CHAPTER X.

SAINT-VENANT, 1850—1886.

SECTION I. *Torsion.*

[1.] WE commence our second volume with some account of the later work of the great French elastician whom we are justified in placing beside Poisson and Cauchy. From the last memoir referred to in our first volume till June 13, 1853 we have nothing to report. A slight note, however, entitled: *Divers résultats relatifs à la torsion,* which was read to the *Société philomathique* (*Bulletin,* February 26, 1853, or *L'Institut,* no. 1002, March 16, 1853), sufficiently indicates that our author had been diligently at work during these years on his new theory of torsion. On the 13th of June, 1853, his epoch-making memoir was read to the Academy (*Comptes rendus,* T. XXXVI. p. 1028). The memoir was inserted in T. XIV. of the *Mémoires des Savants étrangers,* 1855, pp. 233—560, under the title:

Mémoire sur la Torsion des Prismes, avec des considérations sur leur flexion, ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément.

We have referred to it in our first volume as the memoir on *Torsion,* and shall continue to do so.

The memoir was referred by the Academy to a committee consisting of Cauchy, Poncelet, Piobert and Lamé. Their report drawn up by Lamé (*Comptes rendus,* T. XXXVII., December 26, 1853, pp. 984—8) speaks very highly of the memoir. We cite the concluding words:

Le travail dont nous venons de rendre compte, mérite des éloges à plus d'un titre : par les nombres et les résultats nouveaux qu'il offre aux arts industriels, il constate, une fois de plus, l'importance de la théorie de l'équilibre d'élasticité ; par l'emploi de la méthode mixte, il indique comment les ingénieurs, qui veulent s'appuyer sur cette théorie, peuvent utiliser tous les procédés actuellement connus de l'analyse mathématique ; par ses tables, ses épures, et ses modèles en relief[1], il donne la marche qu'il faut nécessairement suivre, dans ce genre de recherches, pour arriver à des résultats immédiatement applicables à la pratique ; enfin, par la variété de ses points de vue, il offre un nouvel exemple de ce que peut faire la science du géomètre, unie à celle de l'ingénieur. (p. 988.)

The report gives a succinct account of the memoir. A second account by Saint-Venant himself will be found in : *Notice sur les travaux et titres scientifiques de M. de Saint-Venant*, Paris, 1858, pp. 19—31, and 71—80. This work together with one of the same title published in 1864, when Saint-Venant was again a candidate for the *Institut*, gives an excellent *résumé* of our author's researches previous to 1864. We shall refer to them briefly as *Notice* I. and *Notice* II.

[2.] The memoir itself is principally occupied with the torsion of *prisms*, a great variety of cross-sections being dealt with. This particular problem in torsion has been termed by Clebsch : *Das de Saint-Venantsche Problem* (*Theorie der Elasticität*, S. 74), and following him we shall term it *Saint-Venant's Problem*. The memoir consists of thirteen chapters.

3. The first chapter occupies pp. 233—236 ; and gives an introductory sketch of the contents of the memoir. If the values of the shifts of the several points of an elastic body are given the stresses can be easily found by simple differentiation. But the inverse problem—to find the shifts when the stresses are given—has not been generally solved, because we do not yet know how to integrate the differential equations which present themselves. Saint-Venant accordingly proposes the adoption of a *mixed method* (*méthode mixte ou semi-inverse*), which consists in assuming a part of the shifts and a part of the stresses, and then determining by an exact analysis what the remaining shifts and the remaining

[1] Copies of these numerous models are at present deposited in the mathematical model cases at University College. They represent much better than the poor woodcuts of the original memoir the distortion of the various cross-sections.

stresses must be. Before proceeding to the torsion of prisms Saint-Venant illustrates this *mixed method* in the third and fourth chapters of his memoir by applying it to simple problems.

