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Active in Alexandria in the third century BCE, Apollonius of Perga ranks as one of the greatest Greek geometers. Building on foundations laid by Euclid, he is famous for defining the parabola, hyperbola and ellipse in his major treatise on conic sections. The dense nature of its text, however, made it inaccessible to most readers. When it was originally published in 1896 by the civil servant and classical scholar Thomas Little Heath (1861–1940), the present work was the first English translation and, more importantly, the first serious effort to standardise the terminology and notation. Along with clear diagrams, Heath includes a thorough introduction to the work and the history of the subject. Seeing the treatise as more than an esoteric artefact, Heath presents it as a valuable tool for modern mathematicians. His works on *Diophantos of Alexandria* (1885) and *Aristarchus of Samos* (1913) are also reissued in this series.

# Treatise on Conic Sections

*Edited in Modern Notation with Introductions, Including an Essay on the Earlier History of the Subject*

Apollonius of Perga
Edited by T.L. Heath

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# APOLLONIUS OF PERGA

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*Ariſtippus Philoſophus Socraticus, naufragio cum ejectus ad Rhodienſium litus animadvertiſſet Geometrica ſchemata deſcripta, exclamaviſſe ad comites ita dicitur,* Bene ſperemus, Hominum enim veſtigia video.
*Vitruv. Architect. lib. 6. Præf.*

# APOLLONIUS OF PERGA

## TREATISE ON CONIC SECTIONS

EDITED IN MODERN NOTATION

WITH INTRODUCTIONS INCLUDING AN ESSAY ON THE EARLIER HISTORY OF THE SUBJECT

BY

T. L. HEATH, M.A.

SOMETIME FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

*ζηλοῦντες τοὺς Πυθαγορείους, οἷς πρόχειρον ἦν καὶ τοῦτο σύμβολον σχᾶμα καὶ βᾶμα, ἀλλ' οὐ σχᾶμα καὶ τριώβολον.* PROCLUS.

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MANIBUS

EDMUNDI HALLEY

D. D. D.

# PREFACE.

IT is not too much to say that, to the great majority of mathematicians at the present time, Apollonius is nothing more than a name and his *Conics*, for all practical purposes, a book unknown. Yet this book, written some twenty-one centuries ago, contains, in the words of Chasles, "the most interesting properties of the conics," to say nothing of such brilliant investigations as those in which, by purely geometrical means, the author arrives at what amounts to the complete determination of the evolute of any conic. The general neglect of the "great geometer," as he was called by his contemporaries on account of this very work, is all the more remarkable from the contrast which it affords to the fate of his predecessor Euclid; for, whereas in this country at least the *Elements* of Euclid are still, both as regards their contents and their order, the accepted basis of elementary geometry, the influence of Apollonius upon modern text-books on conic sections is, so far as form and method are concerned, practically *nil*.

Nor is it hard to find probable reasons for the prevailing absence of knowledge on the subject. In the first place, it could hardly be considered surprising if the average mathematician were apt to show a certain faintheartedness when confronted with seven Books in Greek or Latin which contain 387 propositions in all; and doubtless the apparently portentous bulk of the treatise has deterred many from attempting to make its acquaintance. Again, the form of the propositions is an additional difficulty, because the reader finds in them none of the ordinary aids towards the comprehension of somewhat complicated geometrical work, such as the conventional appropriation, in modern text-books, of definite letters to denote particular points on the various conic sections. On the contrary, the enunciations of propositions which, by the aid of a notation once agreed upon, can now be stated in a few lines, were by Apollonius invariably given in words like the enunciations of Euclid. These latter are often sufficiently unwieldy; but the inconvenience is greatly intensified in Apollonius, where the greater complexity of the conceptions entering into the investigation of conics, as compared with the more elementary notions relating to the line and circle, necessitates in many instances an enunciation extending over a space equal to (say) half a page of this book. Hence it is often a matter of considerable labour even to grasp the enunciation of a proposition. Further, the propositions are, with the exception that separate paragraphs mark the formal divisions, printed continuously; there are no breaks for the purpose of enabling the eye to take in readily the successive steps in the demonstration and so facilitating the comprehension of the argument as a whole. There is no uniformity of notation, but in almost every fresh proposition a different letter is employed to denote the same point: what wonder then if there are the most serious obstacles in the way of even remembering the results of certain propositions? Nevertheless these propositions, though unfamiliar to mathematicians of the present day, are of the very essence of Apollonius' system, are being constantly used, and must therefore necessarily be borne in mind.

