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By the end of the eighteenth century, British mathematics had been stuck in a rut for a hundred years. Calculus was still taught in the style of Newton, with no recognition of the great advances made in continental Europe. The examination system at Cambridge even mandated the use of Newtonian notation. As discontented undergraduates, Charles Babbage (1791–1871) and John Herschel (1792–1871) formed the Analytical Society in 1811. The group, including William Whewell and George Peacock, sought to promote the new continental mathematics. Babbage's preface to the present work, first published in 1813, may be considered the movement's manifesto. He provided the first paper here, and Herschel the two others. Although the group was relatively short-lived, its ideas took root as its erstwhile members rose to prominence. As the society's sole publication, this remains a significant text in the history of British mathematics.

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Memoirs of the Analytical Society

CHARLES BABBAGE
JOHN HERSCHEL

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ANALYTICAL SOCIETY

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PREFACE.

To examine the varied relations of necessary truth, and to trace through its successive developements, the simple principle to its ultimate result, is the peculiar province of Mathematical Analysis. Aided by that refined system, which the ingenuity of modern calculators has elicited, and to which the term Analytics is now almost exclusively appropriated, it pursues trains of reasoning, which, from their length and intricacy, would resist for ever the unassisted efforts of human sagacity. To what cause are we to attribute this surprising advantage? One, undoubtedly the most obvious, consists in the nature of the ideas themselves, whose relations form the object of investigation—and the accuracy with which they are defined. This is equally indeed the property of every branch of Mathematical enquiry. Three causes however chiefly appear to have given so vast a superiority to Analysis, as an instrument of reason. Of these, the accurate simplicity of its language claims the first place. An arbitrary symbol can neither convey, nor excite any idea foreign to its original definition. This immutability, no less than the symmetry of its notation, (which should ever be guarded with a jealousy commensurate to its vital importance,) facilitates the translation of an expression into common language at any stage of an operation,—disburdens the memory of all the load of the previous steps,—and at the same time, affords it a considerable assistance in retaining the results. Another, and perhaps not less considerable cause, is to be found in the conciseness of that notation. Every train of reasoning implies an exercise of the judgement, which, being an operation of the mind, deciding on the agreement or disagreement of ideas successively presented to it, it is reasonable to presume will be more correct, in proportion as the ideas compared follow each other more closely; provided the succession be not so rapid as to cause confusion. Were an Analytical operation of any complexity converted into common language, in all its detail, the mind, after acquiring a clear conception of one part of the related ideas, must suspend its decision until it could obtain an equally perspicuous one of the remainder of the proposition; and in so long an interval as this must occupy, the impression of the former ideas would necessarily have faded in some degree from the memory, unless fixed by an expense of time and attention, sufficient to deter any one from the employment of such means of discovery. It is the spirit of this symbolic language, by that mechanical tact, (so much in unison with all our faculties,) which carries the eye at one glance through the most intricate modifications of quantity, to condense pages into lines, and volumes into pages; shortening the road to discovery, and preserving the mind unfatigued by continued efforts of attention to the minor parts, that it may exert its whole vigor on those which are more important.

The last cause we have occasion to notice is, that Analysis, by separating the difficulties of a question, overcomes those which appear almost insuperable when combined, or at least, reducing each to its least terms, leaves them as the acknowledged landmarks of its progress,—open to approach on all sides, should ulterior discovery furnish any rational hope of their removal. Meanwhile that progress continues unimpeded. Simple relations are found to exist between the most refractory functions, and even when the difficulties themselves prove invincible, their nature at least becomes thoroughly understood, and means of evading them almost universally pointed out.

That the preceding observations are not founded on bare speculation, the whole history of Analytical Science will abundantly evince. It is our intention, in the following pages of this Preface, to give a general outline of that history up to the present time. From the space allotted to it, it is evident that little else than the most prominent points in so wide a field can be selected for observation. Faint, however, as it is, the subject cannot but communicate to it some portion of its interest; as well as the reflection, that, (with the exception of one branch of it *) the history of the more modern discoveries has hitherto unfortunately found little place in our language †.

Symbolic reasoning appears to have been ushered into the world under unfavourable auspices, and to have been regarded in its infancy with an eye of extreme jealousy. And, indeed, if we consider the rudeness of its first attempts, the poverty of its first resources, and the lavish want of œconomy in their employment, we shall find little reason to wonder, that for a long period, the new methods were looked upon as inelegant, although serviceable auxiliaries of the ancient processes, to be regularly discarded after serving their turn. To employ as many symbols of operation and as few of quantity as possible, is a precept which is now found invariably to

* The calculus of variations, the history of whose rise and progress has been ably combined with the exposition of its theory, in a late work, "On Isoperimetrical Problems."

