CHAPTER I.

THE SEVENTEENTH AND EIGHTEENTH CENTURIES.

GALILEI TO GIRARD. 1638—1798.

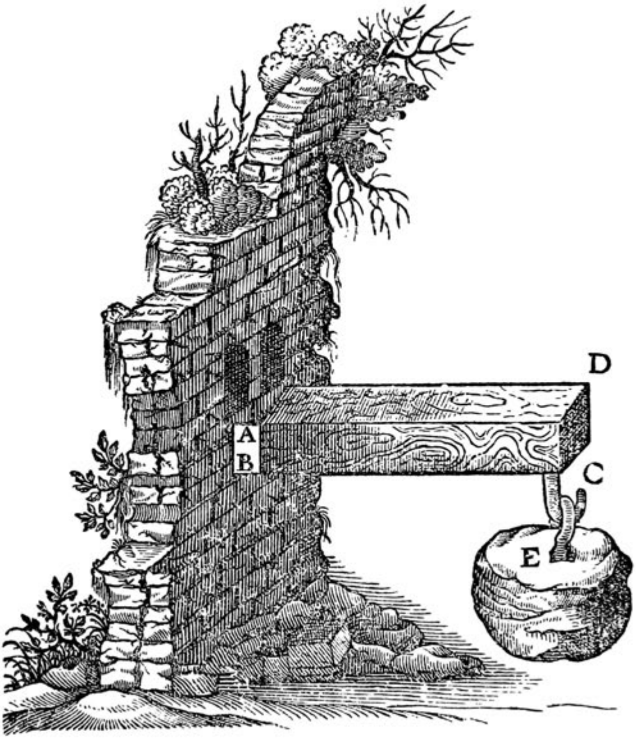
[1.] THE modern theory of elasticity may be considered to have its birth in 1821, when Navier first gave the equations for the equilibrium and motion of elastic solids, but some of the problems which belong to this theory had previously been solved or discussed on special principles, and to understand the growth of our modern conceptions it is needful to investigate the work of the seventeenth and eighteenth centuries.

[2.] The first memoir that requires notice is by Galileo Galilei and forms the second dialogue of the *Discorsi e Dimostrazioni matematiche*, Leiden 1638[1]. This dialogue both from its contents and form is of great historical interest. It not only gave the impulse but determined the direction of all the inquiries concerning the rupture and strength of beams, with which the physicists and mathematicians for the next century principally busied themselves. Galilei gives 17 propositions with regard to the fracture of rods, beams and hollow cylinders. The noteworthy feature about his method of discussion is that he supposed the fibres of a strained beam to be *inextensible*. There are two

[1] There is an English translation in *Thomas Salusbury's Mathematical Collections and Translations*, London, 1665. Tom. II. p. 89.

problems which Galilei discussed, and which form the starting points of many later memoirs. They are the following:

[3.] A beam (*ABCD*) being built horizontally into a wall (at *AB*) and strained by its own or an applied weight (*E*), to find the breaking force upon a section perpendicular to its axis. This problem is always associated by later writers with Galilei's name, and we shall call it in future *Galilei's Problem.*



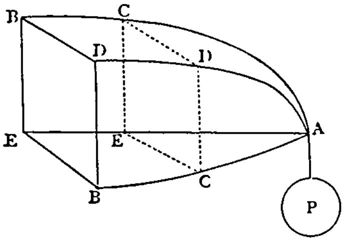
(From the *Discorsi*, Leiden 1638.)

The 'base of fracture' being defined as the section of the beam where it is built into the wall; we have the following results:—

(i) The resistances of the bases of fracture of similar prismatic beams are as the squares of their corresponding dimensions.

In this case the beams are supposed loaded at the free end till the base of fracture is ruptured; the weights of the beams are neglected.

(ii) Among an infinite number of homogeneous and similar beams there is only one, of which the weight is exactly in equilibrium with the resistance of the base of fracture. All others, if of a greater length will break,—if of a less length will have a superfluous resistance in their base of fracture.



