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I

INTRODUCTORY

The Greeks and mathematics.

It is an encouraging sign of the times that more and more effort is being directed to promoting a due appreciation and a clear understanding of the gifts of the Greeks to mankind. What we owe to Greece, what the Greeks have done for civilization, aspects of the Greek genius: such are the themes of many careful studies which have made a wide appeal and will surely produce their effect. In truth all nations, in the West at all events, have been to school to the Greeks, in art, literature, philosophy, and science, the things which are essential to the rational use and enjoyment of human powers and activities, the things which make life worth living to a rational human being. 'Of all peoples the Greeks have dreamed the dream of life the best.' And the Greeks were not merely the pioneers in the branches of knowledge which they invented and to which they gave names. What they began they carried to a height of perfection which has not since been surpassed; if there are exceptions, it is only where a few crowded centuries were not enough to provide the accumulation of experience required, whether for the purpose of correcting hypotheses which at first could only be of the nature of guesswork, or of suggesting new methods and machinery.

Of all the manifestations of the Greek genius none is more impressive and even awe-inspiring than that which is revealed by the history of Greek mathematics. Not only are the range and the sum of what the Greek mathematicians actually accomplished wonderful in themselves; it is necessary to bear in mind that this mass of original work was done in an almost incredibly short space of time, and in spite of the comparative inadequacy (as it would seem to us) of the only methods at their disposal, namely those of pure geometry, supplemented, where necessary, by the ordinary arithmetical operations.

Let us, confining ourselves to the main subject of pure geometry by way of example, anticipate so far as to mark certain definite stages in its development, with the intervals separating them. In Thales's time (about 600 B.C.) we find the first glimmerings of a theory of geometry, in the theorems that a circle is bisected by any diameter, that an isosceles triangle has the angles opposite to the equal sides equal, and (if Thales really discovered this) that the angle in a semicircle is a right angle. Rather more than half a century later Pythagoras was taking the first steps towards the theory of numbers and continuing the work of making geometry a theoretical science; he it was who first made geometry one of the subjects of a liberal education. The Pythagoreans, before the next century was out (i. e. before, say, 450 B.C.), had practically completed the subject-matter of Books I–II, IV, VI (and perhaps III) of Euclid's *Elements*, including all the essentials of the 'geometrical algebra' which remained fundamental in Greek geometry; the only drawback was that their theory of proportion was not applicable to incommensurable but only to commensurable magnitudes, so that it proved inadequate as soon as the incommensurable came to be discovered. In the same fifth century the difficult problems of doubling the cube and trisecting any angle, which are beyond the geometry of the straight line and circle, were not only mooted but solved theoretically, the former problem having been first reduced to that of finding two mean proportionals in continued proportion (Hippocrates of Chios) and then solved by a remarkable construction in three dimensions (Archytas), while the latter was solved by means of the curve of Hippias of Elis known as the *quadratrix*; the problem of squaring the circle was also attempted, and Hippocrates, as a contribution to it, discovered and squared three out of the five lunes which can be squared by means of the straight line and circle. In the fourth century Eudoxus discovered the great theory of proportion expounded in Euclid, Book V, and laid down the principles of the *method of exhaustion* for measuring areas and volumes; the conic sections and their fundamental properties were discovered by Menaechmus; the theory of irrationals (probably discovered, so far as $\sqrt{2}$ is concerned, by the early Pythagoreans) was generalized by Theaetetus; and the geometry of the sphere was worked out in systematic treatises. About the end of the century Euclid wrote his *Elements* in thirteen Books. The next century, the third, is that of Archimedes, who may be said to have anticipated the integral calculus, since, by performing what are practically *integrations*, he found the area of a parabolic segment and of a spiral, the surface and volume of a sphere and a segment of a sphere, the volume of any segment of the solids of revolution of the second degree, the centres of gravity of a semicircle, a parabolic segment, any segment of a paraboloid of revolution, and any segment of a sphere or spheroid. Apollonius of Perga, the 'great geometer', about 200 B.C., completed the theory of geometrical conics, with specialized investigations of normals as maxima and minima leading quite easily to the determination of the circle of curvature at any point of a conic and of the equation of the evolute of the conic, which with us is part of analytical conics. With Apollonius the main body of Greek geometry is complete, and we may therefore fairly say that four centuries sufficed to complete it.

But some one will say, how did all this come about? What special aptitude had the Greeks for mathematics? The answer to this question is that their genius for mathematics was simply one aspect of their genius for philosophy. Their mathematics indeed constituted a large part of their philosophy down to Plato. Both had the same origin.

Conditions favouring the development of philosophy among the Greeks.

