CHAPTER XII.

THE OLDER GERMAN ELASTICIANS: F. NEUMANN, KIRCHHOFF AND CLEBSCH.

SECTION I.

Franz Neumann.

[1192.] WE have already had occasion to deal with three important memoirs of F. Neumann's, which fall into the period occupied by our first volume, and we have now to turn to a work of his which, if only published in 1885, still in substance mainly belongs to the years 1857–8. To Franz Neumann's teaching in Königsberg is due much of the impulse which mathematical physics received in the fifties in Germany; the most distinguished German physicists of the past forty years have been nearly all pupils of Neumann's, and this remark is specially true in the field of elasticity. Of those who attended his lectures on this subject and received probably from him their first stimulus to original investigations, we may name Kirchhoff, Strehlke, Clebsch, Borchardt, Carl Neumann and Voigt as among the more important[1]. Franz Neumann's lectures on elasticity were given in Königsberg at different times from 1857 to 1874, and in 1885 were published under the supervision of O. E. Meyer of Breslau with the title: *Vorlesungen über die Theorie der Elasticität der festen Körper*

[1] O. E. Meyer includes in the list Von der Mühll, Minnigerode, Zöppritz, Gehring, Saalschütz, Wangerin and Baumgarten: see preface to the *Vorlesungen*, S. viii.

und des Lichtäthers. The volume contains xiii + 374 pages, and is based on the notebooks of the brothers L. and O. E. Meyer for the years 1857–60, and those of Baumgarten and W. Voigt for the years 1869–74. According to the Editor the work contains all that was of importance in Neumann's lectures. The exact amount of originality in the several investigations I shall endeavour to point out in the course of my analysis, and I content myself here with the following remarks from the preface:

Zu den Gebieten, mit welchen Professor Neumann sich in jüngeren und späteren Jahren mit besonderer Vorliebe beschäftigt hat, gehört auch die Theorie der Elasticität; es konnte daher nicht fehlen, dass seine Vorlesungen über diesen Gegenstand häufig eigene Arbeiten betrafen. Seinem ausgesprochenen Wunsche, dass alle in verschiedenen Semestern vorgetragenen eigenen Untersuchungen in dieses Werk aufgenommen werden sollten, bin ich gern soweit nachgekommen, als es mir zu erreichen möglich war (S. v–vi).

The work is divided into twenty-one sections of which we note the important points in the following articles.

[1193.] In Section 1, *Einleitung* (S. 1–7), we have first some remarks on the origin of the theory of elasticity. Neumann attributes it not so much to a development from the isolated problems of Bernoulli and Euler as to the impulse given by Fresnel's new theory of light. He says:

Die exacte Beurtheilung seiner Beobachtungen führte Fresnel zu Thatsachen, welche im geraden Widerspruch standen zu den anerkannten Principien der Wellenbewegung in elastischen Medien. In der Schallwelle ist die Bewegung der Theilchen parallel dem Strahl, die Welle eine longitudinale; Fresnel fand, dass in der Lichtwelle jene Bewegung senkrecht gegen den Strahl gerichtet, die Welle also eine transversale ist, und doch soll der Unterschied der Eigenschaften beider Medien, der Luft und des Lichtäthers, nur quantitativ, nicht qualitativ sein. Die Mechaniker jener Zeit läugneten die Möglichkeit einer solchen Bewegung, weil sie unverträglich sei mit den hydrodynamischen Grundgleichungen, welche auf elastische Flüssigkeiten, auf Luft angewandt nur longitudinale Wellen kennen lehren. Fresnel, sich vertheidigend, machte darauf aufmerksam, dass möglicherweise in diesen Gleichungen nicht alle Kräfte berücksichtigt sein möchten, welche in elastischen Medien zur Wirkung kommen können. Er fand in der That, dass in den hydrodynamischen Gleichungen nur solche inneren Kräfte enthalten sind, welche aus einer Verdünnung oder Verdichtung des Mediums entstehen und welche wiederum eine Aenderung der Dichtigkeit hervorbringen. Er stellte sich daher die Frage, ob es in einem elastischen Medium keine anderen Kräfte gebe, ob in einem solchen System, wie es die Theilchen eines elastischen Körpers bilden, nicht auch Kräfte entstehen können aus einer Verschiebung der Theilchen, durch welche die Dichtigkeit nicht geändert wird. Wie jetzt die Sachen liegen, ist es leicht, den Standpunkt, auf den Fresnel sich stellte, klar zu machen (S. 1–2).