The foregoing remarks refer to the editions where Apollonius can be read in the Greek or in a Latin translation, i.e. to those of Halley and Heiberg; but the only attempt which has been made to give a complete view of the substance of Apollonius in a form more accessible to the modern reader is open to much the same objections. This reproduction of the *Conics* in German by H. Balsam (Berlin, 1861) is a work deserving great praise both for its accuracy and the usefulness of the occasional explanatory notes, but perhaps most of all for an admirable set of figures to the number of 400 at the end of the book; the enunciations of the propositions are, however, still in words, there are few breaks in the continuity of the printing, and the notation is not sufficiently modernised to make the book of any more real service to the ordinary reader than the original editions.

An edition is therefore still wanted which shall, while in some places adhering even more closely than Balsam to the original text, at the same time be so entirely remodelled by the aid of accepted modern notation as to be thoroughly readable by any competent mathematician; and this want it is the object of the present work to supply.

In setting myself this task, I made up my mind that any satisfactory reproduction of the *Conics* must fulfil certain essential conditions: (1) it should be Apollonius and nothing but Apollonius, and nothing should be altered either in the substance or in the order of his thought, (2) it should be complete, leaving out nothing of any significance or importance, (3) it should exhibit under different headings the successive divisions of the subject, so that the definite scheme followed by the author may be seen as a whole.

Accordingly I considered it to be the first essential that I should make myself thoroughly familiar with the whole work at first hand. With this object I first wrote out a perfectly literal translation of the whole of the extant seven Books. This was a laborious task, but it was not in other respects difficult, owing to the excellence of the standard editions. Of these editions, Halley's is a monumental work, beyond praise alike in respect of its design and execution; and for Books V—VII it is still the only complete edition. For Books I—IV I used for the most part the new Greek text of Heiberg, a scholar who has earned the undying gratitude of all who are interested in the history of Greek mathematics by successively bringing out a critical text (with Latin translation) of Archimedes, of Euclid's *Elements*, and of all the writings of Apollonius still extant in Greek. The only drawback to Heiberg's Apollonius is the figures, which are poor and not seldom even misleading, so that I found it a great advantage to have Halley's edition, with its admirably executed diagrams, before me even while engaged on Books I—IV.

The real difficulty began with the constructive work of re-writing the book, involving as it did the substitution of a new and uniform notation, the condensation of some propositions, the combination of two or more into one, some slight re-arrangements of order for the purpose of bringing together kindred propositions in cases where their separation was rather a matter of accident than indicative of design, and so on. The result has been (without leaving out anything essential or important) to diminish the bulk of the work by considerably more than one-half and to reduce to a corresponding extent the number of separate propositions.

When the re-editing of the *Conics* was finished, it seemed necessary for completeness to prefix an Introduction for the purposes (1) of showing the relation of Apollonius to his predecessors in the same field both as regards matter and method, (2) of explaining more fully than was possible in the few notes inserted in square brackets in the body of the book the mathematical significance of certain portions of the *Conics* and the probable connexion between this and other smaller treatises of Apollonius about which we have information, (3) of describing and illustrating fully the form and language of the propositions as they stand in the original Greek text. The first of these purposes required that I should give a sketch of the history of conic sections up to the time of Apollonius; and I have accordingly considered it worth while to make this part of the

Introduction as far as possible complete. Thus e.g. in the case of Archimedes I have collected practically all the propositions in conics to be found in his numerous works with the substance of the proofs where given; and I hope that the historical sketch as a whole will be found not only more exhaustive, for the period covered, than any that has yet appeared in English, but also not less interesting than the rest of the book.