All men by nature desire to know, says Aristotle.[1] The Greeks, beyond any other people of antiquity, possessed the love of knowledge for its own sake; with them it amounted to an instinct and a passion.[2] We see this first of all in their love of adventure. It is characteristic that in the *Odyssey* Odysseus is extolled as the hero who had 'seen the cities of many men and learned their mind',[3] often even taking his life in his hand, out of a pure passion for extending his horizon,

[1] Arist. *Metaph.* A. 1, 980 a 21.
[2] Cf. Butcher, *Some Aspects of the Greek Genius*, 1892, p. 1.
[3] *Od.* i. 3.

as when he went to see the Cyclopes in order to ascertain 'what sort of people they were, whether violent and savage, with no sense of justice, or hospitable and godfearing'.[1] Coming nearer to historical times, we find philosophers and statesmen travelling in order to benefit by all the wisdom that other nations with a longer history had gathered during the centuries. Thales travelled in Egypt and spent his time with the priests. Solon, according to Herodotus,[2] travelled 'to see the world' (θεωρίης εἵνεκεν), going to Egypt to the court of Amasis, and visiting Croesus at Sardis. At Sardis it was not till 'after he had seen and examined everything' that he had the famous conversation with Croesus; and Croesus addressed him as the Athenian of whose wisdom and peregrinations he had heard great accounts, proving that he had covered much ground in seeing the world and pursuing philosophy. (Herodotus, also a great traveller, is himself an instance of the capacity of the Greeks for assimilating anything that could be learnt from any other nations whatever; and, although in Herodotus's case the object in view was less the pursuit of philosophy than the collection of interesting information, yet he exhibits in no less degree the Greek passion for seeing things as they are and discerning their meaning and mutual relations; 'he compares his reports, he weighs the evidence, he is conscious of his own office as an inquirer after truth'.) But the same avidity for learning is best of all illustrated by the similar tradition with regard to Pythagoras's travels. Iamblichus, in his account of the life of Pythagoras,[3] says that Thales, admiring his remarkable ability, communicated to him all that he knew, but, pleading his own age and failing strength, advised him for his better instruction to go and study with the Egyptian priests. Pythagoras, visiting Sidon on the way, both because it was his birthplace and because he properly thought that the passage to Egypt would be easier by that route, consorted there with the descendants of Mochus, the natural philosopher and prophet, and with the other Phoenician hierophants, and was initiated into all the rites practised in Biblus, Tyre, and in many parts of Syria, a regimen to which he submitted, not out of religious

[1] *Od.* ix. 174–6.
[2] Herodotus, i. 30.
[3] Iamblichus, *De vita Pythagorica*, cc. 2–4.

enthusiasm, '*as you might think*' (ὡς ἄν τις ἁπλῶς ὑπολάβοι), but much more through love and desire for philosophic inquiry, and in order to secure that he should not overlook any fragment of knowledge worth acquiring that might lie hidden in the mysteries or ceremonies of divine worship; then, understanding that what he found in Phoenicia was in some sort an offshoot or descendant of the wisdom of the priests of Egypt, he concluded that he should acquire learning more pure and more sublime by going to the fountain-head in Egypt itself.

'There', continues the story, 'he studied with the priests and prophets and instructed himself on every possible topic, neglecting no item of the instruction favoured by the best judges, no individual man among those who were famous for their knowledge, no rite practised in the country wherever it was, and leaving no place unexplored where he thought he could discover something more. . . . And so he spent 22 years in the shrines throughout Egypt, pursuing astronomy and geometry and, of set purpose and not by fits and starts or casually, entering into all the rites of divine worship, until he was taken captive by Cambyses's force and carried off to Babylon, where again he consorted with the Magi, a willing pupil of willing masters. By them he was fully instructed in their solemn rites and religious worship, and in their midst he attained to the highest eminence in arithmetic, music, and the other branches of learning. After twelve years more thus spent he returned to Samos, being then about 56 years old.'

Whether these stories are true in their details or not is a matter of no consequence. They represent the traditional and universal view of the Greeks themselves regarding the beginnings of their philosophy, and they reflect throughout the Greek spirit and outlook.

From a scientific point of view a very important advantage possessed by the Greeks was their remarkable capacity for accurate observation. This is attested throughout all periods, by the similes in Homer, by vase-paintings, by the ethnographic data in Herodotus, by the 'Hippocratean' medical books, by the biological treatises of Aristotle, and by the history of Greek astronomy in all its stages. To take two commonplace examples. Any person who examines the under-side of a horse's hoof, which we call a 'frog' and the

Greeks called a 'swallow', will agree that the latter is the more accurate description. Or again, what exactness of perception must have been possessed by the architects and workmen to whom we owe the pillars which, seen from below, appear perfectly straight, but, when measured, are found to bulge out (ἔντασις).

