CHAPTER XIX

LAPLACE'S FIRST THREE MEMOIRS.

741. THE investigations of Laplace on Attractions and the Figure of the Earth fall naturally into five divisions. The first division consists of three memoirs, which treat the subjects without the use of what we now call the *Potential Function*, or of that branch of analysis which we now call *Laplace's Functions*. The second division consists of a separate volume which uses the Potential Function. The third division consists of various memoirs which use both the Potential Function and Laplace's Functions. The fourth division is formed by the republication of the preceding researches in the first and second volumes of the *Mécanique Céleste*. The fifth division consists of researches subsequent to the publication of the second volume of the *Mécanique Céleste*; they are reproduced in the fifth volume of the *Mécanique Céleste*.

We shall consider in the present Chapter Laplace's first three memoirs.

742. We begin with the seventh volume of the *Mémoires de Mathématique...par divers Savans*...1773: the date of publication is 1776. This volume contains two memoirs by Laplace, which among other subjects treat largely of Probability: see pages 473...475 of my *History...of Probability*. The part of the volume with which we are now concerned is entitled *Sur la figure de la Terre*; it occupies pages 524...534. It is not stated when these investigations were sent to the Academy; but from the title of the volume in which they appear we see that Laplace was not a member of the Academy when they were sent.

743. Laplace begins thus on page 524:

Lorsque Newton voulut déterminer la figure de la Terre, il considéra cette Planète comme une masse fluide homogène, et il supposa que la figure qu'elle a prise en vertu de son mouvement de rotation est celle d'un sphéroïde elliptique. Cette supposition étoit fort précaire; les Géomètres en ont ensuite démontré la possibilité; mais si la figure nécessaire pour l'équilibre, au lieu d'être elliptique, eût été d'un autre genre, on auroit été fort embarrassé pour la déterminer, parce qu'il est beaucoup plus facile de s'assurer si une figure donnée convient à l'équilibre, que de chercher immédiatement celles qui peuvent y convenir. Ce dernier Problème est sans contredit un des points les plus intéressans du Système du Monde; voici quelques recherches qui y sont relatives.

744. Thus the following is the problem to be discussed: a mass of homogeneous fluid in the form of a figure of revolution nearly spherical rotates with uniform angular velocity round its axis of figure and remains in relative equilibrium; determine the form. I call this problem Legendre's, because he was the first to solve it with tolerable success.

745. Let there be a circle of radius unity; let ψ be the angle which the radius to any point makes with a fixed radius: so that the ordinate of this point is $\sin\psi$. Produce this ordinate until it becomes $\sin\psi + \frac{\alpha y}{\sin\psi}$, where α is very small, and y is some function of ψ. Put x for $\cos\psi$. Then Laplace arrives at a differential equation between y and x of an infinite order, to determine the required generating curve; that is a differential equation involving $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ and so on *ad infinitum*.

746. The preceding notation does not look very promising; in fact Laplace does not explicitly start with it, but arrives at it as he proceeds. Unless y is very small when ψ is very small the process is not satisfactory. Moreover Laplace in order to form his differential equation expands a function into a series without discussing whether the series is convergent.

747. The main result at which he arrives deserves notice. He wishes to know whether equilibrium would subsist for any other form besides an exact sphere when there is no rotation. He cannot completely solve this problem; but he shews that y cannot consist of a series of the form $ax^\lambda + bx^\mu + cx^\nu + \dots$, where $\lambda, \mu, \nu, \dots$ are numbers in *descending* order of magnitude. That is y cannot consist of a *finite* number of terms each involving a power of x; nor can y be an infinite series of *descending* powers of x: but he does not shew that y cannot be an infinite series of *ascending* powers of x.

When the fluid is supposed to rotate Laplace's demonstration amounts to shewing that among all series, finite or infinite, which can be arranged in descending powers of x, the only admissible form of y is $ax^2 + bx + c$; where a, b, and c are constants.

748. Laplace's demonstration is difficult, but satisfactory; that is to say after the points to which we have drawn attention in Art. 746, no very serious objection will occur to a reader.