This account of the origin of the theory of elasticity, attributing it to the inability of the hydrodynamical equations to offer any explanation of the phenomena of light, has been accepted by several writers (see the review of our first volume in the *Bulletin des sciences mathématiques* T. 12, p. 38, 1888), but it must be distinctly borne in mind that the first propounder of the theory was Navier, an elastician of the old, or Bernoulli-Eulerian school, who both in theory and practice had frequently dealt with elastic stresses by the old methods, and whose memoir of 1827 was preceded not by optical investigations but by researches on the elasticity of rods and plates.

Neumann after briefly referring to the labours of Navier, Poisson and Cauchy concludes his first section by defining stress on their lines, i.e. by supposing inter-molecular force central and a function only of the central distance.

[1194.] The second section is entitled: *Allgemeine Lehrsätze über die Druckkräfte* (S. 8–25) and develops the usual stress equations without regard to any molecular hypothesis. The third section (S. 26–36) discusses Cauchy's and Lamé's ellipsoids of stress and the principal tractions without reference, however, to those writers: see our Arts. 610*, (iv), and 1059*. The fourth section entitled: *Das System der Dilatationen* (S. 37–51) deals with the geometry of small strains, and discusses the ellipsoids of strain and the principal stretches. The fifth section is entitled: *Beziehungen zwischen den Druckkräften und den Verrückungen* (S. 52–9). It deals only with uncrystalline and presumably homogeneous and isotropic bodies. Neumann remarks that experiment shows us that stress and strain vanish and arise coevally; hence he argues that one must be capable of being mathematically expressed as a function of the other. He then states that there can be no doubt that in uncrystalline bodies the axes of principal stretch and principal traction must coincide, and he continues:

> Aus unserer Annahme, dass die Dilatationen kleine Grössen seien, folgt, dass die Druckkräfte, welche wir als Functionen jener anzusehen haben, in der Gestalt einer Entwickelung nach Potenzen der Dilatationen dargestellt werden können. Da ferner nach unserer Annahme die Dilatationen so kleine Grössen sind, dass wir nur ihre erste Potenz zu berücksichtigen brauchen, so müssen die Hauptdruckkräfte lineare Functionen der Dilatationen sein; und zwar werden sie, da sie mit jenen zugleich verschwinden, ohne Hinzufügung eines constanten Gliedes ihnen einfach proportional zu setzen sein (S. 52–3).

Obviously here Neumann falls into the same *non-sequitur* as Cauchy in his memoir of 1827 (see our Art. 614*), as Maxwell in 1850 (see our Art. 1536*), or Lamé in 1852 (see our Art. 1051*). Neumann then obtains by transformation the ordinary stress-strain relations and the body-shift equations for an isotropic elastic solid. He employs Δ for our θ, $A-B$ for our 2μ, and B for our λ. Further he uses pressures not tractions throughout his work.

The Sections 2–5 of Neumann's work form an elementary theory of elasticity, at least so far as isotropic bodies are concerned. They do not possess any particular advantages in the present state of our science.

[1195.] The sixth section of the work (S. 60–6) is entitled *Navier's Differentialgleichungen.* It deduces the body-shift equations directly by Navier's method (see our Art. 266*); this method leads to uni-constant isotropy and avoids all introduction of the stresses. In starting with Navier's investigation Neumann adopts the historical plan. He points out the objections to Navier's process (S. 66: see our Arts. 531*–2*), and then turns to Poisson's and Cauchy's treatment of the problem in his seventh section entitled: *Poisson's Ableitung der allgemeinen Gleichungen* (S. 67–79). Neumann's investigation follows fairly closely Poisson's of 1828. He deduces the shift-equations for the cases of isotropy and of three rectangular axes of elastic symmetry. The latter system he speaks of as crystalline, although it is often produced by working in bodies without crystalline structure. He says:

Zu diesen Krystallen, deren Zahl sehr gross ist, gehören alle Formen des regulären, viergliedrigen zwei- und zweigliedrigen und sechsgliedrigen Systems mit Ausnahme gewisser, hemiëdrischer Formen, bei denen die parallelen Krystallflächen fehlen, z. B. beim regulären Tetraëder. Wir nennen diese Formen die geneigtflächigen Hemiëder. Ferner findet eine solche symmetrische Vertheilung nicht mehr statt bei allen Krystallen des zwei- und eingliedrigen und des ein- und eingliedrigen Systems (S. 75).