For the purposes of the earlier history of conics, and the chapters on the mathematical significance of certain portions of the *Conics* and of the other smaller treatises of Apollonius, I have been constantly indebted to an admirable work by H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum* (German edition, Copenhagen, 1886), which to a large extent covers the same ground, though a great portion of his work, consisting of a mathematical analysis rather than a reproduction of Apollonius, is of course here replaced by the re-edited treatise itself. I have also made constant use of Heiberg's *Litterargeschichtliche Studien über Euklid* (Leipzig, 1882), the original Greek of Euclid's *Elements*, the works of Archimedes, the *συναγωγή* of Pappus and the important *Commentary on Eucl. Book* I. by Proclus (ed. Friedlein, Leipzig, 1873).

The frontispiece to this volume is a reproduction of a quaint picture and attached legend which appeared at the beginning of Halley's edition. The story is also told elsewhere than in Vitruvius, but with less point (cf. Claudii Galeni Pergameni Προτρεπτικὸς ἐπὶ τέχνας c. v. § 8, p. 108, 3–8 ed. I. Marquardt, Leipzig, 1884). The quotation on the title page is from a vigorous and inspiring passage in Proclus' *Commentary on Eucl. Book* I. (p. 84, ed. Friedlein) in which he is describing the scientific purpose of his work and contrasting it with the useless investigations of paltry lemmas, distinctions of cases, and the like, which formed the stock-in-trade of the ordinary Greek commentator. One merit claimed by Proclus for his work I think I may fairly claim for my own, that it at least contains *ὅσα πραγματειωδεστέραν ἔχει θεωρίαν*; and I should indeed be proud if, in the judgment of competent critics, it should be found possible to apply to it the succeeding phrase, *συντελεῖ πρὸς τὴν ὅλην φιλοσοφίαν.*

Lastly, I wish to express my thanks to my brother, Dr R. S. Heath, Principal of Mason College, Birmingham, for his kindness in reading over most of the proof sheets and for the constant interest which he has taken in the progress of the work.

T. L. HEATH.

*March*, 1896.

## LIST OF PRINCIPAL AUTHORITIES.

EDMUND HALLEY, *Apollonii Pergaei Conicorum libri octo et Sereni Antissensis de sectione cylindri et coni libri duo.* (Oxford, 1710.)

EDMUND HALLEY, *Apollonii Pergaei de Sectione Rationis libri duo, ex Arabico versi.* (Oxford, 1706.)

J. L. HEIBERG, *Apollonii Pergaei quae Graece exstant cum commentariis antiquis.* (Leipzig, 1891–3.)

H. BALSAM, *Des Apollonius von Perga sieben Bücher über Kegelschnitte nebst dem durch* Halley *wieder hergestellten achten Buche deutsch bearbeitet.* (Berlin, 1861.)

---

J. L. HEIBERG, *Litterargeschichtliche Studien über Euklid.* (Leipzig, 1882.)

J. L. HEIBERG, *Euclidis elementa.* (Leipzig, 1883–8.)

G. FRIEDLEIN, *Procli Diadochi in primum Euclidis elementorum librum commentarii.* (Leipzig, 1873.)

J. L. HEIBERG, *Quaestiones Archimedeae.* (Copenhagen, 1879.)

J. L. HEIBERG, *Archimedis opera omnia cum commentariis Eutocii.* (Leipzig, 1880–1.)

F. HULTSCH, *Pappi Alexandrini collectionis quae supersunt.* (Berlin, 1876–8.)

C. A. BRETSCHNEIDER, *Die Geometrie und die Geometer vor Euklides.* (Leipzig, 1870.)

M. CANTOR, *Vorlesungen über Geschichte der Mathematik.* (Leipzig, 1880.)

H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum. Deutsche Ausgabe.* (Copenhagen, 1886.)

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# INTRODUCTION.

## PART I.

## THE EARLIER HISTORY OF CONIC SECTIONS AMONG THE GREEKS.

### CHAPTER I.

### THE DISCOVERY OF CONIC SECTIONS: MENAECHMUS.

There is perhaps no question that occupies, comparatively, a larger space in the history of Greek geometry than the problem of the Doubling of the Cube. The tradition concerning its origin is given in a letter from Eratosthenes of Cyrene to King Ptolemy Euergetes quoted by Eutocius in his commentary on the second Book of Archimedes' treatise *On the Sphere and Cylinder**; and the following is a translation of the letter as far as the point where we find mention of **Menaechmus**, with whom the present subject begins.