After finishing his demonstration, Laplace says on his page 534:

.. Je dois observer ici que M. d'Alembert a déjà fait une remarque semblable pour le cas où les exposans de x sont des nombres entiers et positifs (*voyez le tome* V *des Opuscules* de ce grand Géomètre).

These words are quite consistent with the supposition, that Laplace had found the error which we have pointed out in D'Alembert's process; because *to make a remark* is far less than *to demonstrate*. See Art. 576.

749. Laplace concludes with these words:

Il seroit utile d'étendre ces recherches au cas où les couches de la masse fluide sont inégalement denses; c'est ce que je me propose de faire dans un autre Mémoire.

The intention here expressed was not carried into effect until the publication of Laplace's seventh memoir in the Paris *Mémoires* for 1789.

750. All that Laplace's first memoir contains on our subject is reproduced with better notation in his second memoir to which

we shall proceed in our next Article: it is therefore unnecessary to treat the first memoir with much detail. In the second memoir we shall find that the radius vector of the generating curve is denoted by $1 + \alpha f(\cos\psi)$, where α is very small; so that what was called $\sin\psi + \frac{\alpha y}{\sin\psi}$ in Art. 745 is equal to the $\sin\psi\{1+\alpha f(\cos\psi)\}$ of the second memoir: that is $\frac{y}{\sin\psi}$ of the first memoir is $\sin\psi f(\cos\psi)$ of the second memoir, or y of the first memoir is $\sin^2\psi f(\cos\psi)$ of the second memoir. In the second memoir y is put for $f(\cos\psi)$.

751. In the Paris *Mémoires* for 1772, *Seconde Partie*, published in 1776, we have a memoir by Laplace entitled *Recherches sur le Calcul Intégral et sur le Système du Monde;* at a later part of the volume there are some *Additions* to this memoir: among these Additions we have a section entitled *De l'Équilibre des Sphéroïdes homogènes,* which occupies pages 536...554 of the volume.

752. The problem proposed to be discussed is the same as that of the preceding memoir: see Art. 744. Laplace was not able to solve the problem completely; but he reproduced his former demonstration, somewhat improved, that for a large number of figures the relative equilibrium was impossible.

753. But although he did not in this memoir arrive at the necessary form for equilibrium, yet he obtained a very remarkable result: namely, that the law of the variation of gravity, whatever be the form of equilibrium, is the same as for an oblatum. We will give in substance the method by which Laplace obtains this result.

We may remark that Laplace investigates the polar expression for an element of mass, namely in the usual modern notation $r^2 dr \sin\theta\, d\theta\, d\phi$: see his page 539. The investigation is in fact the same as we now have in our elementary books: see *Integral Calculus,* third edition, Art. 207.

In his first memoir Laplace used this polar expression but did not investigate it; he merely says, "on trouvera facilement...": see his page 525. See also Art. 710.

754. Let there be a curve differing very little from a circle, and symmetrical with respect to a diameter. Let half the length of this diameter be unity, and let the length of a radius vector inclined at an angle ψ to the diameter be $1 + \alpha f(\cos\psi)$, where f denotes any function, and α is a very small quantity the square of which we shall neglect. Suppose a solid formed by the revolution of this curve round the diameter which divides it symmetrically; take this diameter for the direction of the axis of x: then the equation to the surface will be

$$\sqrt{(x^2+y^2+z^2)} = 1 + \alpha f\left\{\frac{x}{\sqrt{(x^2+y^2+z^2)}}\right\} \text{ .... (1).}$$

We propose to find the attraction of the solid at a point situated on its surface; this point without loss of generality we may take in the plane of (x, y): let ψ be the angle between the radius vector of this point and the axis of revolution.

