CHAPTER I.

CARDAN. KEPLER. GALILEO.

1. THE practice of games of chance must at all times have directed attention to some of the elementary considerations of the Theory of Probability. Libri finds in a commentary on the *Divina Commedia* of Dante the earliest indication of the different probability of the various throws which can be made with three dice. The passage from the commentary is quoted by Libri; it relates to the first line of the sixth canto of the *Purgatorio*. The commentary was published at Venice in 1477. See Libri, *Histoire des Sciences Mathématiques en Italie*, Vol. II. p. 188.

2. Some other intimations of traces of our subject in older writers are given by Gouraud in the following passage, unfortunately without any precise reference.

Les anciens paraissent avoir entièrement ignoré cette sorte de calcul. L'érudition moderne en a, il est vrai, trouvé quelques traces dans un pöeme en latin barbare intitulé : *De Vetula*, œuvre d'un moine du Bas-Empire, dans un commentaire de Dante de la fin du XV^e siècle, et dans les écrits de plusieurs mathématiciens italiens du moyen âge et de la renaissance, Pacioli, Tartaglia, Peverone;......Gouraud, *Histoire du Calcul des Probabilités*, page 3.

3. A treatise by Cardan entitled *De Ludo Aleæ* next claims our attention. This treatise was published in 1663, in the first volume of the edition of Cardan's collected works, long after Cardan's death, which took place in 1576.

Montmort says, "Jerôme Cardan a donné un Traité De Ludo Aleæ; mais on n'y trouve que de l'érudition et des réflexions morales." *Essai d'Analyse*... p. XL. Libri says, "Cardan a écrit un traité spécial de *Ludo Aleœ*, où se trouvent résolues plusieurs questions d'analyse combinatoire." *Histoire*, Vol. III. p. 176. The former notice ascribes too little and the latter too much to Cardan.

4. Cardan's treatise occupies fifteen folio pages, each containing two columns; it is so badly printed as to be scarcely intelligible. Cardan himself was an inveterate gambler; and his treatise may be best described as a gambler's manual. It contains much miscellaneous matter connected with gambling, such as descriptions of games and an account of the precautions necessary to be employed in order to guard against adversaries disposed to cheat: the discussions relating to chances form but a small portion of the treatise.

5. As a specimen of Cardan's treatise we will indicate the contents of his thirteenth Chapter. He shews the number of cases which are favourable for each throw that can be made with two dice. Thus two and twelve can each be thrown in only one way. Eleven can be thrown in two ways, namely, by six appearing on either of the two dice and five on the other. Ten can be thrown in three ways, namely, by five appearing on each of the dice, or by six appearing on either and four on the other. And so on.

Cardan proceeds, "Sed in Ludo fritilli undecim puncta adjicere decet, quia una Alea potest ostendi."...The meaning apparently is, that the person who throws the two dice is to be considered to have thrown a given number when one of the dice *alone* exhibits that number, as well as when the number is made up by the sum of the numbers on the two dice. Hence, for six or any smaller number eleven more favourable cases arise besides those already considered.

Cardan next exhibits correctly the number of cases which are favourable for each throw that can be made with three dice. Thus three and eighteen can each be thrown in only one way; four and seventeen can each be thrown in three ways; and so on. Cardan also gives the following list of the number of cases *in Fritillo:*

1	2	3	4	5	6	7	8	9	10	11	12
108	111	115	120	126	133	33	36	37	36	33	26

Here we have corrected two misprints by the aid of Cardan's verbal statements. It is not obvious what the table means. It might be supposed, in analogy with what has already been said, that if a person throws three dice he is to be considered to have thrown a given number when one of the dice *alone* exhibits that number, or when two dice together exhibit it as their sum, as well as when all the three dice exhibit it as their sum: and this would agree with Cardan's remark, that for numbers higher than twelve the favourable cases are the same as those already given by him for three dice. But this meaning does not agree with Cardan's table; for with this meaning we should proceed thus to find the cases favourable for an ace: there are 5^3 cases in which no ace appears, and there are 6^3 cases in all, hence there are 6^3-5^3 cases in which we have an ace or aces, that is 91 cases, and not 108 as Cardan gives.