A still more essential fact is that the Greeks were a race of *thinkers*. It was not enough for them to know the fact (the ὅτι); they wanted to know the why and wherefore (the διὰ τί), and they never rested until they were able to give a rational explanation, or what appeared to them to be such, of every fact or phenomenon. The history of Greek astronomy furnishes a good example of this, as well as of the fact that no visible phenomenon escaped their observation. We read in Cleomedes[1] that there were stories of extraordinary lunar eclipses having been observed which 'the more ancient of the mathematicians' had vainly tried to explain; the supposed 'paradoxical' case was that in which, while the sun appears to be still above the western horizon, the *eclipsed* moon is seen to rise in the east. The phenomenon was seemingly inconsistent with the recognized explanation of lunar eclipses as caused by the entrance of the moon into the earth's shadow; how could this be if both bodies were above the horizon at the same time? The 'more ancient' mathematicians tried to argue that it was possible that a spectator standing on an *eminence* of the spherical earth might see along the generators of a *cone*, i.e. a little downwards on all sides instead of merely in the plane of the horizon, and so might see both the sun and the moon although the latter was in the earth's shadow. Cleomedes denies this, and prefers to regard the whole story of such cases as a fiction designed merely for the purpose of plaguing astronomers and philosophers; but it is evident that the cases had actually been observed, and that astronomers did not cease to work at the problem until they had found the real explanation, namely that the phenomenon is due to atmospheric refraction, which makes the sun visible to us though it is actually beneath the horizon. Cleomedes himself gives this explanation, observing that such cases of atmospheric refraction were especially

[1] Cleomedes, *De motu circulari*, ii. 6, pp. 218 sq.

noticeable in the neighbourhood of the Black Sea, and comparing the well-known experiment of the ring at the bottom of a jug, where the ring, just out of sight when the jug is empty, is brought into view when water is poured in. We do not know who the 'more ancient' mathematicians were who were first exercised by the 'paradoxical' case; but it seems not impossible that it was the observation of this phenomenon, and the difficulty of explaining it otherwise, which made Anaxagoras and others adhere to the theory that there are other bodies besides the earth which sometimes, by their interposition, cause lunar eclipses. The story is also a good illustration of the fact that, with the Greeks, pure theory went hand in hand with observation. Observation gave data upon which it was possible to found a theory; but the theory had to be modified from time to time to suit observed new facts; they had continually in mind the necessity of 'saving the phenomena' (to use the stereotyped phrase of Greek astronomy). Experiment played the same part in Greek medicine and biology.

Among the different Greek stocks the Ionians who settled on the coast of Asia Minor were the most favourably situated in respect both of natural gifts and of environment for initiating philosophy and theoretical science. When the colonizing spirit first arises in a nation and fresh fields for activity and development are sought, it is naturally the younger, more enterprising and more courageous spirits who volunteer to leave their homes and try their fortune in new countries; similarly, on the intellectual side, the colonists will be at least the equals of those who stay at home, and, being the least wedded to traditional and antiquated ideas, they will be the most capable of striking out new lines. So it was with the Greeks who founded settlements in Asia Minor. The geographical position of these settlements, connected with the mother country by intervening islands, forming stepping-stones as it were from the one to the other, kept them in continual touch with the mother country; and at the same time their geographical horizon was enormously extended by the development of commerce over the whole of the Mediterranean. The most adventurous seafarers among the Greeks of Asia Minor, the Phocaeans, plied their trade successfully as far as the Pillars of Hercules, after they had explored the Adriatic sea, the west coast of Italy, and the coasts of the Ligurians and Iberians. They are said to have founded Massalia, the most important Greek colony in the western countries, as early as 600 B.C. Cyrene, on the Libyan coast, was founded in the last third of the seventh century. The Milesians had, soon after 800 B.C., made settlements on the east coast of the Black Sea (Sinope was founded in 785); the first Greek settlements in Sicily were made from Euboea and Corinth soon after the middle of the eighth century (Syracuse 734). The ancient acquaintance of the Greeks with the south coast of Asia Minor and with Cyprus, and the establishment of close relations with Egypt, in which the Milesians had a large share, belongs to the time of the reign of Psammetichus I (664–610 B.C.), and many Greeks had settled in that country.

The free communications thus existing with the whole of the known world enabled complete information to be collected with regard to the different conditions, customs and beliefs prevailing in the various countries and races; and, in particular, the Ionian Greeks had the inestimable advantage of being in contact, directly and indirectly, with two ancient civilizations, the Babylonian and the Egyptian.