Put $x = \xi\cos\psi + \eta\sin\psi$, $y = \xi\sin\psi - \eta\cos\psi$; thus (1) becomes

$$\sqrt{(\xi^2+\eta^2+z^2)} = 1 + \alpha f\left\{\frac{\xi\cos\psi + \eta\sin\psi}{\sqrt{(\xi^2+\eta^2+z^2)}}\right\} \text{ .... (2).}$$

We can now pass easily to polar coordinates which have their origin at the attracted point: put

$$\xi = h - r\sin\theta\cos\phi, \quad \eta = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

where $$h = 1 + \alpha f(\cos\psi).$$

[I use θ for Laplace's p, and $\frac{\pi}{2} - \phi$ for his q.]

Thus (2) becomes

$$\sqrt{(h^2 - 2hr\sin\theta\cos\phi + r^2)}$$
$$= 1 + \alpha f\left\{\frac{h\cos\psi - r\sin\theta\cos\phi\cos\psi + r\sin\theta\sin\phi\sin\psi}{\sqrt{(h^2 - 2hr\sin\theta\cos\phi + r^2)}}\right\} \text{...(3).}$$

We proceed to find from (3) the value of r to the order of approximation which we require.

If $\alpha = 0$, we should get $r = 2\sin\theta\cos\phi$; assume then

$$r = 2\sin\theta\cos\phi + \rho,$$

where ρ will be very small.

Substitute in (3) and we obtain

$$\alpha f(\cos\psi) - 2\alpha\sin^2\theta\cos^2\phi f(\cos\psi) + \rho\sin\theta\cos\phi = \alpha f(u),$$

where u stands for $\cos\psi - 2\sin^2\theta\cos\phi\cos(\phi+\psi)$; so that

$$\rho = \frac{\alpha(2\sin^2\theta\cos^2\phi - 1)}{\sin\theta\cos\phi} f(\cos\psi) + \frac{\alpha}{\sin\theta\cos\phi} f(u).$$

We may also arrange the value of ρ thus,

$$\rho = 2\alpha\sin\theta\cos\phi f(\cos\psi) + \alpha\frac{f(u) - f(\cos\psi)}{\sin\theta\cos\phi};$$

and this shews that ρ remains small even when $\sin\theta\cos\phi$ is very small: for then we have $f(u)$ very nearly equal to $f(\cos\psi)$.

Now the attraction at the point resolved along the radius vector

$$= \iint r\sin^2\theta\cos\phi\, d\theta\, d\phi = \iint (2\sin\theta\cos\phi + \rho)\sin^2\theta\cos\phi\, d\theta\, d\phi;$$

the limits for θ are 0 and π; the limits for ϕ are $-\left(\frac{\pi}{2}+\beta\right)$ and $\frac{\pi}{2}-\beta$, where β is some function of ψ, which is of the order of α. It is easy to see that for our approximation we may proceed as if β were zero. Denote this resolved attraction by A: thus

$$A = 2\iint \sin^3\theta\cos^2\phi\, d\theta d\phi + \alpha f(\cos\psi)\iint \sin\theta(2\sin^2\theta\cos^2\phi - 1)\, d\theta d\phi$$
$$+ \alpha\iint \sin\theta f(u)\, d\theta\, d\phi.$$

The first and second integrations may be easily effected; with respect to these it is *exactly* true that we may proceed as if β were zero: and we obtain

$$A = \frac{4\pi}{3} - \frac{2\alpha\pi}{3} f(\cos\psi) + \alpha\iint \sin\theta f(u)\, d\theta\, d\phi.$$

Let B denote the attraction resolved in the meridian plane at right angles to the radius vector; then

$$B = \iint (2\sin\theta\cos\phi + \rho)\sin^2\theta\sin\phi\, d\theta\, d\phi$$

$$= 2 \iint \sin^2\theta \cos\phi \sin\phi \, d\theta \, d\phi + 2\alpha f(\cos\psi) \iint \sin^2\theta \sin\phi \cos\phi \, d\theta \, d\phi$$
$$+ \alpha \iint \frac{f(u) - f(\cos\psi)}{\cos\phi} \sin\theta \sin\phi \, d\theta \, d\phi.$$

It is easy to see that the first and second integrals vanish; so that

$$B = \alpha \iint \frac{f(u) - f(\cos\psi)}{\cos\phi} \sin\theta \sin\phi \, d\theta \, d\phi :$$

this integral is finite, for $f(u) - f(\cos\psi)$ vanishes when $\cos\phi$ vanishes.