The connexion between the numbers in the ordinary mode of using dice and the numbers which Cardan gives appears to be the following. Let n be the number of cases which are favourable to a given throw in the ordinary mode of using three dice, and N the number of cases favourable to the same throw in Cardan's mode; let m be the number of cases favourable to the given throw in the ordinary mode of using *two* dice. Then for any throw not less than thirteen, $N=n$; for any throw between seven and twelve, both inclusive, $N=3m+n$; for any throw not greater than six, $N=108+3m+n$. There is only one deviation from this law; Cardan gives 26 favourable cases for the throw twelve, and our proposed law would give $3+25$, that is 28.

We do not, however, see what simple mode of playing with three dice can be suggested which shall give favourable cases agreeing in number with those determined by the above law.

6. Some further account of Cardan's treatise will be found

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in the *Life of Cardan*, by Henry Morley, Vol. I. pages 92—95. Mr Morley seems to misunderstand the words of Cardan which he quotes on his page 92, in consequence of which he says that Cardan "lays it down coolly and philosophically, as one of his first axioms, that dice and cards ought to be played for money." In the passage quoted by Mr Morley, Cardan seems rather to admit the propriety of moderation in the stake, than to assert that there must be a stake; this moderation Cardan recommends elsewhere, as for example in his second Chapter. Cardan's treatise is briefly noticed in the article *Probability* of the *English Cyclopædia.*

7. Some remarks on the subject of chance were made by Kepler in his work *De Stella Nova in pede Serpentarii*, which was published in 1606. Kepler examines the different opinions on the cause of the appearance of a new star which shone with great splendour in 1604, and among these opinions the Epicurean notion that the star had been produced by the fortuitous concurrence of atoms. The whole passage is curious, but we need not reproduce it, for it is easily accessible in the reprint of Kepler's works now in the course of publication; see *Joannis Kepleri Astronomi Opera Omnia edidit Dr Ch. Frisch*, Vol. II. pp. 714—716. See also the *Life of Kepler* in the *Library of Useful Knowledge*, p. 13. The passage attracted the attention of Dugald Stewart; see his *Works edited by Hamilton*, Vol. I. p. 617.

A few words of Kepler may be quoted as evidence of the soundness of his opinions; he shows that even such events as throws of dice do not happen without a cause. He says,

> Quare hoc jactu Venus cecidit, illo canis? Nimirum lusor hac vice tessellam alio latere arripuit, aliter manu condidit, aliter intus agitavit, alio impetu animi manusve projecit, aliter interflavit aura, alio loco alvei impegit. Nihil hic est, quod sua causa sic caruerit, si quis ista subtilia posset consectari.

8. The next investigation which we have to notice is that by Galileo, entitled *Considerazione sopra il Giuco dei Dadi.* The date of this piece is unknown; Galileo died in 1642. It appears that a friend had consulted Galileo on the following difficulty: with three dice the number 9 and the number 10 can each be produced by six different combinations, and yet experience shows that the

number 10 is oftener thrown than the number 9. Galileo makes a careful and accurate analysis of all the cases which can occur, and he shows that out of 216 possible cases 27 are favourable to the appearance of the number 10, and 25 are favourable to the appearance of the number 9.

The piece will be found in Vol. XIV. pages 293—296, of *Le Opere .... di Galileo Galilei*, Firenze, 1855. From the *Bibliografia Galileiana* given in Vol. XV. of this edition of Galileo's works we learn that the piece first appeared in the edition of the works published at Florence in 1718: here it occurs in Vol. III. pages 119—121.