Dealing, at the beginning of the *Metaphysics*, with the evolution of science, Aristotle observes that science was preceded by the arts. The arts were invented as the result of general notions gathered from experience (which again was derived from the exercise of memory); those arts naturally came first which are directed to supplying the necessities of life, and next came those which look to its amenities. It was only when all such arts had been established that the sciences, which do not aim at supplying the necessities or amenities of life, were in turn discovered, and this happened first in the places where men began to have leisure. This is why the mathematical arts were founded in Egypt; for there the priestly caste was allowed to be at leisure. Aristotle does not here mention Babylon; but, such as it was, Babylonian science also was the monopoly of the priesthood.

It is in fact true, as Gomperz says,[1] that the first steps on the road of scientific inquiry were, so far as we know from

[1] *Griechische Denker*, i, pp. 36, 37.

history, never accomplished except where the existence of an organized caste of priests and scholars secured the necessary industry, with the equally indispensable continuity of tradition. But in those very places the first steps were generally the last also, because the scientific doctrines so attained tend, through their identification with religious prescriptions, to become only too easily, like the latter, mere lifeless dogmas. It was a fortunate chance for the unhindered spiritual development of the Greek people that, while their predecessors in civilization had an organized priesthood, the Greeks never had. To begin with, they could exercise with perfect freedom their power of unerring eclecticism in the assimilation of every kind of lore. 'It remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.'[1] For the same reason, while using the earlier work of Egyptians and Babylonians as a basis, the Greek genius could take an independent upward course free from every kind of restraint and venture on a flight which was destined to carry it to the highest achievements.

The Greeks then, with their 'unclouded clearness of mind' and their freedom of thought, untrammelled by any 'Bible' or its equivalent, were alone capable of creating the sciences as they did create them, i.e. as living things based on sound first principles and capable of indefinite development. It was a great boast, but a true one, which the author of the *Epinomis* made when he said, 'Let us take it as an axiom that, whatever the Greeks take from the barbarians, they bring it to fuller perfection'.[2] He has been speaking of the extent to which the Greeks had been able to explain the relative motions and speeds of the sun, moon and planets, while admitting that there was still much progress to be made before absolute certainty could be achieved. He adds a characteristic sentence, which is very relevant to the above remarks about the Greek's free outlook:

'Let no Greek ever be afraid that we ought not at any time to study things divine because we are mortal. We ought to

[1] Cumont, *Neue Jahrbücher*, xxiv, 1911, p. 4. [2] *Epinomis*, 987 D.

maintain the very contrary view, namely, that God cannot possibly be without intelligence or be ignorant of human nature: rather he knows that, when he teaches them, men will follow him and learn what they are taught. And he is of course perfectly aware that he does teach us, and that we learn, the very subject we are now discussing, number and counting; if he failed to know this, he would show the greatest want of intelligence; the God we speak of would in fact not know himself, if he took it amiss that a man capable of learning should learn, and if he did not rejoice unreservedly with one who became good by divine influence.'[1]

Nothing could well show more clearly the Greek conviction that there could be no opposition between religion and scientific truth, and therefore that there could be no impiety in the pursuit of truth. The passage is a good parallel to the statement attributed to Plato that *θεὸς ἀεὶ γεωμετρεῖ.*

Meaning and classification of mathematics.

The words *μαθήματα* and *μαθηματικός* do not appear to have been definitely appropriated to the special meaning of mathematics and mathematicians or things mathematical until Aristotle's time. With Plato *μάθημα* is quite general, meaning any subject of instruction or study; he speaks of *καλὰ μαθήματα*, good subjects of instruction, as of *καλὰ ἐπιτηδεύματα*, good pursuits, of women's subjects as opposed to men's, of the Sophists hawking sound *μαθήματα*; what, he asks in the *Republic*, are the greatest *μαθήματα*? and he answers that the greatest *μάθημα* is the Idea of the Good.[2] But in the *Laws* he speaks of *τρία μαθήματα*, three subjects, as fit for freeborn men, the subjects being arithmetic, the science of measurement (geometry), and astronomy[3]; and no doubt the pre-eminent place given to mathematical subjects in his scheme of education would have its effect in encouraging the habit of speaking of these subjects exclusively as *μαθήματα*. The Peripatetics, we are told, explained the special use of the word in this way; they pointed out that, whereas such things as rhetoric and poetry and the whole of popular *μουσική* can be understood even by one who has not learnt them, the subjects called by the special name of *μαθήματα* cannot be known

[1] *Epinomis*, 988 A. [2] *Republic*, vi. 505 A. [3] *Laws*, vii. 817 E.