[4.] The second chapter occupies pp. 236—288; it analyses strain and stress and investigates the general formulae for the equilibrium of elastic bodies. In 1868 Saint-Venant contributed to Moigno's *Statique* another elementary discussion of the fundamental formulae of elasticity; the later work is somewhat fuller and contains the more matured views of the author; the earlier is, however, very good. I will note the leading features of the treatment adopted:

(α) On p. 236 Saint-Venant defines the shifts as the *déplacements moyens* or as the *déplacements des centres de gravité de groupes d'un certain nombre de molécules.* He thus starts from the molecular standpoint, but this definition does not appear to be absolutely necessary to the course of his reasoning.

(β) On pp. 237—248 we have the analysis of strain. Here the slides first defined by Navier and Vicat (see our Vol. I. p. 877), and then theoretically considered by Saint-Venant in the *Cours lithographié* (see our Art. 1564*), are for the first time introduced by name and directly from their physical meaning into a general theory of elasticity. The slide of two lines primitively rectangular is defined as the *cosine of the angle between them* after strain (p. 238).

(γ) On p. 239 Saint-Venant carefully limits his researches to very small strains within the elastic limit, so that what he says later (pp. 281—288) on the conditions of *rupture*, must when applied to his torsion problems be interpreted only of the elastic limit. Indeed, as for certain materials, set is produced by any initial loading below the yield-point and is not practically dangerous (i.e. the material is not 'enervated,' to use Saint-Venant's language), we can only look upon the conditions of torsional rupture given in the memoir as of value when either (1) the material is elastic and *follows Hooke's Law* nearly up to rupture (cf. the steel bar H of the plate p. 893 of our Vol. I.), or, (2) the material has a state of ease extending almost up to the yield-point.

(δ) On pp. 242—5 we have the general expressions for s_r and $\sigma_{rr'}$. The first is due to Navier in his memoir of 1821, the second is attributed by Saint-Venant to Lamé (*Leçons...l'élasticité*, 1852, p. 46) but as we have seen it had been previously given by Hopkins in 1847 (see our Art. 1368*). From the second flows naturally a discussion of principal and maximum slide, together with a proof of Saint-Venant's theorem that a slide is equal to a stretch and a squeeze of half the magnitude of the slide in the bisectors of the slide angles (see our Art. 1570*).

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Finally the strain is expressed for *small* shifts in terms of the shift-fluxions (pp. 246—8). There is reference in a footnote to the strain-values for *large* shifts (see our Art. 1618*).

(ϵ) We next pass to an analysis of stress on pp. 248—254. Stress is defined from the molecular standpoint as follows:

> Nous appellerons donc en général *Pression, sur un des deux côtes d'une petite face plane imaginée à l'intérieur d'un corps ou à la limite de séparation de deux corps, la résultante de toutes les actions des molécules situées de ce côté sur les molécules du côté opposé, et dont les directions traversent cette face;* toutes ces forces étant supposées transportées parallèlement à elles-mêmes sur un même point pour les composer ensemble. (p. 248.)

The reader will find it interesting to follow the evolution of the stress-definition by comparing this with Arts. 426*, 440*, 546*, 616*, 678—9* and 1563*.

From this definition Saint-Venant deduces Cauchy's theorems (see our Arts. 606* and 610*) and an expression for $\widehat{rr'}$. On p. 253 p_{rr} is erroneously printed for $p_{rr'}$.