The resulting equations involving six independent constants agree with those which would be obtained by substituting the stress-strain relations of our Art. 117 (a) with the rari-constant conditions $d = d'$, $e = e'$, $f = f'$, in the usual body stress-equations.

The seven sections with which we have already dealt belong to the 1857–8 notebooks. Section 8 is taken from a notebook of 1859–60, and is entitled: *Entwickelung der Gleichungen aus dem Princip der virtuellen Geschwindigkeit* (S. 80–106). This is a reproduction of the method of Carl Neumann's memoir of 1860: see our Art. 667. F. Neumann, I think, supposes the first application of the principle of virtual moments to the theory of elasticity to have been made in the above memoir, but this is hardly correct: see our Arts. 268* and 759*. The method of the *Vorlesungen* is somewhat clearer and briefer than that of C. Neumann; it is also applied to bodies with three axes of elastic symmetry.

[1196.] Section 9 (S. 107–20), taken from a notebook of 1857–8, deals with the thermo-elastic equations in the method previously adopted by Duhamel and Neumann himself. We have seen that Neumann in 1841 (see our Art. 1196*) claimed priority in the deduction of these equations, and the Editor of the *Vorlesungen* (S. vi) apparently looks upon this section as an original part of the present work. The results do not seem to be more general than those of Duhamel (1838, see our Arts. 868* and 877*) and in all cases of doubt, priority of publication must be decisive.

Neumann like Duhamel limits his equations to the range in which extension is proportional to rise in temperature. His body-stress-equations involving thermal effect (2) and (3), S. 113, are equivalent to Equations (2) of our Art. 883*; his surface stress-equations (1) and (2), S. 114, to Equation (3) of the same article; his remarks on the relations between temperature and normal pressure, and between the thermo-elastic-constant, the stretch-modulus and the thermal stretch-coefficient are equivalent to those of Duhamel in our Arts. 875* and 888*.

[1197.] § 58 (S. 115–8) is entitled: *Krystallinische Körper.* In it Neumann questions whether the thermo-elastic constant is in crystalline bodies the same for all directions. He suggests equations of the form (see our Art. 883*):

$$\rho\left(\frac{d^2u}{dt^2}-X\right)=\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{zx}}{dz}-\beta_x\frac{dq}{dx},$$

$$\rho\left(\frac{d^2v}{dt^2}-Y\right)=\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}-\beta_y\frac{dq}{dy},$$

$$\rho\left(\frac{d^2w}{dt^2}-Z\right)=\frac{d\widehat{zx}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}-\beta_z\frac{dq}{dz},$$

in which he assumes, I suppose, the body to have three rectangular axes of elastic symmetry, coinciding with the thermal axes. The surface stress-equations will now be given by:

$$X'=(\widehat{xx}-\beta_x q)\cos l+\widehat{xy}\cos m+\widehat{zx}\cos n,$$

$$Y'=\widehat{xy}\cos l+(\widehat{yy}-\beta_y q)\cos m+\widehat{yz}\cos n,$$

$$Z'=\widehat{zx}\cos l+\widehat{yz}\cos m+(\widehat{zz}-\beta_z q)\cos n,$$

so that it is obvious that a rise of temperature is no longer equivalent to a uniform surface traction: see our Arts. 684–5.

Hierauf beruht die Entscheidung durch die Beobachtung. Man bestimmt durch directe Messung die Aenderung der Winkel, wenn der Druck auf die Oberfläche des Krystalls geändert wird, wenn man ihn z. B. aus dem Drucke einer Atmosphäre in den von 10 Atmosphären oder in den luftleeren Raum bringt. Auf dieselbe Weise misst man die Winkeländerung, welche durch eine Erhöhung der Temperatur, z. B. von 0° auf 100°, hervorgebracht wird. Erhält man beide Male ein entsprechendes System von Winkeländerungen, so sind alle drei Werthe von β unter sich gleich; befolgen die Aenderungen verschiedene Gesetze, so sind sie verschieden (S. 116–7).

Neumann then describes a method of making the needful measurements. He cites some experiments of Mitscherlich's (*Abhandlungen der Berliner Akademie*, 1825, S. 212) upon calcspar. This material expands in the direction of its axis owing to a rise of temperature and contracts perpendicular to the axis. The stretch for 100° C. increase of temperature was found to be ·00286 and the squeeze – ·00056. Thus the dilatation was ·00174. A similar result was exhibited by gypsum which in three different directions had different stretches or squeezes.

