CHAPTER I.

NEWTON.

1. NEARLY two centuries have passed away since the publication of the greatest work known in the history of science. Newton's *Philosophiæ Naturalis Principia Mathematica* appeared in 1687. The volume is in quarto; it contains a title-leaf, a dedication to the Royal Society on another leaf, a preface on two pages, some Latin verses by Halley on two pages, then the text consisting apparently of 510 pages, followed by errata on one leaf. I say the text consists apparently of 510 pages; there are, however, no pages numbered from 384 to 399 inclusive: the third Book begins on page 401, and so perhaps some of this was struck off before the second Book was finished, and a gap was left in the number of pages which proved too large.

The second edition of the *Principia* appeared in 1713, edited by Cotes; the third in 1726, edited by Pemberton. Newton was born in 1642, and died in 1727.

2. Newton's researches on Attractions form Sections XII. and XIII. of the first Book of the *Principia*. Section XII. contains Propositions 70...84; it relates to the attraction of spherical bodies. Section XIII. contains Propositions 85...93; it relates to the attraction of bodies which are not spherical. These Sections remain unchanged in the other two editions of the *Principia*.

3. In his Proposition 70, Newton shews that a particle will be in equilibrium if placed at any point of the hollow part of an indefinitely thin spherical shell, which attracts according to

the law of the inverse square of the distance. Newton's demonstration is remarkable for its simplicity. Let any indefinitely small double cone be described with the position of the attracted particle as vertex; the areas of the indefinitely small surfaces which the cone intercepts on the shell are ultimately as the squares of the distances of the elements from the vertex: thus the elements exert equal attractions in opposite directions. Therefore the entire shell exerts no action in any direction.

We assume here and in the other propositions that the attracting body is homogeneous unless the contrary is stated.

4. In his Proposition 71, Newton shews that an indefinitely thin spherical shell attracts an external particle towards the centre of the shell, with a force which varies inversely as the square of the distance of the particle from the centre of the shell. Newton's demonstration is geometrical; it can, however, be easily translated into an analytical form.

Let a be the radius of the shell, c the distance of the particle from the centre of the shell, ds an element of the length of the circle which by revolution round the straight line joining the particle to the centre generates the surface of the shell, r the distance of this element from the particle, y its distance from the axis of revolution. Then the element of surface generated by the revolution of ds is $2\pi y ds$; and the attraction of this element along the axis is $\frac{2\pi k\rho y ds}{r^2}\cos\theta$; where k is the thickness of the shell, ρ is the density, and θ is the angle between the direction of r and the axis. Let p denote the perpendicular from the centre of the shell on the direction of r. We have

$$p = c\sin\theta, \qquad r^2 - 2rc\cos\theta + c^2 = a^2;$$

hence $$\frac{dr}{d\theta} = -\frac{rc\sin\theta}{r - c\cos\theta}, \qquad \frac{ds}{d\theta} = \frac{ar}{r - c\cos\theta}.$$

Thus $$\frac{2\pi k\rho y ds}{r^2}\cos\theta = \frac{2\pi k\rho y\cos\theta}{r^2}\cdot\frac{ar d\theta}{r - c\cos\theta}$$

$$= \frac{2\pi k\rho y a dp}{cr(r - c\cos\theta)} = \frac{2\pi k\rho a p dp}{c^2\sqrt{(a^2 - p^2)}}.$$

Hence the resultant attraction of the shell will be found by integrating this expression between appropriate limits. If we take 0 and a as the limits of p, we obtain the attraction of either of the two parts into which the shell is divided by the curve of contact of straight lines drawn from the particle to *touch* the shell; hence these two parts exert equal attractions, and the attraction of the whole shell is

$$2 \times \frac{2\pi k\rho a}{c^2} \int_0^a \frac{pdp}{\sqrt{(a^2 - p^2)}},$$

which varies inversely as c^2.

The value of the definite integral is a; and thus the attraction of the whole shell is $\frac{4\pi k\rho a^2}{c^2}$

We see from this investigation that if any right cone be taken having its vertex at the position of the particle, and its axis coincident with the straight line drawn from the particle to the centre of the shell, we can determine the attraction which is exerted by the portion of the shell cut off by the cone: we have only to give an appropriate value to the upper limit of p in the integration. We may observe too that if any indefinitely small cone be taken having its vertex at the position of the particle, the two distinct portions of the shell which it intercepts exert equal attractions.