In a footnote to p. 254 a generalisation of the expression for $\widehat{rr'}$ is obtained. Suppose x, y, z to be any three concurrent but non-rectangular lines, and let x', y', z' be lines normal respectively to the planes yz, zx, xy. Then in our notation:

$$\begin{aligned}\widehat{rr'} = {} & \frac{\cos rx'}{\cos xx'}\left(\widehat{xx}\,\frac{\cos r'x'}{\cos xx'} + \widehat{xy}\,\frac{\cos r'y'}{\cos yy'} + \widehat{xz}\,\frac{\cos r'z'}{\cos zz'}\right)\\ & + \frac{\cos ry'}{\cos yy'}\left(\widehat{yx}\,\frac{\cos r'x'}{\cos xx'} + \widehat{yy}\,\frac{\cos r'y'}{\cos yy'} + \widehat{yz}\,\frac{\cos r'z'}{\cos zz'}\right)\\ & + \frac{\cos rz'}{\cos zz'}\left(\widehat{zx}\,\frac{\cos r'x'}{\cos xx'} + \widehat{zy}\,\frac{\cos r'y'}{\cos yy'} + \widehat{zz}\,\frac{\cos r'z'}{\cos zz'}\right).\end{aligned}$$

The proof is easily obtained by the orthogonal projection of areas.

(ζ) Saint-Venant next proceeds to express the relations between stress and strain (pp. 255—262). It cannot be said that this portion of his work is so satisfactory as the later treatment in Moigno's *Statique* (see p. 268 *et seq.*) or the full discussion of the generalised Hooke's Law in his edition of *Clebsch* (pp. 39—41). In fact the linearity of the stress-strain relations is obtained in the text by assumption: *Admettons donc avec tout le monde que les pressions sont fonctions linéaires des dilatations et des glissements tant qu'ils sont très-petits* (p. 257). A long footnote (pp. 257—261) treats the matter from the standpoint of central intermolecular action. Appeal is made to Cauchy (*Exercices de mathématiques* t. IV. p. 2: see our Art. 656*) for the reduction of the 36 coefficients to 15. Saint-Venant, however,—consistent rari-constant elastician as he has always been—retains the multi-constant formulae, remarking:

Mais des doutes ont été élevés sur le principe de cette réductibilité des 36 coefficients à 15 inégaux. Bien que ce doute ait pour motif principal une autre manière de l'établir, et qu'il ne paraisse atteindre, tout au plus, que les corps régulièrement cristallisés dont nous n'aurons pas à nous occuper dans la suite de ce mémoire, et, même, ceux seulement de ces corps où des groupes atomiques éprouveraient des rotations ou des déformations particulières lorsque l'on déforme l'ensemble, nous conserverons en général, à l'exemple de M. Lamé, l'indépendance des coefficients, ce qui, comme il l'a remarqué, ne rend pas plus compliquées les solutions analytiques des problèmes.

The reference to atomic rotations was suggested by Cauchy's paper of 1851: see our Art. 681*.

(η) We have next to deal with the reduction in the number of coefficients which arises in certain symmetrical distributions of homogeneity or in cases of isotropy. Saint-Venant adopts Cauchy's definitions of homogeneity and isotropy, which should have found a place in our first volume under Art. 606* (see the *Exercices* t. IV. p. 2):

On dit alors que le corps est *homogène*, ou que *l'élasticité y est la même dans les mêmes directions en tous ses points* (p. 263).

On the other hand a body is *isotrope* when it has *une élasticité constante ou égale en tous sens autour du point* (p. 272).

Saint-Venant refers to a *semi-polaire* distribution of elastic homogeneity as an example of elastic distribution. He has, as we shall see later, thoroughly treated the entire subject in a memoir of May 21, 1860.

The various cases in which one or more planes of symmetry exist are worked out, but I think brevity as well as uniformity of method are gained by adopting Green's expression for the internal work due to the strains.

(θ) As an example of Saint-Venant's method in this section we may take the following problem. He has shewn that in the case of one plane of symmetry, that of yz, the shears perpendicular to this plane reduce to:

$$\widehat{xy} = f\sigma_{xy} + h\sigma_{zx}, \quad \widehat{zx} = e\sigma_{zx} + h\sigma_{xy} \quad \text{.................. (i)},$$

where

$$f = |xyxy| \quad h = |xyzx| = |zxxy|$$
$$e = |zxzx|,$$

in the umbral coefficient notation: see Vol. I. p. 885.