9. Libri in his *Histoire des Sciences Mathématiques en Italie*, Vol. IV. page 288, has the following remark relating to Galileo: ..."l'on voit, par ses lettres, qu'il s'était longtemps occupé d'une question délicate et non encore résolue, relative à la manière de compter les erreurs en raison géométrique ou en proportion arithmétique, question qui touche également au calcul des probabilités et à l'arithmétique politique." Libri refers to Vol. II. page 55, of the edition of Galileo's works published at Florence in 1718; there can, however, be no doubt, that he means Vol. III. The letters will be found in Vol. XIV. pages 231—284 of *Le Opere...di Galileo Galilei*, Firenze, 1855; they are entitled *Lettere intorno la stima di un cavallo.* We are informed that in those days the Florentine gentlemen, instead of wasting their time in attention to ladies, or in the stables, or in excessive gaming, were accustomed to improve themselves by learned conversation in cultivated society. In one of their meetings the following question was proposed; a horse is really worth a hundred crowns, one person estimated it at ten crowns and another at a thousand; which of the two made the more extravagant estimate? Among the persons who were consulted was Galileo; he pronounced the two estimates to be equally extravagant, because the ratio of a thousand to a hundred is the same as the ratio of a hundred to ten. On the other hand, a priest named Nozzolini, who was also consulted, pronounced the higher estimate to be more extravagant than the other, because the excess of a thousand above a hundred is greater than that of a hundred above ten. Various letters of

Galileo and Nozzolini are printed, and also a letter of Benedetto Castelli, who took the same side as Galileo; it appears that Galileo had the same notion as Nozzolini when the question was first proposed to him, but afterwards changed his mind. The matter is discussed by the disputants in a very lively manner, and some amusing illustrations are introduced. It does not appear, however, that the discussion is of any scientific interest or value, and the terms in which Libri refers to it attribute much more importance to Galileo's letters than they deserve. The Florentine gentlemen when they renounced the frivolities already mentioned might have investigated questions of greater moment than that which is here brought under our notice.

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CHAPTER II.

PASCAL AND FERMAT.

10. THE indications which we have given in the preceding Chapter of the subsequent Theory of Probability are extremely slight; and we find that writers on the subject have shewn a justifiable pride in connecting the true origin of their science with the great name of Pascal. Thus,

Elle doit la naissance à deux Géomètres français du dix-septième siècle, si fécond en grands hommes et en grandes découvertes, et peut-être de tous les siècles celui qui fait le plus d'honneur à l'esprit humain. Pascal et Fermat se proposèrent et résolurent quelques problèmes sur les probabilités...Laplace, *Théorie...des Prob.* 1st edition, page 3.

Un problême relatif aux jeux de hasard, proposé à un austère janséniste par un homme du monde a été l'origine du calcul des probabilités. Poisson, *Recherches sur la Prob.* page 1.

The problem which the Chevalier de Méré (a reputed gamester) proposed to the recluse of Port Royal (not yet withdrawn from the interests of science by the more distracting contemplation of the "greatness and the misery of man"), was the first of a long series of problems, destined to call into existence new methods in mathematical analysis, and to render valuable service in the practical concerns of life." Boole, *Laws of Thought,* page 243.

11. It appears then that the Chevalier de Méré proposed certain questions to Pascal; and Pascal corresponded with Fermat on the subject of these questions. Unfortunately only a portion of the correspondence is now accessible. Three letters

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of Pascal to Fermat on this subject, which were all written in 1654, were published in the *Varia Opera Mathematica D. Petri de Fermat*...Tolosæ, 1679, pages 179—188. These letters are reprinted in Pascal's works; in the edition of Paris, 1819, they occur in Vol. IV. pages 360—388. This volume of Pascal's works also contains some letters written by Fermat to Pascal, which are not given in Fermat's works; two of these relate to Probabilities, one of them is in reply to the second of Pascal's three letters, and the other apparently is in reply to a letter from Pascal which has not been preserved; see pages 385—388 of the volume.

We will quote from the edition of Pascal's works just named. Pascal's first letter indicates that some previous correspondence had occurred which we do not possess; the letter is dated July 29, 1654. He begins,

Monsieur, L'impatience me prend aussi-bien qú'à vous; et quoique je sois encore au lit, je ne puis m'empêcher de vous dire que je reçus hier au soir, de la part de M. de Carcavi, votre lettre sur les partis, que j'admire si fort, que je ne puis vous le dire. Je n'ai pas le loisir de m'étendre; mais en un mot vous avez trouvé les deux partis des dés et des parties dans la parfaite justesse: j'en suis tout satisfait; car je ne doute plus maintenant que je ne sois dans la vérité, après la rencontre admirable où je me trouve avec vous. J'admire bien davantage la méthode des parties que celle des dés; j'avois vu plusieurs personnes trouver celle des dés, comme M. le chevalier de Meré, qui est celui qui m'a proposé ces questions, et aussi M. de Roberval; mais M. de Meré n'avoit jamais pu trouver la juste valeur des parties, ni de biais pour y arriver: de sorte que je me trouvois seul qui eusse connu cette proportion.

