XII

ARISTARCHUS OF SAMOS

HISTORIANS of mathematics have, as a rule, given too little attention to Aristarchus of Samos. The reason is no doubt that he was an astronomer, and therefore it might be supposed that his work would have no sufficient interest for the mathematician. The Greeks knew better; they called him Aristarchus 'the mathematician', to distinguish him from the host of other Aristarchuses; he is also included by Vitruvius among the few great men who possessed an equally profound knowledge of all branches of science, geometry, astronomy, music, &c.

> 'Men of this type are rare, men such as were, in times past, Aristarchus of Samos, Philolaus and Archytas of Tarentum, Apollonius of Perga, Eratosthenes of Cyrene, Archimedes and Scopinas of Syracuse, who left to posterity many mechanical and gnomonic appliances which they invented and explained on mathematical (lit. 'numerical') principles.'[1]

That Aristarchus was a very capable geometer is proved by his extant work *On the sizes and distances of the Sun and Moon* which will be noticed later in this chapter: in the mechanical line he is credited with the discovery of an improved sun-dial, the so-called σκάφη, which had, not a plane, but a concave hemispherical surface, with a pointer erected vertically in the middle throwing shadows and so enabling the direction and the height of the sun to be read off by means of lines marked on the surface of the hemisphere. He also wrote on vision, light and colours. His views on the latter subjects were no doubt largely influenced by his master, Strato of Lampsacus; thus Strato held that colours were emanations from bodies, material molecules, as it were, which imparted to the intervening air the same colour as that possessed by the body, while Aristarchus said that colours are 'shapes or forms

[1] Vitruvius, *De architectura*, i. 1. 16.

stamping the air with impressions like themselves, as it were', that 'colours in darkness have no colouring', and that 'light is the colour impinging on a substratum'.

Two facts enable us to fix Aristarchus's date approximately. In 281/280 B.C. he made an observation of the summer solstice; and a book of his, presently to be mentioned, was published before the date of Archimedes's *Psammites* or *Sand-reckoner*, a work written before 216 B.C. Aristarchus, therefore, probably lived *circa* 310–230 B.C., that is, he was older than Archimedes by about 25 years.

To Aristarchus belongs the high honour of having been the first to formulate the Copernican hypothesis, which was then abandoned again until it was revived by Copernicus himself. His claim to the title of 'the ancient Copernicus' is still, in my opinion, quite unshaken, notwithstanding the ingenious and elaborate arguments brought forward by Schiaparelli to prove that it was Heraclides of Pontus who first conceived the heliocentric idea. Heraclides is (along with one Ecphantus, a Pythagorean) credited with having been the first to hold that the earth revolves about its own axis every 24 hours, and he was the first to discover that Mercury and Venus revolve, like satellites, about the sun. But though this proves that Heraclides came near, if he did not actually reach, the hypothesis of Tycho Brahe, according to which the earth was in the centre and the rest of the system, the sun with the planets revolving round it, revolved round the earth, it does not suggest that he moved the earth away from the centre. The contrary is indeed stated by Aëtius, who says that 'Heraclides and Ecphantus make the earth move, *not in the sense of translation*, but by way of turning on an axle, like a wheel, from west to east, about its own centre'.[1] None of the champions of Heraclides have been able to meet this positive statement. But we have conclusive evidence in favour of the claim of Aristarchus; indeed, ancient testimony is unanimous on the point. Not only does Plutarch tell us that Cleanthes held that Aristarchus ought to be indicted for the impiety of 'putting the Hearth of the Universe in motion'[2]; we have the best possible testimony in the precise statement of a great

[1] Aët. iii. 13. 3, *Vors.* i^3, p. 341. 8.
[2] Plutarch, *De facie in orbe lunae*, c. 6, pp. 922 F–923 A.

contemporary, Archimedes. In the *Sand-reckoner* Archimedes has this passage.

'You [King Gelon] are aware that "universe" is the name given by most astronomers to the sphere the centre of which is the centre of the earth, while its radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account, as you have heard from astronomers. But Aristarchus brought out *a book consisting of certain hypotheses*, wherein it appears, as a consequence of the assumptions made, that the universe is many times greater than the "universe" just mentioned. His hypotheses are that *the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit*, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.'

(The last statement is a variation of a traditional phrase, for which there are many parallels (cf. Aristarchus's Hypothesis 2 'that the earth is in the relation of a point and centre to the sphere in which the moon moves'), and is a method of saying that the 'universe' is infinitely great in relation not merely to the size of the sun but even to the orbit of the earth in its revolution about it; the assumption was necessary to Aristarchus in order that he might not have to take account of parallax.)