Neumann does not cite any experiments to determine how far the thermal results for these crystals are in accordance with those which would be produced by uniform surface tractions. He merely remarks that rods might be cut in certain directions from such crystals so that they would not change their length with change of temperature:

Hier löst also eine krystallinische Substanz ein Problem, dessen Lösung oft sehr gewünscht wird (S. 118).

The section concludes with a paragraph deducing the amplified form of Fourier's differential equation for the conduction of heat. This is in accord with Duhamel's results cited in our Art. 883*, Equation (i).

[1198.] The tenth section of the *Vorlesungen* is entitled *Kirchhoffs allgemeine Lehrsätze* (S. 121–32). Of this section § 60 reproduces Kirchhoff's proof of the uniqueness of the solution of the equations for the equilibrium of an elastic solid: see our Art. 1255: § 61 (S. 125–8) extends the proof of the uniqueness of the solution to the case of vibrations. This, I think, had not been done by either Kirchhoff or Clebsch and is original[1] Neumann, as in the previous paragraph, supposes isotropy. We will indicate his method of proof. If there be two solutions, then their difference, given say by the shifts U, V, W, must satisfy the body- and surface-equations with abstraction of body-force and surface-load.

Consider the quadruple integral

$$\iiiint dt\,dx\,dy\,dz\left\{\left(\rho\frac{d^2U}{dt^2}+\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{zx}}{dz}\right)\frac{dU}{dt}\right.$$
$$+\left(\rho\frac{d^2V}{dt^2}+\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}\right)\frac{dV}{dt}$$
$$\left.+\left(\rho\frac{d^2W}{dt^2}+\frac{d\widehat{zx}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}\right)\frac{dW}{dt}\right\},$$

which is zero owing to the body stress-equations. Integrate the stress terms by parts; the surface integrals then vanish owing to the surface stress-equations. Substitute for the stresses from the stress-strain relations, and the whole will be found a complete differential with regard to the time. Integrating out with regard to the time we find:

[1] The whole of this section is due to the lectures of 1859–60, and thus precedes Clebsch's *Treatise*. Kirchhoff's investigation was first given in the memoir of 1858: see our Art. 1255.

$$0 = \iiint dx\,dy\,dz \left\{ \left(\frac{dU}{dt}\right)^2 + \left(\frac{dV}{dt}\right)^2 + \left(\frac{dW}{dt}\right)^2 \right\}$$

$$+ \iiint dx\,dy\,dz \left[2\mu \left\{ \left(\frac{dU}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dW}{dz}\right)^2 \right\} + \lambda\theta^2 \right.$$

$$\left. + \mu \left\{ \left(\frac{dV}{dz} + \frac{dW}{dy}\right)^2 + \left(\frac{dW}{dx} + \frac{dU}{dy}\right)^2 + \left(\frac{dU}{dy} + \frac{dV}{dx}\right)^2 \right\} \right].$$

Hence it follows that all the squared terms must separately vanish at all points of the body. We see then that U, V, W are not functions of the time and that they can only express a translation and rotation of the body as a whole.

[1199.] § 62 of the *Vorlesungen* is entitled: *Verallgemeinerung des Beweises für Krystalle.* It is a not very satisfactory extension of the proof of the preceding section to bodies for which the stress-strain relations are of the form:

$$\widehat{xx} = as_x + fs_y + es_z, \qquad \widehat{yz} = d\sigma_{yz},$$
$$\widehat{yy} = fs_x + bs_y + ds_z, \qquad \widehat{zx} = e\sigma_{zx},$$
$$\widehat{zz} = es_x + ds_y + cs_z, \qquad \widehat{xy} = f\sigma_{xy};$$

i.e. to bodies for which *we can assume rari-constancy and which possess three rectangular axes of elastic symmetry.* Even if we suppose rari-constancy, such bodies are by no means the only existing type of crystal. Further Neumann's proof depends on the conditions that

$$a > e + f, \qquad b > f + d, \qquad c > d + e \quad \text{..................(i)}.$$