Now by a suitable change of axes these shears can be expressed each in terms of a single slide. This problem is not reproduced in Moigno's *Statique*.

Turn the axes of yz round x through an angle β, then we easily find:

$$\left.\begin{aligned} \widehat{z'x} &= -\widehat{xy}\sin\beta + \widehat{zx}\cos\beta \\ \widehat{xy'} &= \widehat{xy}\cos\beta + \widehat{xz}\sin\beta \end{aligned}\right\} \text{.................... (ii)}.$$

$$\left.\begin{aligned} \sigma_{xy} &= \sigma_{xy'}\cos\beta - \sigma_{xz'}\sin\beta \\ \sigma_{xz} &= \sigma_{xy'}\sin\beta + \sigma_{xz'}\cos\beta \end{aligned}\right\} \text{.................... (iii)}.$$

Substitute from (iii) in (i) and then the values so deduced in (ii). We obtain

$$\left.\begin{aligned}\widehat{xy'} &= \left(\frac{f+e}{2} + \frac{f-e}{2}\cos 2\beta + h\sin 2\beta\right)\sigma_{xy'}\\ &\quad + \left(-\frac{f-e}{2}\sin 2\beta + h\cos 2\beta\right)\sigma_{xz'}\\ \widehat{xz'} &= \left(\frac{f+e}{2} - \frac{f-e}{2}\cos 2\beta - h\sin 2\beta\right)\sigma_{xz'}\\ &\quad + \left(-\frac{f-e}{2}\sin 2\beta + h\cos 2\beta\right)\sigma_{xy'}\end{aligned}\right\}\text{.........(iv).}$$

Obviously, if we take $\tan 2\beta = \dfrac{2h}{f-e}$ we reduce this last pair of equations to

$$\left.\begin{aligned}\widehat{xy'} &= f_1\,\sigma_{xy'}\\ \widehat{xz'} &= e_1\,\sigma_{xz'}\end{aligned}\right\}\text{.............................(v),}$$

where f_1 and e_1 are roots of the quadratic $\mu^2 - (f+e)\,\mu + fe - h^2 = 0$.

Such is substantially Saint-Venant's reduction. It is obvious, however, that this result follows at once when a known problem as to the invariants of a conic is applied to the work-function.

(ι) A remark as to isotropy on p. 272 may be reproduced as bearing on the uni-constant controversy:

> Mais l'isotropie paraît rare. Non-seulement les corps fibreux, tels que bois, les fers étirés ou forgés, mais même les corps grenus ou vitreux, refroidis de la surface au centre après leur fusion, peuvent présenter des élasticités différentes en divers sens.

Saint-Venant refers to the experiments and remarks of Regnault, Savart and Poncelet already noted in our first volume: see Arts. 332*, 978* and 1227*

(κ) On pp. 272—8 we have deductions of the body-stress equations, the body-shift equations and the surface-stress equations.

On p. 276 Saint-Venant deduces the body-shift equation for a planar distribution of elasticity such as he requires for his torsion problem.

He takes for the shears the expressions found in Equation (v) above, and for the traction $\widehat{xx}$ perpendicular to the planar system the expression

$$\widehat{xx} = as_x + bs_y + cs_z + d\sigma_{yz} + e\sigma_{zx} + f\sigma_{xy},$$

with *six independent* constants. Substituting in the body-stress equation $\dfrac{d\widehat{xx}}{dx} + \dfrac{d\widehat{xy}}{dy} + \dfrac{d\widehat{xz}}{dz} = X$, and expressing the strain in terms of the shift-fluxions, he finds:

$$a\frac{d^2u}{dx^2}+f_1\frac{d^2u}{dy^2}+e_1\frac{d^2u}{dz^2}+f\frac{d^2u}{dxdy}+e\frac{d^2u}{dxdz}$$
$$+(f_1+b)\frac{d^2v}{dxdy}+(e_1+c)\frac{d^2w}{dxdz}+d\left(\frac{d^2v}{dxdz}+\frac{d^2w}{dxdy}\right)+f\frac{d^2v}{dx^2}+e\frac{d^2w}{dx^2}=X.$$

C'est la seule équation dont nous aurons besoin pour les problèmes sur la torsion, comme on verra.