[4.] The second problem with which Galilei particularly busied himself, was the discovery of 'solids of equal resistance.' The problem in its simplest form may be thus stated; ACB, $AC'B'$ are two curves in vertical and horizontal planes respectively, a solid is generated by treating ACB and $AC'B'$ as the bases of cylinders with generators perpendicular to the bases. This solid $BEB'DA$ is then treated as a beam built in at the base $BEB'D$ and from A a weight is suspended. The problem is to find the form of the generating curves so that the resistance of a section $CE'C'D$ may be exactly equal to the tendency to rupture at that place Obviously the problem may take a more complex form by supposing any system of forces to act upon the beam As we have stated it, it still remains indeterminate, for we must either be given one of the generating curves or else a relation between them. Galilei supposed the curve $AC'B'$ to be replaced by a line parallel to AE, so that all vertical sections of his beam parallel to $ACBE$ were curves equal to ACB. In this case he easily determined that the 'solid of equal resistance' must have a parabola for its generating curve.

[5.] This problem of solids of equal resistance led to a memorable controversy in the scientific world. It was discussed by P. Wurtz, François Blondel (*Galilaeus Promotus* 1649 (?); *Sur la résistance des solides, Mém. Acad. Paris, Tom.* I. 1692), Alex. Marchetti (*De resistentia solidorum*, Florence 1669), V. Viviani (*Opere Galilei*, Bologna 1655), Guido Grandi (*La controversia contro dal Sig. A. Marchetti*, and *Risposta apologetica...alle opposizioni dal Sig. A. Marchetti*; both Lucca 1712), and still more fully later, in memoirs to be referred to, of Varignon (1702) and Parent (1710). An interesting account of the controversy and also of writings on the same subject will be found in Girard's work (*cf infra* Art. 124).

Closely as the problem of solids of equal resistance is associated with the growth of the mathematical theory of elasticity, it is nevertheless the problem of the flexure of a horizontal beam which may be said to have produced the entire theory.

[6] While the continental scientists were thus busy with problems, which were treated without any conception of elasticity, and yet were to lead ultimately to the problem of the elastic curve, their English contemporaries seem to have been discussing hypotheses as to the nature of elastic bodies. One of the earliest memoirs in this direction which I have met with is due to Sir William Petty, and is entitled:

The Discourse made before the Royal Society concerning the use of Duplicate Proportion; together with a new Hypothesis of Springing or Elastique Motions, London 12*mo.* 1674.

Although absolutely without *scientific* value, this little work throws a flood of light on the state of scientific investigation at the time. On p. 114 we are treated to an 'instance' of duplicate proportion in the "*Compression of Yielding and Elastic Bodies as Wooll*, &c." There is an appendix (p. 121) on the new hypothesis as to elasticity The writer explains it by a complicated system of atoms to which he gives not only polar properties, but also *sexual* characteristics, remarking in justification that the statement of Genesis i. 27:—"male and female created he them"—must be taken to refer to the very ultimate parts of nature, or, to atoms as well as to mankind! (p. 131.)

Much more scientific value must be granted to the work of the next English writer.

[7] The discovery apparently of the modern conception of elasticity seems due to Robert Hooke, who in his work *De potentiâ restitutiva*, London 1678, states that 18 years before the date of that publication he had first found out the theory of springs, but had omitted to publish it because he was anxious to obtain a patent for a particular application of it. He continues:—

About three years since His Majesty was pleased to see the Experiment that made out this theory tried at *White Hall*, as also my Spring Watch.

About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, *viz. ceiiinosssttuu, id est, Ut Tensio sic vis;* That is, The Power of any spring is in the same proportion with the Tension thereof.

By 'spring' Hooke does not merely denote a spiral wire, or a bent rod of metal or wood, but any "springy body" whatever. Thus after describing his experiments he writes:

From all which it is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore it self to its natural position is always proportionate to the Distance or space it is removed therefrom, whether it be by rarefaction, or separation of its parts the one from the other, or by a Condensation, or crowding of those parts nearer together. Nor is it observable in these bodies only, but in all other springy bodies whatsoever, whether Metal, Wood, Stones, baked Earths, Hair, Horns, Silk, Bones, Sinews, Glass and the like. Respect being had to the particular figures of the bodies bended, and to the advantageous or disadvantageous ways of bending them.

[8.] The modern expression of the six components of stress as linear functions of the strain components may perhaps be *physically* regarded as a generalised form of Hooke's Law (See the remark made on this point by Saint-Venant in his *Mémoire sur la Torsion des Prismes*, pp. 256—7, and compare the same physicist's valuable note in his translation of Clebsch's *Theorie der Elasticität fester Korper*, pp. 39—40).

[9.] The principles of the *Congruity* and *Incongruity* of bodies and of the 'fluid subtil matter' or *menstruum* by which all bodies near the earth are incompassed—wherewith Hooke sought to theoretically ground his experimental law will no more satisfy the modern mathematician than the above-mentioned researches of Galilei. They are however very characteristic of the mathematical metaphysics of the period[1].