The last integral may be transformed. By integration by parts we have

$$\int \{f(u) - f(\cos\psi)\} \sin\theta \, d\theta$$
$$= -\cos\theta \{f(u) - f(\cos\psi)\} - 4 \int \cos^2\theta \sin\theta f'(u) \cos\phi \cos(\phi + \psi) \, d\theta ;$$

when this is taken between the limits 0 and π the first term vanishes; so that we have

$$B = -4\alpha \iint \cos^2\theta \sin\theta f'(u) \sin\phi \cos(\phi + \psi) \, d\theta \, d\phi$$
$$= -2\alpha \iint \cos^2\theta \sin\theta f'(u) \{\sin(2\phi + \psi) - \sin\psi\} \, d\theta \, d\phi.$$

Now $\quad u = \cos\psi - \sin^2\theta \{\cos(2\phi + \psi) + \cos\psi\}$;

so that $\quad \int \sin(2\phi + \psi) f'(u) \, d\phi = \dfrac{f(u)}{2\sin^2\theta}$,

and this vanishes when taken between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Thus finally

$$B = 2\alpha \sin\psi \iint \cos^2\theta \sin\theta f'(u) \, d\theta \, d\phi \text{ ....(4).}$$

755. We shall now shew that

$$\frac{dA}{d\psi} = \frac{2\alpha\pi}{3} f'(\cos\psi) \sin\psi - \frac{B}{2} \text{ .... (5).}$$

We have $\frac{dA}{d\psi} = \frac{2\alpha\pi}{3} f'(\cos\psi)\sin\psi + \alpha\iint \sin\theta f'(u)\frac{du}{d\psi}\,d\theta\,d\phi,$

and $$u = \cos^2\theta\cos\psi - \sin^2\theta\cos(2\phi+\psi),$$

so that $$\frac{du}{d\psi} = -\cos^2\theta\sin\psi + \sin^2\theta\sin(2\phi+\psi).$$

Hence

$$\frac{dA}{d\psi} = \frac{2\alpha\pi}{3} f'(\cos\psi)\sin\psi - \alpha\sin\psi\iint\cos^2\theta\sin\theta f'(u)\,d\theta\,d\phi$$

$$= \frac{2\alpha\pi}{3} f'(\cos\psi)\sin\psi - \frac{B}{2}.$$

This is the first appearance in Laplace's writings of a theorem which he seems to have valued highly: see Art. 652. We shall meet the theorem again several times: it appears in a different form in the *Mécanique Céleste*, Livre III. § 10.

756. Now let us suppose that the attracting body is a fluid, or at least that there is a superficial stratum of fluid. Then for relative equilibrium the resolved part of the force along the tangent to the meridian must vanish. This part consists of the resolved parts of A and B, together with the centrifugal force.

The direction of A is nearly at right angles to the tangent; the cosine of the angle between the directions is $-\alpha f'(\cos\psi)\sin\psi$. The direction of B makes only an indefinitely small angle with the tangent. Hence, denoting the angular velocity by ω, we have

$$B - A\alpha f'(\cos\psi)\sin\psi - \omega^2\sin\psi\cos\psi = 0;$$

that is, neglecting the square of α,

$$B = \frac{4\pi\alpha}{3} f'(\cos\psi)\sin\psi + \omega^2\sin\psi\cos\psi \;\;\ldots\ldots\ldots(6).$$

Let P denote the gravity at the point considered;

then approximately $\quad P = A - \omega^2\sin^2\psi,$

therefore $\dfrac{dP}{d\psi} = \dfrac{dA}{d\psi} - 2\omega^2 \sin\psi \cos\psi$

$$= \frac{2a\pi}{3} f'(\cos\psi)\sin\psi - \frac{B}{2} - 2\omega^2 \sin\psi\cos\psi \text{ by (5)}$$

$$= -\frac{5}{2}\omega^2 \sin\psi\cos\psi \text{ by (6).}$$

Therefore $P = \text{constant} - \frac{5}{4}\omega^2\sin^2\psi = P_0 - \frac{5}{4}\omega^2\sin^2\psi$, where P_0 denotes the force of gravity at the pole. This is the result which, as we stated in Art. 753, Laplace established.