"Eratosthenes to King Ptolemy greeting.

"There is a story that one of the old tragedians represented Minos as wishing to erect a tomb for Glaucus and as saying, when he heard that it was a hundred feet every way,

Too small thy plan to bound a royal tomb.
Let it be double; yet of its fair form
Fail not, but haste to double every side†.

* In quotations from Archimedes or the commentaries of Eutocius on his works the references are throughout to Heiberg's edition (*Archimedis opera omnia cum commentariis Eutocii*. 3 vols. Leipzig, 1880–1). The reference here is III. p. 102.

† μικρόν γ' ἔλεξας βασιλικοῦ σηκὸν τάφου·
διπλάσιος ἔστω· τοῦ καλοῦ δὲ μὴ σφαλεὶς
δίπλαζ' ἕκαστον κῶλον ἐν τάχει τάφου.

Valckenaer (*Diatribe de fragm. Eurip.*) suggests that the verses are from the

But he was clearly in error; for, when the sides are doubled, the area becomes four times as great, and the solid content eight times as great. Geometers also continued to investigate the question in what manner one might double a given solid while it remained in the same form. And a problem of this kind was called the doubling of the cube; for they started from a cube and sought to double it. While then for a long time everyone was at a loss, Hippocrates of Chios was the first to observe that, if between two straight lines of which the greater is double of the less it were discovered how to find two mean proportionals in continued proportion, the cube would be doubled; and thus he turned the difficulty in the original problem* into another difficulty no less than the former. Afterwards, they say, some Delians attempting, in accordance with an oracle, to double one of the altars fell into the same difficulty. And they sent and begged the geometers who were with Plato in the Academy to find for them the required solution. And while they set themselves energetically to work and sought to find two means between two given straight lines, Archytas of Tarentum is said to have discovered them by means of half-cylinders, and Eudoxus by means of the so-called curved lines. It is, however, characteristic of them all that they indeed gave demonstrations, but were unable to make the actual construction or to reach the point of practical application, except to a small extent Menaechmus and that with difficulty."

Some verses at the end of the letter, in commending Eratosthenes' own solution, suggest that there need be no resort to Archytas' unwieldy contrivances of cylinders or to "cutting the cone in the triads of Menaechmus†." This last phrase of Eratosthenes appears

*Polyidus* of Euripides, but that the words after σφαλεὶς (or σφαλῇς) are Eratosthenes' own, and that the verses from the tragedy are simply

μικρόν γ' ἔλεξας βασιλικοῦ σηκὸν τάφου·
διπλάσιος ἔστω· τοῦ κύβου δὲ μὴ σφαλῇς.

It would, however, be strange if Eratosthenes had added words merely for the purpose of correcting them again: and Nauck (*Tragicorum Graecorum Fragmenta*, Leipzig, 1889, p. 874) gives the three verses as above, but holds that they do not belong to the *Polyidus*, adding that they are no doubt from an earlier poet than Euripides, perhaps Aeschylus.

* τὸ ἀπόρημα αὐτοῦ is translated by Heiberg "haesitatio eius," which no doubt means "*his* difficulty." I think it is better to regard αὐτοῦ as neuter, and as referring to the problem of doubling the cube.

† μηδὲ Μενεχμείους κωνοτομεῖν τριάδας.

again, by way of confirmatory evidence, in a passage of Proclus*, where, quoting Geminus, he says that the conic sections were discovered by Menaechmus.

Thus the evidence so far shows (1) that Menaechmus (a pupil of Eudoxus and a contemporary of Plato) was the discoverer of the conic sections, and (2) that he used them as a means of solving the problem of the doubling of the cube. We learn further from Eutocius† that Menaechmus gave two solutions of the problem of the two mean proportionals, to which Hippocrates had reduced the original problem, obtaining the two means first by the intersection of a certain parabola and a certain rectangular hyperbola, and secondly by the intersection of two parabolas‡. Assuming that $a$, $b$ are the two given unequal straight lines and $x$, $y$ the two required mean proportionals, the discovery of Hippocrates amounted to the discovery of the fact that from the relation

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b} \quad\quad (1)$$

it follows that $$\left(\frac{a}{x}\right)^3 = \frac{a}{b},$$

and, if $a = 2b$, $$a^3 = 2x^3.$$

The equations (1) are equivalent to the three equations

$$x^2 = ay, \quad y^2 = bx, \quad xy = ab \quad\quad (2),$$

and the solutions of Menaechmus described by Eutocius amount to the determination of a point as the intersection of the curves represented in a rectangular system of Cartesian coordinates by any two of the equations (2).