We may observe that Proposition 71 has been very well treated by Professor Thomson: see *Cambridge and Dublin Mathematical Journal*, Vol. III. page 146.

5. Propositions 72...76 extend the conclusions obtained respecting indefinitely thin spherical shells to spheres.

It appears that Newton arrived at his theorems respecting the attraction of spheres in 1685. See the *Mécanique Céleste*, Vol. V., page 87; Rigaud's *Historical Essay on the first publication of the Principia*, page 27 of the Appendix.

6. Newton's Propositions 77 and 78 relate to the case in which the law of attraction is that of the direct distance.

Between Propositions 78 and 79 a Lemma occurs.

Let x and y be the co-ordinates of a point on a circle; r the distance of the point from any fixed origin. We have

$$r^2 = x^2 + y^2;$$

therefore $$rdr = xdx + ydy = (x - c)\,dx + ydy + cdx.$$

Let c be the distance of the centre of the circle from the origin, the centre being on the axis of x. Then $(x-c)\,dx+ydy=0$; therefore $rdr = cdx$. This result constitutes the Lemma; it is of course demonstrated geometrically by Newton. Throughout this Chapter we shall translate Newton's geometrical processes into modern mathematical language.

7. In his Proposition 79, Newton finds the attraction of a zone of an indefinitely thin spherical shell on a particle at the centre of the shell.

Take the axis of the zone for that of x, and a line at right angles to this through the centre of the shell for the axis of y; let a be the radius of the sphere. Then $2\pi a dx$ represents an element of the zone; and the attraction of this element will be $kf.\,2\pi a \cdot \frac{x}{a}\,dx$, where k denotes the thickness of the shell, and f is a constant which denotes the attraction of a unit of matter, condensed at a point, on a particle at the distance a. Hence the attraction of the zone $= kf.\,2\pi \int xdx$, the integral being taken between proper limits. If the zone be the segment cut off by the plane $x = x_1$, we have to integrate between the limits x_1 and a. Thus we obtain $kf\pi\,(a^2 - x_1^2)$, that is $kf\pi y_1^2$, where y_1 is the radius of the base of the segment.

8. Newton's Proposition 80 investigates the attraction of a sphere on an external particle, the law of attraction being expressed by any function of the distance.

Divide the sphere into elements by describing spherical surfaces indefinitely close to each other from the external particle as centre. Let r be the radius of one of the surfaces of one of the segments of shells thus obtained, and y the radius of the base of the segment; let $\phi(r)$ denote the law of attraction. Then by Art. 7

we have $\pi dr\phi(r)y^2$ for the attraction of the segment. Let c be the distance of the external particle from the centre of the sphere; then by Art. 6 we have $dr = \frac{cdx}{r}$; thus the attraction becomes $\frac{c\pi}{r}\phi(r)y^2dx$. Hence the resultant attraction of the sphere is $c\pi\int_{c-a}^{c+a}\frac{\phi(r)}{r}y^2dx$, where a is the radius of the sphere.

9. Newton's Proposition 81 amounts to a transformation of the integral obtained in Art. 8.

We have $y^2 = a^2 - (c-x)^2$, and also $y^2 = r^2 - x^2$;

therefore $$r^2 = a^2 - c^2 + 2cx.$$

Put $\frac{c^2 - a^2}{2c} = b$, and $x - b = x'$; thus

$$r^2 = 2c(x-b) = 2cx',\ y^2 = -2bc + 2cx - x^2 = 2cx' - (x'+b)^2.$$

Hence the resultant attraction

$$= c\pi\int\frac{2(c-b)x' - x'^2 - b^2}{r}\phi(r)\,dx',$$

the limits of x' being $c-a-b$ and $c+a-b$.

As soon as $\phi(r)$ is known we can substitute for r in terms of x', and effect the integration. Newton gives three examples:

$$(1)\quad \phi(r) = \frac{\mu}{r}, \qquad (2)\quad \phi(r) = \frac{\mu}{r^3}, \qquad (3)\quad \phi(r) = \frac{\mu}{r^4},$$

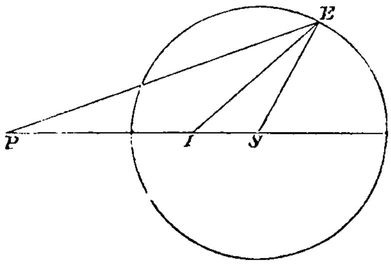
where μ in each case is a constant.

10. Newton's Proposition 82 shews that the calculation of the attraction of a sphere on an internal particle may be made to depend on the calculation of the attraction on an external particle.

