AN ESSAY

ON THE APPLICATION OF MATHEMATICAL ANALYSIS

TO THE THEORIES

OF ELECTRICITY AND MAGNETISM.*

* *Published at Nottingham, in* 1828.

PREFACE.

AFTER I had composed the following Essay, I naturally felt anxious to become acquainted with what had been effected by former writers on the same subject, and, had it been practicable, I should have been glad to have given, in this place, an historical sketch of its progress; my limited sources of information, however, will by no means permit me to do so; but probably I may here be allowed to make one or two observations on the few works which have fallen in my way, more particularly as an opportunity will thus offer itself, of noticing an excellent paper, presented to the Royal Society by one of the most illustrious members of that learned body, which appears to have attracted little attention, but which, on examination, will be found not unworthy the man who was able to lay the foundations of pneumatic chymistry, and to discover that water, far from being according to the opinions then received, an elementary substance, was a compound of two of the most important gases in nature.

It is almost needless to say the author just alluded to is the celebrated CAVENDISH, who, having confined himself to such simple methods, as may readily be understood by any one possessed of an elementary knowledge of geometry and fluxions, has rendered his paper accessible to a great number of readers; and although, from subsequent remarks, he appears dissatisfied with an hypothesis which enabled him to draw some important conclusions, it will readily be perceived, on an attentive perusal of his paper, that a trifling alteration will suffice to render the whole perfectly legitimate*.

* In order to make this quite clear, let us select one of CAVENDISH'S propositions, the twentieth for instance, and examine with some attention the method

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Little appears to have been effected in the mathematical theory of electricity, except immediate deductions from known formulæ, that first presented themselves in researches on the figure of the earth, of which the principal are,—the determination of the law of the electric density on the surfaces of conducting bodies differing little from a sphere, and on those of ellipsoids, from 1771, the date of CAVENDISH'S paper, until about 1812, when M. POISSON presented to the French Institute two memoirs of singular elegance, relative to the distribution of electricity on the surfaces of conducting spheres, previously electrified and put in presence of each other. It would be quite

there employed. The object of this proposition is to show, that when two similar conducting bodies communicate by means of a long slender canal, and are charged with electricity, the respective quantities of redundant fluid contained in them, will be proportional to the $n-1$ power of their corresponding diameters: supposing the electric repulsion to vary inversely as the n power of the distance. This is proved by considering the canal as cylindrical, and filled with incompressible fluid of uniform density: then the quantities of electricity in the interior of the two bodies are determined by a very simple geometrical construction, so that the total action exerted on the whole canal by one of them, shall exactly balance that arising from the other; and from some remarks in the 27th proposition, it appears the results thus obtained, agree very well with experiments in which real canals are employed, whether they are straight or crooked, provided, as has since been shown by COULOMB, n is equal to two. The author however confesses he is by no means able to demonstrate this, although, as we shall see immediately, it may very easily be deduced from the propositions contained in this paper.

For this purpose, let us conceive an incompressible fluid of uniform density, whose particles do not act on each other, but which are subject to the same actions from all the electricity in their vicinity, as real electric fluid of like density would be; then supposing an infinitely thin canal of this hypothetical fluid, whose perpendicular sections are all equal and similar, to pass from a point a on the surface of one of the bodies, through a portion of its mass, along the interior of the real canal, and through a part of the other body, so as to reach a point A on its surface, and then proceed from A to a in a right line, forming thus a closed circuit, it is evident from the principles of hydrostatics, and may be proved from our author's 23d proposition, that the whole of the hypothetical canal will be in equilibrium, and as every particle of the portion contained within the system is necessarily so, the rectilinear portion aA must therefore be in equilibrium. This simple consideration serves to complete CAVENDISH'S demonstration, whatever may be the form or thickness of the real canal, provided the quantity of electricity in it is very small compared with that contained in the bodies. An analogous application of it will render the demonstration of the 22d proposition complete, when the two coatings of the glass plate communicate with their respective conducting bodies, by fine metallic wires of any form.

impossible to give any idea of them here: to be duly appretiated they must be read. It will therefore only be remarked, that they are in fact founded upon the consideration of what have, in this Essay, been termed potential functions, and by means of an equation in variable differences, which may immediately be obtained from the one given in our tenth article, serving to express the relation between the two potential functions arising from any spherical surface, the author deduces the values of these functions belonging to each of the two spheres under consideration, and thence the general expression of the electric density on the surface of either, together with their actions on any exterior point.