Neumann demonstrates this as follows. Crystals, he states, do not according to experiment differ widely from isotropic bodies, hence we must have:

$$3\lambda = a - \kappa_1 = b - \kappa_2 = c - \kappa_3,$$
$$\lambda = d - \varpi_1 = e - \varpi_2 = f - \varpi_3,$$

where κ_1, κ_2, κ_3, ϖ_1, ϖ_2, ϖ_3 are very small quantities as compared with λ. Hence it follows that the relations (i) above must be true. This supposes again the limit to be uni-constant isotropy. Now the objection to this sort of proof is that relations akin to (i) may hold, and certainly the uniqueness of the solution must hold, for wood and other materials, in which there is no approach to isotropy at all. Neumann's concluding words would seem to suggest that he considered the proposition proved for *all* bodies which occur in nature. In a footnote the remark is made that the laws of double refraction require that in the case of the ether we should have

$$a = 3(e + f - d), \quad b = 3(f + d - e), \quad c = 3(d + e - f) \quad \text{......(ii)},$$

and that since d, e, f differ only slightly, relations (i) must also be satisfied for the ether. That relations (ii) are *not* absolutely necessary on the elastic jelly theory of the ether has been indicated in our Art. 148. A more complete proof of the uniqueness of the solution of the equations of elasticity is given in Kirchhoff's *Vorlesungen*[1]: see our Arts. 1240, 1255 and 1278.

[1200.] § 63 (S. 129–32) belongs to the lectures of 1873–4. It is an investigation of the elastic energy of the stresses for an isotropic solid; it is so far more general than that to be found in the usual text-books, in that it regards possible changes of temperature due to the strain.

Let X, Y, Z be the body-forces at the point x, y, z of the solid, and X', Y', Z' the surface-load at the element dS of the surface. Then we can deduce from the thermo-elastic equations (see our Art. 1197) the following relation:

$$\frac{1}{2}\frac{d}{dt}\iiint\rho\left\{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2+\left(\frac{dw}{dt}\right)^2\right\}dx\,dy\,dz$$

$$=\frac{d}{dt}\iiint\rho(Xu+Yv+Zw)\,dx\,dy\,dz$$

$$+\iint\left(X'\frac{du}{dt}+Y'\frac{dv}{dt}+Z'\frac{dw}{dt}\right)dS$$

$$-\frac{1}{2}\frac{d}{dt}\iiint\{\lambda\theta^2+2\mu(s_x^2+s_y^2+s_z^2)+\mu(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2)\}\,dx\,dy\,dz$$

$$+\iiint\beta q\frac{d\theta}{dt}\,dx\,dy\,dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

where $$\frac{dq}{dt}=\frac{k}{c_v\rho}\nabla^2 q-\frac{\gamma-1}{\delta}\frac{d\theta}{dt}\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

(see our Art. 885*).

Now if X', Y', Z' are independent of t, i.e. *if the surface load be always the same*, we may integrate the whole of this with regard to t except the last term of the last line. This last can be integrated easily in two cases:

(i) Steady temperature, or q no function of t. We have:

$$\frac{1}{2}\iiint\rho\left\{\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2+\left(\frac{dw}{dt}\right)^2\right\}dx\,dy\,dz+\text{constant}$$

$$=\iiint\rho(Xu+Yv+Zw)\,dx\,dy\,dz+\iint(X'u+Y'v+Z'w)\,dS$$

$$-\tfrac{1}{2}\iiint\{\lambda\theta^2+2\mu(s_x^2+s_y^2+s_z^2)+\mu(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2)\}\,dx\,dy\,dz$$

$$+\iiint\beta q\theta\,dx\,dy\,dz\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

[1] The importance of this proposition lies in the result, that if any *particular* solution be found which satisfies all the conditions of an elastic problem, this solution is the only admissible one.

(ii) Suppose we neglect the first term on the right-hand-side of equation (iv), as for example in Newton's hypothesis as to the velocity of sound, then we have:

$$\frac{1}{2}\iiint \rho \left\{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right\} dx\,dy\,dz + \text{constant}$$
$$= \iiint \rho\,(Xu + Yv + Zw)\,dx\,dy\,dz + \iint (X'u + Y'v + Z'w)\,dS$$
$$- \tfrac{1}{2}\iiint \{\lambda\theta^2 + 2\mu\,(s_x{}^2 + s_y{}^2 + s_z{}^2) + \mu\,(\sigma_{yz}{}^2 + \sigma_{zx}{}^2 + \sigma_{xy}{}^2)\}\,dx\,dy\,dz$$
$$- \frac{1}{2}\iiint \frac{\delta}{\gamma - 1}\,\beta q^2\,dx\,dy\,dz \qquad \text{(vi).}$$

[1201.] The eleventh section (S. 133–163) is entitled: *Anwendungen auf unkrystallinische Körper*, and is occupied with the application of the equations of bi-constant isotropic elasticity to certain simple problems. The object of this section, we are told, is to clear up the doubtful points of those theories which starting from the molecular hypothesis reach uni-constant isotropy. Neumann here, however, does not seem to lay sufficient stress on the possibility of various distributions of elastic homogeneity in the rods, wires, hollow cylinders and spheres of which he treats. We may note one or two points.