We have found in Art. 8 for the attraction of an element of the sphere $\pi dr\,\phi(r)\,y^2$, where r is the distance of the particle from every point of the element. In the same manner $\pi dr'\,\phi(r')\,y^2$ will express the attraction of the corresponding element on another particle which is at the distance r' from every point of the element. The two particles and the centre of the sphere are of course on the same straight line. Suppose the second particle within the sphere; let c be the distance of the first particle from the centre of the sphere, c' that of the second, a the radius of the sphere. Let c and c' be taken so that $cc' = a^2$.

In the diagram let

$$SP = c, \quad SI = c', \quad EP = r, \quad EI = r'.$$



As $cc' = a^2$ the triangles PSE and ESI are similar; thus we have

$$\frac{r'}{r} = \frac{c'}{a}.$$

In finding the attraction on the internal particle, we may if we please suppose the matter to be removed which forms a sphere having its centre at the internal particle and radius equal to $a - c'$: thus the limits of integration become $r' = a - c'$ and $r' = a + c'$.

Suppose $\phi(r) = \frac{\mu}{r^n}$; the attraction on the internal particle

$$= \pi \int y^2 \phi(r') \, dr' = \pi\mu \int \frac{y^2}{r'^n} dr';$$

the limits being $a - c'$ and $a + c'$. Now put $\frac{rc'}{a}$ for r'; thus we get $\pi\mu \left(\frac{a}{c'}\right)^{n-1} \int \frac{y^2}{r^n} dr$, and the limits of r are $\frac{a^2}{c'} - a$ and $\frac{a^2}{c'} + a$, that is, $c - a$ and $c + a$.

Hence the attraction on the internal particle at the distance c' from the centre is equal to the product of $\left(\frac{a}{c'}\right)^{n-1}$ into the attraction on the external particle at the distance c from the centre.

And $$\left(\frac{a}{c'}\right)^{n-1} = \frac{(cc')^{\frac{n-1}{2}}}{c'^{n-1}} = \left(\frac{c'}{c}\right)^{\frac{1}{2}} \left(\frac{c}{c'}\right)^{\frac{n}{2}}$$

This is the result which Newton intended to give. He says that the attraction on the particle at I is to the attraction on the particle at P, in ratione composita ex subduplicatâ ratione distantiarum a centro IS et PS, et subduplicatâ ratione virium centripetarum, in locis illis P et I, ad centrum tendentium. It seems to me that instead of P et I we ought to read I et P.

11. Newton's Propositions 83 and 84 shew briefly that there would be no difficulty in calculating the attraction of a homogeneous segment of a sphere on a particle situated on the axis of the segment.

12. Newton's Propositions 85, 86, and 87 involve simple general statements, which need not be repeated here.

Propositions 88 and 89 shew that if the law of attraction is that of the direct distance, the resultant attraction exerted by a body or a system of bodies is the same as if the body or system were collected at its centre of gravity.

13. Proposition 90 finds the attraction of a circular lamina on a particle which is situated on the straight line drawn through the centre of the lamina at right angles to its plane. Then Proposition 91 shews how from this we can deduce the attraction of a solid of revolution on a particle situated at any point of the axis. Newton makes this depend on the problem of finding the area of a certain curve; that is, in modern language, he leaves only a single integration to be effected. He takes the case of a right cylinder for an example; and he also states the result for the case of an ellipsoid of revolution, which he calls a spheroid. He shews by a special investigation that a shell bounded by two concentric similar and similarly situated ellipsoidal surfaces of revolution exerts no attraction on a particle placed at any point within the hollow part; the demonstration is very striking and well known: see *Statics*, Chapter XIII. Of course this result includes Newton's Proposition 70 as a particular case; but the demonstrations differ and should be carefully compared.

Hence follows the important result that along the same radius vector from the centre the attraction of an ellipsoid of revolution on an internal particle varies as the distance from the centre.

Newton contented himself with considering ellipsoids of revolution; but the processes and results of Proposition 91, as we now know, may be easily extended to ellipsoids which are not solids of revolution.

14. Proposition 92 shews how we may find experimentally the law of attraction of given matter. Form the given matter into such a shape that the resultant attraction can be obtained when the law of attraction is assumed; for example, the shape of a sphere. Then ascertain by experiment what the resultant attraction really is at various distances; and thus we shall be guided in assuming a law of attraction and verifying the assumption.