Plutarch, in the passage referred to above, also makes it clear that Aristarchus followed Heraclides in attributing to the earth the daily rotation about its axis. The bold hypothesis of Aristarchus found few adherents. Seleucus, of Seleucia on the Tigris, is the only convinced supporter of it of whom we hear, and it was speedily abandoned altogether, mainly owing to the great authority of Hipparchus. Nor do we find any trace of the heliocentric hypothesis in Aristarchus's extant work *On the sizes and distances of the Sun and Moon*. This is presumably because that work was written before the hypothesis was formulated in the book referred to by Archimedes. The geometry of the treatise is, however, unaffected by the difference between the hypotheses.

Archimedes also says that it was Aristarchus who discovered that the apparent angular diameter of the sun is about 1/720th part of the zodiac circle, that is to say, half a degree. We do not know how he arrived at this pretty accurate figure: but, as he is credited with the invention of the σκάφη, he may have used this instrument for the purpose. But here again the discovery must apparently have been later than the treatise *On sizes and distances*, for the value of the subtended angle is there assumed to be 2° (Hypothesis 6). How Aristarchus came to assume a value so excessive is uncertain. As the mathematics of his treatise is not dependent on the actual value taken, 2° may have been assumed merely by way of illustration; or it may have been a guess at the apparent diameter made before he had thought of attempting to measure it. Aristarchus assumed that the angular diameters of the sun and moon at the centre of the earth are equal.

The method of the treatise depends on the just observation, which is Aristarchus's third 'hypothesis', that 'when the moon appears to us halved, the great circle which divides the dark and the bright portions of the moon is in the direction of our eye'; the effect of this (since the moon receives its light from the sun), is that at the time of the dichotomy the centres of the sun, moon and earth form a triangle right-angled at the centre of the moon. Two other assumptions were necessary: first, an estimate of the size of the angle of the latter triangle at the centre of the earth at the moment of dichotomy: this Aristarchus assumed (Hypothesis 4) to be 'less than a quadrant by one-thirtieth of a quadrant', i.e. 87°, again an inaccurate estimate, the true value being 89° 50′; secondly, an estimate of the breadth of the earth's shadow where the moon traverses it: this he assumed to be 'the breadth of two moons' (Hypothesis 5).

The inaccuracy of the assumptions does not, however, detract from the mathematical interest of the succeeding investigation. Here we find the logical sequence of propositions and the absolute rigour of demonstration characteristic of Greek geometry; the only remaining drawback would be the practical difficulty of determining the exact moment when the moon 'appears to us halved'. The form and style of the book are thoroughly classical, as befits the period between Euclid and Archimedes;

the Greek is even remarkably attractive. The content from the mathematical point of view is no less interesting, for we have here the first specimen extant of pure geometry used with a *trigonometrical* object, in which respect it is a sort of forerunner of Archimedes's *Measurement of a Circle*. Aristarchus does not actually evaluate the trigonometrical ratios on which the ratios of the sizes and distances to be obtained depend; he finds limits between which they lie, and that by means of certain propositions which he assumes without proof, and which therefore must have been generally known to mathematicians of his day. These propositions are the equivalents of the statements that,

(1) if α is what we call the circular measure of an angle and α is less than $\frac{1}{2}\pi$, then the ratio $\sin\alpha/\alpha$ *decreases*, and the ratio $\tan\alpha/\alpha$ *increases*, as α increases from 0 to $\frac{1}{2}\pi$;

(2) if β be the circular measure of another angle less than $\frac{1}{2}\pi$, and $\alpha > \beta$, then

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}.$$

Aristarchus of course deals, not with actual circular measures, sines and tangents, but with angles (expressed not in degrees but as fractions of right angles), arcs of circles and their chords. Particular results obtained by Aristarchus are the equivalent of the following:

$$\tfrac{1}{18} > \sin 3^\circ > \tfrac{1}{20}, \qquad \text{[Prop. 7]}$$

$$\tfrac{1}{45} > \sin 1^\circ > \tfrac{1}{60}, \qquad \text{[Prop. 11]}$$

$$1 > \cos 1^\circ > \tfrac{89}{90}, \qquad \text{[Prop. 12]}$$

$$1 > \cos^2 1^\circ > \tfrac{44}{45}. \qquad \text{[Prop. 13]}$$

The book consists of eighteen propositions. Beginning with six hypotheses to the effect already indicated, Aristarchus declares that he is now in a position to prove

(1) that the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon from the earth;

(2) that the diameter of the sun has the same ratio as aforesaid to the diameter of the moon;

(3) that the diameter of the sun has to the diameter of the earth a ratio greater than 19 : 3, but less than 43 : 6.

The propositions containing these results are Props. 7, 9 and 15.