It will be noted that it contains *eight* independent constants, and that X is a body-*force*, not a body-*acceleration*, and acts *towards* the origin. It is needless to say that Saint-Venant much reduces the number of his constants before he applies this equation to his problem. In Moigno's *Statique* (p. 637) he adopts in place of X the more usual notation of $-\rho X$ where ρ is the density.

[5.] The concluding pages of this chapter (pp. 278—288) contain matter which appears here for the first time, and which, as it is of considerable interest, deserves an article to itself. The section is entitled: *Conditions de résistance à la rupture éloignée ou à une altération progressive et dangereuse de la contexture des corps.*

(*a*) We have already noted the misleading character of this title: see Art. 4. (γ). In the first place initial loads frequently produce set which although neither progressive nor dangerous may alter the shape or elastic homogeneity of the body; and in the second place, if the body be in a state of ease, still in many cases the generalised Hooke's law will be far from holding even approximately up to the elastic limit. Saint-Venant recognises the first point by distinguishing between small sets, "qui ne font qu'*écrouir* le corps ou rendre plus stable l'arrangement de ses parties" (p. 278) and large sets, which he holds either augment progressively so that "la matière *s'énervera* bientôt" (p. 239), or else by change of form destroy the value of a structure. But he hardly seems to have taken note of the second point, for he does not hesitate on pp. 280 and 286 to use stretch- and slide- moduli which connote a proportionality of stress and strain. The same point recurs in almost each torsion problem, where a *condition de non-rupture ou de stabilité de la cohésion* is given (e.g. pp. 351, 396 etc.). It is essentially a limit to the proportionality of stress and strain which is in each case given, but this limit in many materials has no sensible existence or may in the case of a material which does not possess an extended state of ease be safely passed.

(*b*) One further remark before we proceed to Saint-Venant's process. He starts from the formula (p. 280)

$$s_r = s_x\cos^2\alpha + s_y\cos^2\beta + s_z\cos^2\gamma + \sigma_{yz}\cos\beta\cos\gamma + \sigma_{zx}\cos\gamma\cos\alpha$$
$$+\sigma_{xy}\cos\alpha\cos\beta \ldots\ldots\ldots\ldots\text{(i)},$$

but on p. 242 he has obtained this by supposing the stretches and slides to be so small that their squares may be neglected. It is conceivable that in some materials before rupture and, possibly, before a dangerous set is reached, this might not be allowable.

(*c*) Our author begins by noticing that the proper limit to be taken for the stability of a material is a *stretch* and not a *traction* limit. He attributes to Mariotte[1] the first recognition of this fact "que c'est le degré d'extension qui fait rompre les corps" and remarks that although it is legitimate, and occasionally convenient, to take a traction limit given by $T = E\bar{s}$ where $\bar{s}$ is the stretch-limit and E the stretch-modulus, T need not be the stress across any plane, whatever, at the point in question.

Et cette sorte de notation est sans inconvénient si l'on n'oublie pas que T *représente simplement le produit* $E\bar{s}$, ou la force capable de donner (aussi par unité superficielle) à ce même petit prisme supposé isolé, la dilatation limite $\bar{s}$ relative à sa situation dans le corps, mais qu'il ne représente que *quelquefois* et *non toujours* l'effort intérieur ou la pression supportée normalement par sa section transversale pendant qu'il fait partie du corps. (p. 280.)

This remark is all the more important as the distinction has been neglected by Lamé, Clebsch and more recent elasticians: see our Arts. 1013*, 1016* footnotes and 1567*.