Let $AO$, $BO$ be straight lines placed so as to form a right angle at $O$, and of length $a$, $b$ respectively§. Produce $BO$ to $x$ and $AO$ to $y$.

* *Comm. on Eucl.* I., p. 111 (ed. Friedlein). The passage is quoted, with the context, in the work of Bretschneider, *Die Geometrie und die Geometer vor Euklides*, p. 177.

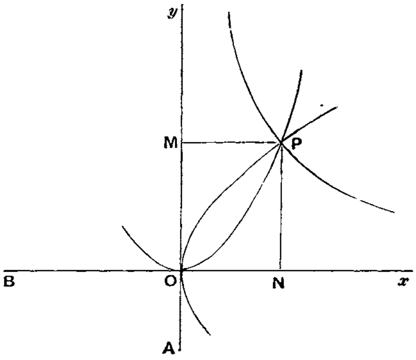
† *Commentary on Archimedes* (ed. Heiberg, III. p. 92—98).

‡ It must be borne in mind that the words *parabola* and *hyperbola* could not have been used by Menaechmus, as will be seen later on; but the phraseology is that of Eutocius himself.

§ One figure has been substituted for the two given by Eutocius, so as to make it serve for both solutions. The figure is identical with that attached to the *second* solution, with the sole addition of the portion of the rectangular hyperbola used in the first solution.

It is a curious circumstance that in Eutocius' second figure the straight line

The *first* solution now consists in drawing a parabola, with vertex $O$ and axis $Ox$, such that its parameter is equal to $BO$ or $b$, and a hyperbola with $Ox$, $Oy$ as asymptotes such that the rectangle under the distances of any point on the curve from $Ox$, $Oy$ respectively is equal to the rectangle under $AO$, $BO$, i.e. to $ab$. If $P$ be




the point of intersection of the parabola and hyperbola, and $PN$, $PM$ be drawn perpendicular to $Ox$, $Oy$, i.e. if $PN$, $PM$ be denoted by $y$, $x$, the coordinates of the point $P$, we shall have

$$\left.\begin{aligned} y^2 &= b\,.\,ON = b\,.\,PM = bx \\ \text{and}\qquad xy &= PM\,.\,PN = ab \end{aligned}\right\},$$

whence
$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

In the *second* solution of Menaechmus we are to draw the parabola described in the first solution and also the parabola whose

representing the length of the parameter of each parabola is drawn in the same straight line with the axis of the parabola, whereas Apollonius always draws the parameter as a line starting from the vertex (or the end of a diameter) and perpendicular to the axis (or diameter). It is possible that we may have here an additional indication that the idea of the parameter as ὀρθία or the *latus rectum* originated with Apollonius; though it is also possible that the selection of the directions of $AO$, $BO$ was due to nothing more than accident, or may have been made in order that the successive terms in the continued proportion might appear in the figure in cyclic order, which corresponds moreover to their relative positions in the mechanical solution attributed to Plato. For this solution see the same passage of Eutocius (*Archimedes*, ed. Heiberg, III. p. 66—70).

vertex is $O$, axis $Oy$ and parameter equal to $a$. The point $P$ where the two parabolas intersect is given by