Pascal's letter then proceeds to discuss the problem to which it appears from the above extract he attached the greatest importance. It is called in English the Problem of Points, and is thus enunciated: two players want each a given number of points in order to win; if they separate without playing out the game, how should the stakes be divided between them?

The question amounts to asking what is the probability which each player has, at any given stage of the game, of winning the game. In the discussion between Pascal and Fermat it is supposed that the players have equal chances of winning a single point.

12. We will now give an account of Pascal's investigations on the Problem of Points; in substance we translate his words.

The following is my method for determining the share of each player, when, for example, two players play a game of three points and each player has staked 32 pistoles.

Suppose that the first player has gained two points and the second player one point; they have now to play for a point on this condition, that if the first player gains he takes all the money which is at stake, namely 64 pistoles, and if the second player gains each player has two points, so that they are on terms of equality, and if they leave off playing each ought to take 32 pistoles. Thus, if the first player gains, 64 pistoles belong to him, and if he loses, 32 pistoles belong to him. If, then, the players do not wish to play this game, but to separate without playing it, the first player would say to the second "I am certain of 32 pistoles even if I lose this game, and as for the other 32 pistoles perhaps I shall have them and perhaps you will have them; the chances are equal. Let us then divide these 32 pistoles equally and give me also the 32 pistoles of which I am certain." Thus the first player will have 48 pistoles and the second 16 pistoles.

Next, suppose that the first player has gained two points and the second player none, and that they are about to play for a point; the condition then is that if the first player gains this point he secures the game and takes the 64 pistoles, and if the second player gains this point the players will then be in the situation already examined, in which the first player is entitled to 48 pistoles, and the second to 16 pistoles. Thus if they do not wish to play, the first player would say to the second "If I gain the point I gain 64 pistoles; if I lose it I am entitled to 48 pistoles. Give me then the 48 pistoles of which I am certain, and divide the other 16 equally, since our chances of gaining the point are equal." Thus the first player will have 56 pistoles and the second player 8 pistoles.

Finally, suppose that the first player has gained one point and the second player none. If they proceed to play for a point the condition is that if the first player gains it the players will be in the situation first examined, in which the first player is entitled to 56 pistoles; if the first player loses the point each player has then a point, and each is entitled to 32 pistoles. Thus if they do not wish to play, the first player would say to the second "Give me the 32 pistoles of which I am certain and divide the remainder of the 56 pistoles equally, that is, divide 24 pistoles equally." Thus the first player will have the sum of 32 and 12 pistoles, that is 44 pistoles, and consequently the second will have 20 pistoles.

13. Pascal then proceeds to enunciate two general results without demonstrations. We will give them in modern notation.

(1) Suppose each player to have staked a sum of money denoted by A; let the number of points in the game be $n+1$, and suppose the first player to have gained n points and the second player none. If the players agree to separate without playing any more the first player is entitled to $2A - \frac{A}{2^n}$.

(2) Suppose the stakes and the number of points in the game as before, and suppose that the first player has gained one point and the second player none. If the players agree to separate without playing any more, the first player is entitled to

$$A + A\,\frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,\quad 2n}\,.$$

Pascal intimates that the second theorem is difficult to prove. He says it depends on two propositions, the first of which is purely arithmetical and the second of which relates to chances. The first amounts in fact to the proposition in modern works on Algebra which gives the sum of the co-efficients of the terms in the Binomial Theorem. The second consists of a statement of the value of the first player's chance by means of combinations, from which by the aid of the arithmetical proposition the value above given is deduced. The demonstrations of these two results may be obtained from a general theorem which will be given later in the present Chapter; see Art. 23. Pascal adds a table which