I am not aware of any material accessions to the theory of electricity, strictly so called, except those before noticed; but since the electric and magnetic fluids are subject to one common law of action, and their theory, considered in a mathematical point of view, consists merely in developing the consequences which flow from this law, modified only by considerations arising from the peculiar constitution of natural bodies with respect to these two kinds of fluid, it is evident the mathematical theory of the latter, must be very intimately connected with that of the former; nevertheless, because it is here necessary to consider bodies as formed of an immense number of insulated particles, all acting upon each other mutually, it is easy to conceive that superior difficulties must, on this account, present themselves, and indeed, until within the last four or five years, no successful attempt to overcome them had been published. For this farther extension of the domain of analysis, we are again indebted to M. Poisson, who has already furnished us with three memoirs on magnetism: the first two contain the general equations on which the magnetic state of a body depends, whatever may be its form, together with their complete solution in case the body under consideration is a hollow spherical shell, of uniform thickness, acted upon by any exterior forces, and also when it is a solid ellipsoid subject to the influence of the earth's action. By supposing magnetic changes to require time, although an exceedingly short one, to complete them, it had been suggested that M. Arago's discovery relative to the magnetic effects developed in copper, wood, glass, *etc.*, by rotation, might be explained. On this hypothesis M. POISSON has founded his third memoir, and thence deduced formulæ applicable to magnetism in a state of motion. Whether the preceding hypothesis will serve to explain the singular phenomena observed by M. ARAGO or not, it would ill become me to decide; but it is probably quite adequate to account for those produced by the rapid rotation of iron bodies.

We have just taken a cursory view of what has hitherto been written, to the best of my knowledge, on subjects connected with the mathematical theory of electricity; and although many of the artifices employed in the works before mentioned are remarkable for their elegance, it is easy to see they are adapted only to particular objects, and that some general method, capable of being employed in every case, is still wanting. Indeed M. POISSON, in the commencement of his first memoir (*Mém. de l'Institut*, 1811), has incidentally given a method for determining the distribution of electricity on the surface of a spheroid of any form, which would naturally present itself to a person occupied in these researches, being in fact nothing more than the ordinary one noticed in our introductory observations, as requiring the resolution of the equation (a). Instead however of supposing, as we have done, that the point p must be upon the surface, in order that the equation may subsist, M. POISSON availing himself of a general fact, which was then supported by experiment only, has conceived the equation to hold good wherever this point may be situated, provided it is within the spheroid, but even with this extension the method is liable to the same objection as before.

Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispensing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely to that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it. The advantages LAPLACE had derived in the third book of the *Mécanique Céleste*, from the use of a partial differential equation of the second order, there given, were too marked to escape the notice of any one engaged with the present subject, and naturally served to suggest that this equation might be made subservient to the object I had in view. Recollecting, after some attempts to accomplish it, that previous researches on partial differential equations, had shown me the necessity of attending to what have, in this Essay, been denominated the singular values of functions, I found, by combining this consideration with the preceding, that the resulting method was capable of being applied with great advantage to the electrical theory, and was thus, in a short time, enabled to demonstrate the general formulæ contained in the preliminary part of the Essay. The remaining part ought to be regarded principally as furnishing particular examples of the use of these general formulæ; their number might with great ease have been increased, but those which are given, it is hoped, will suffice to point out to mathematicians, the mode of applying the preliminary results to any case they may wish to investigate. The hypotheses on which the received theory of magnetism is founded, are by no means so certain as the facts on which the electrical theory rests; it is however not the less necessary to have the means of submitting them to calculation, for the only way that appears open to us in the investigation of these subjects, which seem as it were desirous to conceal themselves from our view, is to form the most probable hypotheses we can, to deduce rigorously the consequences which flow from them, and to examine whether such consequences agree numerically with accurate experiments.