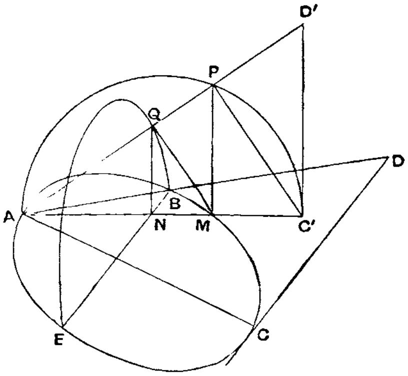
$$\left.\begin{array}{l} y^2 = bx \\ x^2 = ay \end{array}\right\},$$

whence, as before, $$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

We have therefore, in these two solutions, the parabola and the rectangular hyperbola in the aspect of *loci* any points of which respectively fulfil the conditions expressed by the equations in (2); and it is more than probable that the discovery of Menaechmus was due to efforts to determine *loci* possessing these characteristic properties rather than to any idea of a systematic investigation of the sections of a cone as such. This supposition is confirmed by the very special way in which, as will be seen presently, the conic sections were originally produced from the right circular cone; indeed the special method is difficult to explain on any other assumption. It is moreover natural to suppose that, after the discovery of the convertibility of the cube-problem into that of finding two mean proportionals, the two forms of the resulting equations would be made the subject of the most minute and searching investigation. The form (1) expressing the equality of three ratios led naturally to the solution attributed to Plato, in which the four lines representing the successive terms of the continued proportion are placed mutually at right angles and in cyclic order round a fixed point, and the extremities of the lines are found by means of a rectangular frame, three sides of which are fixed, while the fourth side can move freely parallel to itself. The investigation of the form (2) of the equations led to the attempt of Menaechmus to determine the loci corresponding thereto. It was known that the locus represented by $y^2 = x_1 x_2$, where $y$ is the perpendicular from any point on a fixed straight line of given length, and $x_1$, $x_2$ are the segments into which the line is divided by the perpendicular, was a circle; and it would be natural to assume that the equation $y^2 = bx$, differing from the other only in the fact that a constant is substituted for one of the variable magnitudes, would be capable of representation as a locus or a continuous curve. The only difficulty would be to discover its form, and it was here that the cone was introduced.

If an explanation is needed of the circumstance that Menaechmus should have had recourse to any solid figure, and to a cone in particular, for the purpose of producing a plane locus, we find it in the fact that solid geometry had already reached a high state of development, as is shown by the solution of the problem of the two mean proportionals by Archytas of Tarentum (born about 430 B.C.). This solution, in itself perhaps more remarkable than any other, determines a certain point as the intersection of three surfaces of revolution, (1) a right cone, (2) a right cylinder whose base is a circle on the axis of the cone as diameter and passing through the apex of the cone, (3) the surface formed by causing a semicircle, whose diameter is the same as that of the circular base of the cylinder and whose plane is perpendicular to that of the circle, to revolve about the apex of the cone as a fixed point so that the diameter of the semicircle moves always in the plane of the circle, in other words, the surface consisting of half a *split ring* whose centre is the apex of the cone and whose inner diameter is indefinitely small. We find that in the course of the solution (*a*) the intersection of the surfaces (2) and (3) is said to be a certain curve (γραμμήν τινα), being in fact a curve of double curvature, (*b*) a circular section of the right cone is used in the proof, and (*c*), as the penultimate step, two mean proportionals are found in one and the same plane (triangular) section of the cone*.

* The solution of Archytas is, like the others, given by Eutocius (p. 98—102) and is so instructive that I cannot forbear to quote it. Suppose that $AC$, $AB$ are the straight lines between which two mean proportionals are to be found. $AC$ is then made the diameter of a circle, and $AB$ is placed as a chord in the circle.




A semicircle is drawn with diameter $AC$ but in a plane perpendicular to that of $ABC$, and revolves about an axis through $A$ perpendicular to the plane of $ABC$.

Thus the introduction of cones by Menaechmus should not in itself be a matter for surprise.

Concerning Menaechmus' actual method of deducing the properties of the conic sections from the cone we have no definite information; but we may form some idea of his probable procedure

A half-cylinder (right) is now erected with $ABC$ as base: this will cut the surface described by the moving semicircle $APC$ in a certain curve.

Lastly let $CD$, the tangent to the circle $ABC$ at the point $C$, meet $AB$ produced in $D$; and suppose the triangle $ACD$ to revolve about $AC$ as axis. This will generate the surface of a right circular cone, and the point $B$ will describe a semicircle $BQE$ perpendicular to the plane of $ABC$ and having its diameter $BE$ at right angles to $AC$. The surface of the cone will meet in some point $P$ the curve described on the cylinder. Let $APC'$ be the corresponding position of the revolving semicircle, and let $AC'$ meet the circle $ABC$ in $M$.