[10.] Mariotte seems to have been the earliest investigator who applied anything corresponding to the elasticity of Hooke to the fibres of the beam in Galilei's problem. In his *Traité du mouvement des eaux*, Paris 1686 *Partie V. Disc.* 2, pp. 370—400, he publishes the results of experiments made by him in 1680 and shows that Galilei's theory does not accord with experience. He remarks that some of the fibres of the beam extend before rupture, while others again are compressed. He assumes however without the least attempt at proof ("on peut concevoir") that half the fibres are compressed, half extended.

[11.] G. W. Leibniz: *Demonstrationes novae de Resistentiâ solidorum. Acta Eruditorum Lipsiae July* 1684. The stir created by Mariotte's experiments and his rejection of the views of the great Italian seem to have brought the German philosopher into the field. He treats the subject in a rather *ex cathedrâ* fashion, as if his opinion would finally settle the matter. He examines the hypotheses of Galilei and Mariotte, and finding that there is always flexure before rupture, he concludes that the fibres are really extensible. Their resistance is, he states, in proportion to their extension. In other words he applies "Hooke's Law" to the individual fibres. As to the application of his results to special problems, he will leave that to those who have leisure for such matters. The hypothesis of extensible fibres resisting as their extension is usually termed by the writers of this period the Mariotte-Leibniz theory.

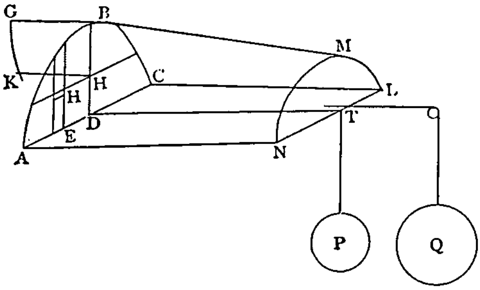
[1] A suggestion which occurs in the tract that one of his newly invented spring scales should be carried to the Pike of Teneriffe to test "whether bodies at a further distance from the centre of the earth do not lose somewhat of their powers or tendency towards it," is of much interest as occurring shortly before Newton's enunciation of the law of gravitation.

[12.] De la Hire: *Traité de Mécanique*, Paris 1695. Proposition CXXVI. of this work is entitled *De la résistance des solides.* The author is acquainted with Mariotte's theory and considers that it approaches the actual state of things closer than that of Galilei. At the same time notwithstanding certain concluding words of his preface, he does little but repeat Galilei's theorems regarding beams and the solid of equal resistance.

[13.] Varignon: *De la Résistance des Solides en général pour tout ce qu'on peut faire d'hypothèses touchant la force ou la ténacité des Fibres des Corps à rompre; Et en particulier pour les hypothèses de Galilée & de M. Mariotte. Mémoires de l'Académie*, Paris 1702.

This author considers that it is possible to state a general formula which will include the hypotheses of both Galilei and Mariotte, but to apply his formula it will in nearly all practical cases which may arise be necessary to assume some definite relation between the extension and resistance of the fibres. As Varignon's method of treating the problem is of some interest, being generally adopted by later writers (although in conjunction with either Galilei's or the Mariotte-Leibniz hypothesis), we shall briefly consider it here, without however retaining his notation.

[14.] Let $ABCNML$ be a beam built into a vertical wall at the section ABC, and supposed to consist of a number of parallel



fibres perpendicular to the wall (it is somewhat difficult to see how this is possible in the figure given, which is copied from Varignon) and equal to AN in length. Let H' be a point on the 'base of fracture,' and $H'E$ perpendicular to $AC=y$, $AE=x$. Then if a weight Q be attached by means of a pulley to the extremity of the beam, and be supposed to produce a uniform horizontal force over the whole section NML, $Q = r . \int y dx$ where r is the resistance of a fibre of unit sectional area and the integration is to extend over the whole base of fracture. Q is by later writers termed the *absolute resistance* and is given by the above formula. Now suppose the beam to be acted upon at its extremity by a vertical force P instead of the horizontal force Q. All the fibres in a horizontal line through H' will have equal resistance, this may be measured by a line HK drawn through H in any fixed direction where H is the point of intersection of the horizontal line through H' and the central vertical BD of the base. As H moves from B to D, K will trace out a curve GK which gives the resistance of the corresponding fibres. Take moments for the equilibrium of the beam about AC

$$P . l = \iint uy dx dy,$$

where l = length of the beam DT and $u = HK$.