(*d*) The stretch in any direction being given by the equation (i) above, we have next to ask what in an aeolotropic body is the distribution of limiting stretch? Saint-Venant having regard to equation (i) *assumes* it to be *ellipsoidal* in character; in other words he takes

$$\bar{s} = \bar{s}_x \cos^2\alpha + \bar{s}_y \cos^2\beta + \bar{s}_z \cos^2\gamma,$$

where $\bar{s}_x$, $\bar{s}_y$, $\bar{s}_z$ are three constants to be determined by experiment, and the axes of ellipsoidal distribution are chosen as those of coordinates. The condition of safety now reduces to the maximum value of $s/\bar{s}$ being = or < 1. By the ordinary max.-min. processes of the Differential Calculus we obtain for $s/\bar{s}$ the equation:

$$4\bar{s}_x\bar{s}_y\bar{s}_z\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \sigma^2_{yz}\bar{s}_x\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)$$
$$- \sigma^2_{zx}\bar{s}_y\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right) - \sigma^2_{xy}\bar{s}_z\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \sigma_{yz}\sigma_{zx}\sigma_{xy} = 0 \ldots\ldots\text{(ii)}.$$

The roots of this equation are known to be real and we must have the greatest of them = or < 1.

Suppose the material is subject only to a sliding strain, then $s_x = s_y = s_z = \sigma_{zx} = \sigma_{xy} = 0$. Hence it follows that

$$\frac{s}{\bar{s}} = \frac{\sigma_{yz}}{2\sqrt{\bar{s}_y\bar{s}_z}}.$$

In other words if $\bar{s}$ is the limit of s, then $2\sqrt{\bar{s}_y\bar{s}_z}$ is the limit of σ_{yz} or gives the slide-limit. Let us represent it by $\bar{\sigma}_{yz}$.

[1] *Traité du mouvement des eaux*, sixième et troisième alinéa du *second discours*.

Similarly we have $\bar{\sigma}_{zx} = 2\sqrt{\bar{s}_z\,\bar{s}_x}$ and $\bar{\sigma}_{xy} = 2\sqrt{\bar{s}_x\,\bar{s}_y}$.
Saint-Venant then rewrites his equation (ii) as:

$$\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \left(\frac{\sigma_{yz}}{\bar{\sigma}_{yz}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right) - \left(\frac{\sigma_{zx}}{\bar{\sigma}_{zx}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)$$
$$- \left(\frac{\sigma_{xy}}{\bar{\sigma}_{xy}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - 2\,\frac{\sigma_{yz}\,\sigma_{zx}\,\sigma_{xy}}{\bar{\sigma}_{yz}\,\bar{\sigma}_{zx}\,\bar{\sigma}_{xy}} = 0 \quad \text{.........(iii)}.$$

He remarks that this equation may be adopted as if the six limiting strains $\bar{s}_x$, $\bar{s}_y$, $\bar{s}_z$, $\bar{\sigma}_{yz}$, $\bar{\sigma}_{zx}$, $\bar{\sigma}_{xy}$, were all independent, and the values of the slide-limits $\bar{\sigma}$ had to be found by experiment. At any rate equations of the form $\bar{\sigma}_{yz} = 2\sqrt{\bar{s}_y\,\bar{s}_z}$ need only be used when there is an absence of experimental data. (p. 284.)

(*e*) In the following paragraph (25) Saint-Venant explains how we are to find $s/\bar{s}$ for every point in the body and then take its maximum value for all these points,

> l'on obtiendra, en l'égalant à l'unité, la condition nécessaire et justement suffisante de la résistance du corps à la rupture (p. 284).