The applications of analysis to the physical Sciences, have the double advantage of manifesting the extraordinary powers of this wonderful instrument of thought, and at the same time of serving to increase them; numberless are the instances of the truth of this assertion. To select one we may remark, that M. FOURIER, by his investigations relative to heat, has not only discovered the general equations on which its motion depends, but has likewise been led to new analytical formulæ, by whose aid MM. CAUCHY and POISSON have been enabled to give the complete theory of the motion of the waves in an indefinitely extended fluid. The same formulæ have also put us in possession of the solutions of many other interesting problems, too numerous to be detailed here. It must certainly be regarded as a pleasing prospect to analysts, that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for farther applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it; and, amongst other things, probably the theory that supposes light to depend on the undulations of a luminiferous fluid, and to which the celebrated Dr T. YOUNG has given such plausibility, may furnish a useful subject of research, by affording new opportunities of applying the general theory of the motion of fluids. The number of these opportunities can scarcely be too great, as it must be evident to those who have examined the subject, that, although we have long been in possession of the general equations on which this kind of motion depends, we are not yet well acquainted with the various limitations it will be necessary to introduce, in order to adapt them to the different physical circumstances which may occur.

Should the present Essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labour he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded.

INTRODUCTORY OBSERVATIONS.

THE object of this Essay is to submit to Mathematical Analysis the phenomena of the equilibrium of the Electric and Magnetic Fluids, and to lay down some general principles equally applicable to perfect and imperfect conductors; but, before entering upon the calculus, it may not be amiss to give a general idea of the method that has enabled us to arrive at results, remarkable for their simplicity and generality, which it would be very difficult if not impossible to demonstrate in the ordinary way.

It is well known, that nearly all the attractive and repulsive forces existing in nature are such, that if we consider any material point p, the effect, in a given direction, of all the forces acting upon that point, arising from any system of bodies S under consideration, will be expressed by a partial differential of a certain function of the co-ordinates which serve to define the point's position in space. The consideration of this function is of great importance in many inquiries, and probably there are none in which its utility is more marked than in those about to engage our attention. In the sequel we shall often have occasion to speak of this function, and will therefore, for abridgement, call it the potential function arising from the system S. If p be a particle of positive electricity under the influence of forces arising from any electrified body, the function in question, as is well known, will be obtained by dividing the quantity of electricity in each element of the body, by its distance from the particle p, and taking the total sum of these quotients for the whole body, the quantities of electricity in those elements which are negatively electrified, being regarded as negative.

It is by considering the relations existing between the density of the electricity in any system, and the potential functions thence arising, that we have been enabled to submit many electrical phenomena to calculation, which had hitherto resisted the attempts of analysts; and the generality of the consideration here employed, ought necessarily, and does, in fact, introduce a great generality into the results obtained from it. There is one consideration peculiar to the analysis itself, the nature and utility of which will be best illustrated by the following sketch.

Suppose it were required to determine the law of the distribution of the electricity on a closed conducting surface A without thickness, when placed under the influence of any electrical forces whatever: these forces, for greater simplicity, being reduced to three, X, Y, and Z, in the direction of the rectangular co-ordinates, and tending to increase them. Then ρ representing the density of the electricity on an element $d\sigma$ of the surface, and r the distance between $d\sigma$ and p, any other point of the surface, the equation for determining ρ which would be employed in the ordinary method, when the problem is reduced to its simplest form, is known to be

$$\text{cons} = a = \int \frac{\rho d\sigma}{r} - \int (Xdx + Ydy + Zdz) \quad \ldots\ldots\ldots \ (a);$$

the first integral relative to $d\sigma$ extending over the whole surface A, and the second representing the function whose complete differential is $Xdx + Ydy + Zdz$, x, y and z being the co-ordinates of p.

This equation is supposed to subsist, whatever may be the position of p, provided it is situate upon A. But we have no general theory of equations of this description, and whenever we are enabled to resolve one of them, it is because some consideration peculiar to the problem renders, in that particular case, the solution comparatively simple, and must be looked upon as the effect of chance, rather than of any regular and scientific procedure.

We will now take a cursory view of the method it is proposed to substitute in the place of the one just mentioned.