This quantity $\iint uy dx dy$ was termed the *relative resistance* of the beam or the *resistance of the base of fracture.* The meaning of these terms is important for the understanding of these early memoirs. (Varignon speaks of *Résistance absolue* and *Résistance respective*, cf. § XIII.) So far there is little to complain of in Varignon's formulae except that it is necessary to know u before we can make use of it. He then proceeds to apply it to Galilei's and the Mariotte Leibniz hypotheses.

[15.] In Galilei's hypothesis of inextensible fibres u is supposed constant $= r$ and the resistance of the base of fracture becomes

$$= r \int y dx dy = \frac{r}{2} \int y^2 dx$$

On the supposition that the fibres are extensible we ought to consider their extension by finding what is now termed the *neutral line or surface.* Varignon however, and he is followed by later writers *assumes that the fibres in the base ACLN are not extended;* and that the extension of the fibre through H' varies as DH, in other words he makes the curve GK a straight line passing through D. Hence if r' be the resistance of the fibre at B, and $DB = a$, the resistance of the fibre at $H = r'y/a$ or the *resistance of the base of fracture* on this hypothesis becomes

$$\frac{r'}{3a}\int y^3 dx.$$

This resistance in the case of a rectangular beam of breadth b and height a becomes on the two hypotheses

$$\frac{ra^2b}{2} \text{ and } \frac{r'a^2b}{3} \text{ respectively.}$$

Hence in calculating the form of "solids of equal resistance" where the resistance of any section of the beam is taken proportional to the breaking moment at the section, it will be indifferent which hypothesis we make use of (Cf. § XXI. of the memoir.)

[16.] Varignon calculates the forms of various solids of resistance, but it is unnecessary to follow him, his results are practically vitiated when applying the true (Leibniz-Mariotte) theory by his assumption of the position of the neutral surface, but in this error he is followed by so great a mathematician as Euler himself. (See Art. 75.)

[17.] Before entering on the more important work of James Bernoulli we may refer to a memoir by A. Parent entitled *Des points de rupture des figures:...des figures retenués par un de leurs bouts et tirées par telles et tant de puissances qu'on voudra. Mémoires de l'Académie.* Paris 1710, Tom. I. p. 235. I mention this memoir as it practically concludes the theory of solids of equal resistance. The author refers to two of his own earlier memoirs (1704 and 1707) which I have not thought it needful to examine. The point of rupture is deduced from the solid of equal resistance in the following not altogether satisfactory fashion. Consider the case of a beam loaded in any fashion; then retaining the horizontal generating curve of the beam (supposed formed by two cylinders with generators respectively horizontal and vertical in the manner described in our Art. 4,) we may replace the beam by a solid of equal resistance of the same length by changing its vertical generating curve, or again we may invert the process, retaining the vertical and varying the horizontal generating curve. In either case the 'relative resistance' of these two solids at any point (i e. section) will be the same, and a point at which the difference between the relative resistances of the actual solid and of the hypothetical solids of equal resistance is a minimum will be a point of minimum resistance. A point at which this difference vanishes or is negative will be a point of rupture. Parent considers a variety of cases of solids of equal resistance and their points of rupture.

18. The first work of genuine mathematical value on our subject is due to James Bernoulli, who considered the form of a bent elastic lamina in a paper entitled *Curvatura Laminae Elasticae*, printed in the *Acta Eruditorum Lipsiae* for June 1694, p. 262, with *Annotationes et Additiones* thereto in the same *Acta*, Dec. 1695, p. 537. The method of this first examination of the elastic curve did not satisfy Bernoulli, and these memoirs were replaced by another entitled:

Véritable hypothèse de la résistance des Solides, avec la démonstration de la Courbure des Corps qui font ressort. This occupies Vol. II. pp. 976—989 of the collected works of the author, published at Geneva, 1744. The date of the memoir is 12th of March 1705; and as the memoirs which follow it in the collected works are entitled *Varia Posthuma*, we may take it to be the last which appeared during the life of this famous mathematician: he died on the 16th of August 1705.

19. The memoir begins by brief notices of what had been already done with respect to the problem by Galilei, Leibniz, and Mariotte; James Bernoulli claims for himself that he first introduced the consideration of the *compression* of parts of the body, whereas previous writers had paid attention to the *extension* alone[1]

20. Three Lemmas which present no difficulty are given and demonstrated:

[1] [As we have seen this remark does not apply to Mariotte.] Ed.