic equation with roots $\frac{\alpha}{3}, \frac{\beta}{3}$.

Answer

$y = \frac{x}{3} \Rightarrow x = 3y$ — Rearrange to make x the subject.

$(3y)^2 - 2(3y) + 12 = 0$ — Substitute for x.

$3y^2 - 2y + 4 = 0$ — Multiply out terms and simplify.