We have noted that this language is hardly exact. The point where this maximum takes place is called after Poncelet *point dangereux*, a name which it is convenient to render by *fail-point*. This term will not necessarily connote rupture, but merely a point at which 'linear elasticity[1]' first *fails*. The consideration of this point leads Saint-Venant to a concise definition of the solid of equal resistance:

> Souvent il y a plusieurs *points dangereux*, ou plusieurs points pour lesquels la plus grande valeur de $s/\bar{s}$ est la même, d'après la manière dont les forces sont appliquées. Lorsque, dans un corps de forme allongée, il y a un pareil point à chacune de ses sections transversales, ce corps est dit d'*égale résistance*: tels sont les prismes lorsqu'ils sont simplement étendus ou tordus par des forces appliquées aux extrémités.

(*f*) We have next the application of (iii) to the case of torsion about x as axis. Here

$$s_x = s_y = s_z = \sigma_{yz} = 0,$$

whence it follows

$$s/\bar{s} = \sqrt{(\sigma_{xy}/\bar{\sigma}_{xy})^2 + (\sigma_{xz}/\bar{\sigma}_{xz})^2}.$$

We have thus the limiting condition

$$1 = \text{or} > \left(\frac{\sigma_{xy}}{\bar{\sigma}_{xy}}\right)^2 + \left(\frac{\sigma_{xz}}{\bar{\sigma}_{xz}}\right)^2.$$

It is obvious that the principal slide in any direction $\sqrt{\sigma_{xy}{}^2 + \sigma_{xz}{}^2}$ is given by the ray of an ellipse of which $\bar{\sigma}_{xy}$ and $\bar{\sigma}_{xz}$ are the

[1] I use the words 'linear elasticity' in the sense in which 'perfect elasticity' has been used by the writers of mathematical text-books, i.e. to connote the elasticity which obeys the generalised Hooke's Law or the linearity of the stress-strain relation.

semi-axes. Saint-Venant uses throughout his memoir a slightly different form. Let μ_1, μ_2 be slide-coefficients and S_1, S_2 the shears capable of producing the slides $\bar{\sigma}_{xy}$ and $\bar{\sigma}_{xz}$; then the condition of *non-rupture par glissement* (i.e. of no *failure* of linear elasticity) is expressed by

$$1 = \text{or} > \left(\frac{\mu_1 \sigma_{xy}}{S_1}\right)^2 + \left(\frac{\mu_2 \sigma_{xz}}{S_2}\right)^2$$

The chapter concludes with a few general remarks on the physical characteristics of rupture by torsion.

[6.] The third chapter occupies pp. 288—99: it relates to the simple case of a prism on any base, whose terminal faces and sides are subjected to any uniform tractive loads. Lamé and Clapeyron in their memoir of 1828 (see our Art. 1011*) had treated the simple case of isotropy. Saint-Venant as an example of the *mixed* or *semi-inverse* method gives the solution for the case when there are three planes of elastic symmetry, the intersection x of one pair being parallel to the axis of the prism. He assumes that the tractions are constant and the shears zero throughout. This satisfies the body stress-equations; the constant values of the tractions are in this case given by the surface stress-equations. The stress-strain relations then give in terms of the elastic constants and the loads the values of the shift-fluxions. We thus arrive at a system of simple linear partial differential equations, whose solution is extremely easy. The complete solution gives for each shift a part proportional to the corresponding coordinate and a general integral which is only the resolved part of the most general displacement of the prism treated as a rigid body. On p. 292 Saint-Venant determines the value of the stretch-modulus when the tractive load on the sides of the prism is zero, and on p. 293 he considers the simple cases of (1) the axis of the prism being an axis of elastic symmetry, and (2) the material being isotropic: see our Art. 1066* On p. 293 we have a remark that some writers have doubted the exactness of the above results, considering them only as plausible but not necessarily unique. Saint-Venant asserts that they are unique, which is undoubtedly true in this case, but I am not quite satisfied with the nature of his proof, for it would at first sight apply to any elastic body. It depends essentially on the following line of reasoning: Take any particular integrals of the equations of elasticity u_0, v_0, w_0, put the shifts equal