Prop. 1 is preliminary, proving that two equal spheres are comprehended by one cylinder, and two unequal spheres by one cone with its vertex in the direction of the lesser sphere, and the cylinder or cone touches the spheres in circles at right angles to the line of centres. Prop. 2 proves that, if a sphere be illuminated by another sphere larger than itself, the illuminated portion is greater than a hemisphere. Prop. 3 shows that the circle in the moon which divides the dark from the bright portion is least when the cone comprehending the sun and the moon has its vertex at our eye. The 'dividing circle', as we shall call it for short, which was in Hypothesis 3 spoken of as a great circle, is proved in Prop. 4 to be, not a great circle, but a small circle not perceptibly different from a great circle. The proof is typical and is worth giving along with that of some connected propositions (11 and 12).

B is the centre of the moon, A that of the earth, CD the diameter of the 'dividing circle in the moon', EF the parallel diameter in the moon. BA meets the circular section of the moon through A and EF in G, and CD in L. GH, GK are arcs each of which is equal to half the arc CE. By Hypothesis 6 the angle CAD is 'one-fifteenth of a sign' $= 2°$, and the angle $BAC = 1°$.

Now, says Aristarchus,

$$1° : 45° \; [> \tan 1° : \tan 45°]$$
$$> BC : CA,$$

and, *a fortiori*,

$$BC : BA \text{ or } BG : BA$$
$$< 1 : 45\,;$$

that is,

$$BG < \tfrac{1}{45} BA$$
$$< \tfrac{1}{44} GA\,;$$

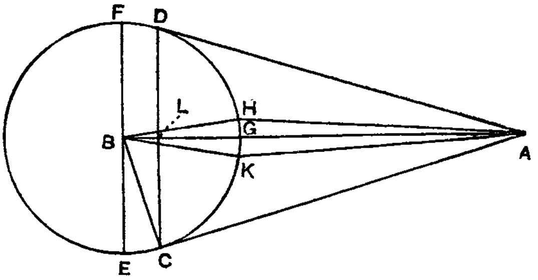
therefore, *a fortiori*,

$$BH < \tfrac{1}{44} HA.$$

Now $BH : HA$ [$= \sin HAB : \sin HBA$]

$> \angle HAB : \angle HBA$,

whence $\angle HAB < \frac{1}{44} \angle HBA$,



and (taking the doubles) $\angle HAK < \frac{1}{44} \angle HBK$.

But $\angle HBK = \angle EBC = \frac{1}{90} R$ (where R is a right angle);

therefore $\angle HAK < \frac{1}{3960} R$.

But 'a magnitude (arc HK) seen under such an angle is imperceptible to our eye';
therefore, *a fortiori*, the arcs CE, DF are severally imperceptible to our eye. Q. E. D.

An easy deduction from the same figure is Prop. 12, which shows that the ratio of CD, the diameter of the 'dividing circle', to EF, the diameter of the moon, is < 1 but $> \frac{89}{90}$.

We have $\angle EBC = \angle BAC = 1°$;

therefore (arc EC) $= \frac{1}{90}$ (arc EG),

and accordingly (arc CG) : (arc GE) $= 89 : 90$.

Doubling the arcs, we have

(arc CGD) : (arc EGF) $= 89 : 90$.

But $CD : EF >$ (arc CGD) : (arc EGF)

[equivalent to $\sin\alpha / \sin\beta > \alpha/\beta$, where $\angle CBD = 2\alpha$, and $2\beta = \pi$];

therefore $CD : EF$ [$= \cos 1°$] $> 89 : 90$,

while obviously $CD : EF < 1$.

Prop. 11 finds limits to the ratio $EF : BA$ (the ratio of the diameter of the moon to the distance of its centre from the centre of the earth); the ratio is $< 2 : 45$ but $> 1 : 30$.

The first part follows from the relation found in Prop. 4, namely $$BC:BA < 1:45,$$ for $$EF = 2\,BC.$$

The second part requires the use of the circle drawn with centre A and radius AC. This circle is that on which the ends of the diameter of the 'dividing circle' move as the moon moves in a circle about the earth. If r is the radius AC of this circle, a chord in it equal to r subtends at the centre A an angle of $\frac{2}{3}R$ or $60°$; and the arc CD subtends at A an angle of $2°$

But (arc subtended by CD) : (arc subtended by r)

$$< CD:r,$$

or $$2:60 < CD:r;$$

that is, $$CD:CA > 1:30.$$

And, by similar triangles,

$$CL:CA = CB:BA, \text{ or } CD:CA = 2\,CB:BA = FE:BA.$$

Therefore $$FE:BA > 1:30.$$

The proposition which is of the greatest interest on the whole is Prop. 7, to the effect that *the distance of the sun from the earth is greater than 18 times, but less than 20 times, the distance of the moon from the earth.* This result represents a great improvement on all previous attempts to estimate the relative distances. The first speculation on the subject was that of Anaximander (*circa* 611–545 B.C.), who seems to have made the distances of the sun and moon from the earth to be in the ratio 3 : 2. Eudoxus, according to Archimedes, made the diameter of the sun 9 times that of the moon, and Phidias, Archimedes's father, 12 times; and, assuming that the angular diameters of the two bodies are equal, the ratio of their distances would be the same.