Drawing $PM$ perpendicular to the plane of $ABC$, we see that it must meet the circumference of the circle $ABC$ because $P$ is on the cylinder which stands on $ABC$ as base. Let $AP$ meet the circumference of the semicircle $BQE$ in $Q$, and let $AC'$ meet its diameter $BE$ in $N$. Join $PC'$, $QM$, $QN$.

Then, since both semicircles are perpendicular to the plane $ABC$, so is their line of intersection $QN$. Therefore $QN$ is perpendicular to $BE$.

Hence $$QN^2 = BN \,.\, NE = AN \,.\, NM.$$

Therefore the angle $AQM$ is a right angle.

But the angle $C'PA$ is also right: therefore $MQ$ is parallel to $C'P$.

It follows, by similar triangles, that

$$C'A : AP = AP : AM = AM : AQ.$$

That is, $$AC : AP = AP : AM = AM : AB,$$

and $AB$, $AM$, $AP$, $AC$ are in continued proportion.

In the language of analytical geometry, if $AC$ is the axis of $x$, a line through $A$ perpendicular to $AC$ in the plane of $ABC$ the axis of $y$, and a line through $A$ parallel to $PM$ the axis of $z$, then $P$ is determined as the intersection of the surfaces

$$x^2+y^2+z^2 = \frac{a^2}{b^2}x^2 \quad \ldots\ldots (1),$$

$$x^2+y^2 = ax \quad \ldots\ldots (2),$$

$$x^2+y^2+z^2 = a\sqrt{x^2+y^2} \quad \ldots\ldots (3),$$

where $$AC = a, \; AB = b.$$

From the first two equations

$$x^2+y^2+z^2 = \frac{(x^2+y^2)^2}{b^2},$$

and from this equation and (3) we have

$$\frac{a}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{b},$$

or $$AC : AP = AP : AM = AM : AB.$$

if we bear in mind (1) what we are told of the manner in which the earlier writers on conics produced the three curves from particular kinds of right circular cones, and (2) the course followed by Apollonius (and Archimedes) in dealing with sections of *any* circular cone, whether right or oblique.

Eutocius, in his commentary on the *Conics* of Apollonius, quotes with approval a statement of Geminus to the effect that the ancients defined a cone as the surface described by the revolution of a right-angled triangle about one of the sides containing the right angle, and that they knew no other cones than right cones. Of these they distinguished three kinds according as the vertical angle of the cone was less than, equal to, or greater than, a right angle. Further they produced only one of the three sections from each kind of cone, always cutting it by a plane perpendicular to one of the generating lines, and calling the respective curves by names corresponding to the particular kind of cone; thus the "section of a right-angled cone" was their name for a parabola, the "section of an acute-angled cone" for an ellipse, and the "section of an obtuse-angled cone" for a hyperbola. The sections are so described by Archimedes.

Now clearly the parabola is the one of the three sections for the production of which the use of a right-angled cone and a section at right angles to a generator gave the readiest means. If $N$ be a point on the diameter $BC$ of any circular section in such a cone, and if $NP$ be a straight line drawn in the plane of the section and perpendicular to $BC$, meeting the circumference of the circle (and therefore the surface of the cone) in $P$,

$$PN^2 = BN \cdot NC.$$

Draw $AN$ in the plane of the axial triangle $OBC$ meeting the generator $OB$ at right angles in $A$, and draw $AD$ parallel to $BC$ meeting $OC$ in $D$; let $DEF$, perpendicular to $AD$ or $BC$, meet $BC$ in $E$ and $AN$ produced in $F$.

Then $AD$ is bisected by the axis of the cone, and therefore $AF$ is likewise bisected by it. Draw $CG$ perpendicular to $BC$ meeting $AF$ produced in $G$.

Now the angles $BAN$, $BCG$ are right; therefore $B$, $A$, $C$, $G$ are concyclic, and

$$BN \cdot NC = AN \cdot NG.$$

But $$AN = CD = FG;$$