15. Proposition 93 treats of the attraction of an infinite plane lamina, deducing it from Proposition 90. A scholium to this Proposition gives some interesting remarks relating to the motion of a particle acted on by a force the direction of which is always parallel to a fixed straight line.

16. Newton's Propositions on Attractions are illustrated by a good commentary in the edition of the *Principia* which is known as the Jesuits' edition. They had been previously discussed by Maupertuis, as we shall see in another Chapter. Notes by Plana on some of the Propositions will be found in the *Memorie della Reale Accademia...di Torino*, second series, Vol. XI., 1851.

17. We pass now to the investigations made by Newton with respect to the Figure of the Earth; they are contained in Propositions XVIII., XIX., and XX. of the third Book of the *Principia:* these Propositions remain substantially the same in the second and third editions as in the first, but modifications occur arising from additional information as to the facts involved.

Before we consider these Propositions we ought to advert to Newton's remarkable conjecture which is contained in Proposition X. Newton here suggests that the mean density of the Earth may be five or six times that of water:...verisimile est quod copia materiæ totius in Terrâ quasi quintuplo vel sextuplo major sit quàm si tota ex aqua constaret. We may now consider it certain that the mean density is between five and six times that of water. Laplace draws attention to Newton's remarkable conjecture in the *Connaissance des Tems* for 1823, page 328.

It will be convenient to give the enunciations of Newton's Propositions XVIII., XIX. and XX.

XVIII. Axes Planetarum quæ ad eosdem axes normaliter ducuntur minores esse.

XIX. Invenire proportionem axis Planetæ ad diametros eidem perpendiculares.

XX. Invenire et inter se comparare pondera corporum in Terræ hujus regionibus diversis.

18. Proposition XVIII. contains a general statement that the planets are not accurately spherical. In the first edition Cassini and Flamsteed are quoted as authorities for this statement with respect to Jupiter; in the second edition instead of these names we are referred to astronomers in general.

19. Proposition XIX. undertakes to determine the ratio of the axes of a planet. This important process consists of various steps. In the first edition Newton begins by saying briefly he finds from calculation that the centrifugal force at the equator is to the force of attraction there as 1 to $290\frac{4}{5}$. In the second edition the details of the calculation are supplied, and the ratio obtained is that of 1 to 289: this ratio is that which is now usually given in our elementary books, and it will be convenient to adopt it as we proceed with an account of Newton's investigation.

Suppose two slender canals of homogeneous fluid, one along the polar radius of the earth, and the other along an equatorial radius. The resultant attraction on the equatorial canal must be greater than that on the polar canal in the ratio of 289 to 288 in order that there may be relative equilibrium. For in proceeding along any given radius inside the earth the attraction varies as the distance, and the centrifugal force varies as the distance; hence the ratio of the latter to the former is constant along the equatorial radius; so that the effect of the centrifugal force may be considered equivalent to removing $\frac{1}{289}$ of the force of attraction.

20. Newton's next step is to compare the attraction of an oblate ellipsoid of revolution on a particle at its pole with the attraction of the same body on a particle at its equator, the ellipticity being supposed very small. He states his results without giving his process at full. It will be remembered that he had found an expression for the attraction of an ellipsoid of revolution at any point of its axis: see Art. 13.

I. Suppose an oblate ellipsoid of revolution formed from an ellipse, such that the major semi-axis CA is to the minor semi-axis CQ as 101 is to 100. The reader can easily draw the diagram for himself. Newton says that the attraction at Q would be to the attraction of a sphere having C for centre and CQ for radius, as 126 is to 125. If ϵ denote the ellipticity we know from our modern works that this ratio is that of $1 + \frac{4\epsilon}{5}$ to 1; see *Statics*, Chapter XIII.; this agrees closely with Newton's numerical example.

II. Suppose a prolate ellipsoid of revolution formed from the same ellipse. Newton says that the attraction at A would be to the attraction of a sphere having C for centre and CA for radius, as 125 is to 126. If ϵ denote the ellipticity we know from our modern works that this ratio is that of $1 - \frac{4\epsilon}{5}$ to 1; see *Statics*, Chapter XIII.: this agrees with Newton's numerical example.

In the first edition Newton put the fraction $\frac{2}{15}$ after 126 and 125 in I. and II. The fraction was removed by Cotes: see the *Correspondence of Newton and Cotes*, page 69.

III. Now return to the oblate ellipsoid of revolution. Suppose a particle at A: Newton says that the attraction on it will be a mean proportional between the attractions of the sphere and of the