Aristarchus's proof is shortly as follows. A is the centre of the sun, B that of the earth, and C that of the moon at the moment of dichotomy, so that the angle ACB is right. $ABEF$ is a square, and AE is a quadrant of the sun's circular orbit. Join BF, and bisect the angle FBE by BG, so that

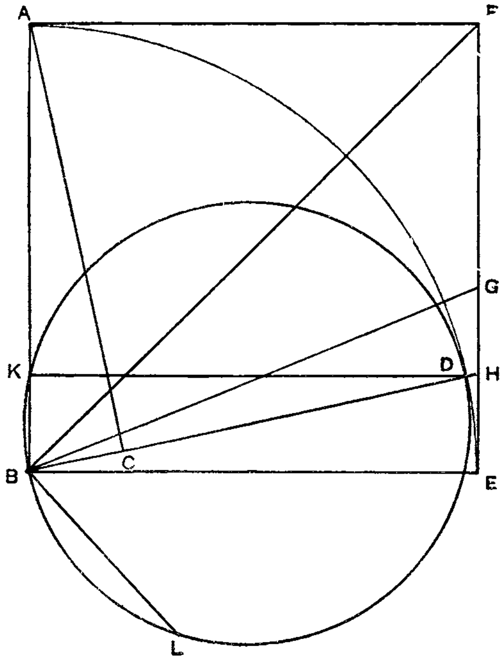
$$\angle GBE = \tfrac{1}{4}R \text{ or } 22\tfrac{1}{2}°$$

I. Now, by Hypothesis 4, $\angle ABC = 87°$,

so that $\angle HBE = \angle BAC = 3°$;

therefore $\angle GBE : \angle HBE = \frac{1}{4} R : \frac{1}{30} R$

$= 15 : 2$,



so that $GE : HE \,[= \tan GBE : \tan HBE] > \angle GBE : \angle HBE$

$> 15 : 2.$ (1)

The ratio which has to be proved $> 18 : 1$ is $AB : BC$ or $FE : EH$.

Now $FG : GE = FB : BE$,

whence $FG^2 : GE^2 = FB^2 : BE^2 = 2 : 1$,

and $FG : GE = \sqrt{2} : 1$

$> 7 : 5$

(this is the approximation to $\sqrt{2}$ mentioned by Plato and known to the Pythagoreans).

Therefore $FE:EG > 12:5$ or $36:15$.

Compounding this with (1) above, we have

$$FE:EH > 36:2 \text{ or } 18:1.$$

II. To prove $BA < 20\,BC$.

Let BH meet the circle AE in D, and draw DK parallel to EB. Circumscribe a circle about the triangle BKD, and let the chord BL be equal to the radius (r) of the circle.

Now $\angle BDK = \angle DBE = \frac{1}{30}R$,

so that arc $BK = \frac{1}{60}$ (circumference of circle).

Thus $(\text{arc } BK):(\text{arc } BL) = \frac{1}{60}:\frac{1}{6}$,

$$= 1:10.$$

And $(\text{arc } BK):(\text{arc } BL) < BK:r$

[this is equivalent to $\alpha/\beta < \sin\alpha/\sin\beta$, where $\alpha < \beta < \frac{1}{2}\pi$],

so that $r < 10\,BK$,

and $BD < 20\,BK$.

But $BD:BK = AB:BC$;

therefore $AB < 20\,BC$. Q. E. D.

The remaining results obtained in the treatise can be visualized by means of the three figures annexed, which have reference to the positions of the sun (centre A), the earth (centre B) and the moon (centre C) during an eclipse. Fig. 1 shows the middle position of the moon relatively to the earth's shadow which is bounded by the cone comprehending the sun and the earth. ON is the arc with centre B along which move the extremities of the diameter of the dividing circle in the moon. Fig. 3 shows the same position of the moon in the middle of the shadow, but on a larger scale. Fig. 2 shows the moon at the moment when it has just entered the shadow; and, as the breadth of the earth's shadow is that of two moons (Hypothesis 5), the moon in the position shown touches BN at N and BL at L, where L is the middle point of the arc ON. It should be added that, in Fig. 1, UV is the diameter of the circle in which the sun is touched by the double cone with B as vertex, which comprehends both the sun and the moon,