757. We shall now form the differential equation of an infinite order at which Laplace arrives.

Put $f(\cos\psi) = y$, and $\cos\psi = x$, so that $f'(\cos\psi) = \dfrac{dy}{dx}$.

Then $f'(u) = f'\{\cos\psi - 2\sin^2\theta\cos\phi\cos(\phi+\psi)\}$

$= f'(\cos\psi - z)$ say;

hence expanding by Taylor's Theorem this becomes

$$\frac{dy}{dx} - z\frac{d^2y}{dx^2} + \frac{z^2}{\lfloor 2}\frac{d^3y}{dx^3} - \frac{z^3}{\lfloor 3}\frac{d^4y}{dx^4} + \dots$$

Then equating the values of B given by (4) and (6), and dividing by $\sin\psi$, we obtain

$$2a\iint \cos^2\theta\sin\theta\left\{\frac{dy}{dx} - z\frac{d^2y}{dx^2} + \frac{z^2}{\lfloor 2}\frac{d^3y}{dx^3} - \dots\right\} d\theta\, d\phi$$

$$= \frac{4\pi a}{3}\frac{dy}{dx} + \omega^2 x.$$

The term $\dfrac{dy}{dx}$ will disappear from this equation, because $\iint \cos^2\theta\sin\theta\, d\theta\, d\phi$ between the proper limits $= \dfrac{2\pi}{3}$. Thus we have

$$\iint \cos^2\theta\sin\theta\left\{z\frac{d^2y}{dx^2} - \frac{z^2}{\lfloor 2}\frac{d^3y}{dx^3} + \frac{z^3}{\lfloor 3}\frac{d^4y}{dx^4} - \dots\right\} d\theta\, d\phi = -\frac{\omega^2 x}{2a} \dots (7).$$

758. The integrations with respect to θ and ϕ in (7) can be easily effected. Consider first the integration with respect to ϕ. We have

$$z = \{2 \sin^2 \theta \cos \phi \cos (\phi + \psi)\}^n = \sin^{2n} \theta \{\cos \psi + \cos (2\phi + \psi)\}^n.$$

Thus we require $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\cos \psi + \cos (2\phi + \psi)\}^n \, d\phi$.

Now $\{\cos \psi + \cos (2\phi + \psi)\}^n = \cos^n \psi + n \cos^{n-1} \psi \cos (2\phi + \psi)$

$$+ \frac{n(n-1)}{\lfloor 2} \cos^{n-2} \psi \cos^2 (2\phi + \psi) + \dots$$

When we integrate between the limits the terms which involve *odd* powers of $\cos (2\phi + \psi)$ disappear, and those which involve even powers are easily obtained. For example, take

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 (2\phi + \psi) \, d\phi;$$

put t for $2\phi + \psi$, then we get

$$\frac{1}{2} \int_{-\pi+\psi}^{\pi+\psi} \cos^4 t \, dt,$$

and this obviously $= \frac{1}{2} \int_0^{2\pi} \cos^4 t \, dt$

$$= \frac{4}{2} \int_0^{\frac{\pi}{2}} \cos^4 t \, dt = 2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3 \cdot 1}{4 \cdot 2} \pi.$$

In this way we easily see that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\cos \psi + \cos (2\phi + \psi)\}^n \, d\phi$

$$= \pi \left\{ x^n + \frac{n(n-1)}{2^2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 4^2} x^{n-4} \right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2^2 \cdot 4^2 \cdot 6^2} x^{n-6} + \dots \right\}.$$