(*a*) He refers (S. 136–8) to the experiments of Cagniard de la Tour, Regnault, Wertheim and himself on the magnitude of the stretch-squeeze ratio: see our Arts. 368*, 1321*, 1358* and 736. He himself had found that for iron-wire $\eta = 1/4$ nearly, but that it was nearer 1/3 for other substances, which he unfortunately does not specify.

(*b*) On S. 141–2 Neumann gives a theory of Wertheim's cylinder method of determining η: see our Art. 802. He remarks on the extreme importance of ascertaining the value of η for truly isotropic bodies, as the development of the molecular theory depends so entirely upon it. In investigating on S. 144–5 the stress in a hollow cylinder due to internal pressure, Neumann takes a stress-limit of strength and applies the theory of elasticity to rupture. Both steps seem to me unjustifiable: see our Arts. 5 (*a*) and (*c*), 169 (*c*) and 320–1.

(*c*) S. 146–153 deal with the oft-considered problem of the hollow spherical shell. Neumann discusses Oersted's theory of the piezometer, and shows how Colladon and Sturm were correct in supposing that a hollow spherical shell with equal internal and external pressures contracts as a solid sphere would do under the same external pressure: see our Arts. 686*–690*. He applies the theory to thermometer bulbs, and in particular shows how the reading of the thermometer is lower with the tube in a vertical than with the tube in a horizontal position owing to the internal pressure of the quicksilver on the bulb being greater in the former case. He shows by a numerical example that the difference of the reading in the two positions might amount to ·2° C.

Bei Thermometern, welche Cylinder statt Kugeln haben, ist dieser Fehler nicht so bedeutend, weil sie in der Regel eine stärkere Wand besitzen. Hierin liegt einer der Vorzüge der Cylinderthermometer (S. 151).

The consideration (S. 151–3) of the strength of an isotropic spherical shell and its comparison with the strength of a cylindrical one, is for reasons we have frequently referred to, very questionable when applied to glass vessels: see our Arts. 1358* and 119.

(*d*) § 74 (S. 153–5) deals with the problem of an isotropic solid elastic sphere surrounded by a shell of different isotropic elastic material, to the outer surface of which is applied a uniform pressure. Neumann finds that the solid core will contract more or less than it would do, if the external pressure were directly applied to it, according as $3\lambda + 2\mu$ for the core is greater or less than it is for the shell.

(*e*) §§ 75–76 (S. 155–61) are introduced by the Editor, and give methods of determining the elastic constants by torsion and *uniform flexure* (i.e. flexure by a couple). These are practically the methods adopted by Kirchhoff and Okatow to determine η: see our Arts. 1271–3. The final paragraphs of this section (S. 161–3) entitled: *Beobachtungen zur Bestimmung des Verhältnisses der beiden Elasticitätsconstanten* are also mainly due to the Editor and give a short résumé of the various experimental determinations of η due to Cornu, Mallock, Kirchhoff, Okatow, Schneebeli, Kohlrausch, Loomis, Baumeister, Röntgen, Amagat, W. Voigt, Littmann, and Everett. Accounts of the researches of these writers will be found under their names in our index, and the results of later researches under the title *stretch-squeeze ratio*. We can only remark here, that several of them still leave open to question the true isotropy of the materials experimented on, and they cannot thus be said to have finally settled the elastic constant controversy: see our Arts. 925*, 932*, 192, 800, and 1271.

[1202.] The twelfth section is entitled: *Elasticität krystallinischer Stoffe* and occupies S. 164–202. This section is taken from lecture notes of the years 1873–4. Neumann here rejects the rari-constant equations for crystals with three axes of elastic symmetry such as he had previously adopted in his work, and on S. 165 expresses the stresses in terms of the strains by linear relations involving 36 constants. Thus he writes:

Diese 36 Elasticitätsconstanten lassen sich im Allgemeinen nicht auf eine geringere Anzahl zurückführen. Jedoch verringert sich in den allermeisten Fällen ihre Zahl sehr erheblich, wenn der Krystall in Bezug auf eine oder mehrere Ebenen symmetrisch gebildet ist. Nur in den seltener vorkommen-