† The admirable review of the Mecanique Celeste (Ed. Rev. N° 22.) will still be fresh in the minds of our readers. But it should be recollected, that the Author of that Essay confines his attention entirely to the subject of Analytical dynamics; referring to the discoveries in the integral calculus merely as connected with that subject, and that too very cursorily. *Our business is exclusively with the pure Analytics.*

ensure elegance and brevity. The very reverse of this principle forms the character of symbolic analysis, up to within fifty years of the present date.

The first and most natural object of research in the Algebraic calculus, was the resolution of equations, involving simply the powers of the unknown quantity. As far as the fourth degree no difficulty occurred, but beyond this, not a step has yet been made. Almost every Analyst of eminence has applied his ingenuity to the accomplishment of the *general* problem, but without success; and after more than two centuries, during which every other branch of Analytics has been advancing with unrivalled rapidity, no progress whatever has been made in this. This, it must be allowed, presents little prospect of success to future researches on the subject; yet ought not the difficulty to be considered insurmountable, until opinion has been confirmed by demonstration. Delambre notices a Memoir presented to the National Institute by M. Ruffini, in which he proposes a proof of the impossibility of the resolution of equations above the fourth degree. If this demonstration be correct, it will render an important service to Algebraists, by diverting them from a pursuit which must necessarily be unsuccessful. The work, however, if yet published, has not arrived in this country. Recent French publications are not easily procured, nor is it surprising that to obtain those of the German Analysts is almost impossible, when Delambre regrets their scarcity even in France.

Although to express in finite algebraic terms, the root of any proposed equation be impracticable, yet the inverse function of any expression, such as

$$a + b\,x + c\,x^2 + d\,x^3 + \&c.$$

may readily be exhibited in an infinite series. When the difficulty of solving equations above the fourth degree was perceived, it was natural to seek rapid and convenient approximations, and accordingly, three of our countrymen, Newton, Ralphson, and Halley, produced nearly at the same time, modes of approximation which have since received various improvements. All such researches, when symbolically conducted, and without regard to the numerical value of the symbols, lead at the bottom to series of greater or less complexity. It seems to have been in following up this idea, that Lagrange was first conducted to that very general resolution of all equations in the series which bears his name; a series which has been productive of discovery wherever it has been applied, and whose fecundity appears yet far from being exhausted. It is thus that the most distant parts of Analysis hang together, nor is it possible to assign the point, however remote or unexpected, in which any proposed career of research may not ultimately terminate. This series made its first appearance in the Mem. de l'Acad. Berlin, 1767-8 *. together with

* The demonstration there given is defective in rigour. A better was given in Note XI. of the "Traitè de la resolution des equations numeriques." Lagrange. But the most elegant is that of Laplace, to

the most systematic method of approximating to the roots of numerical equations which has yet been given. The method has this considerable advantage over all others, that, in all cases where the root is an integer, the formula of approximation will give it exactly, and in many where it is a surd, the continued fraction employed will point out the rational number of which it is the root. Though the complete solutions of equations is nearly hopeless, it might perhaps be of some advantage, and certainly of little difficulty, supposing the roots of equations known, to investigate what change would take place if one or more of the coefficients were augmented or diminished by any small quantity.

To trace the history of the differential calculus through the cloud of dispute and national acrimony, which has been thrown over its origin, would answer little purpose. It is a lamentable consideration, that that discovery which has most of any done honour to the genius of man, should nevertheless bring with it a train of reflections so little to the credit of his heart.

Discovered by Fermat, concinnated and rendered analytical by Newton, and enriched by Leibnitz with a powerful and comprehensive notation, it was presently seen that the new calculus might aspire to the loftiest ends. But, as if the soil of this country were unfavourable to its cultivation, it soon drooped and almost faded into neglect, and we have now to re-import the exotic, with nearly a century of foreign improvement, and to render it once more indigenous among us.

The most prominent feature of this calculus, is the theory of the developement of functions. The theorem which has immortalized the name of Brook Taylor, forms its foundation. Elicited by its Author from a formula which at first sight seemed independent of it, by a method not remarkable for its rigour, it seems to have been long considered in the light of a very general formula of interpolation. Lagrange and Arbogast have, as it were, invented it anew, and established it as the true basis of the differential calculus. The theory of Lagrange is to be found in a Memoir among those of the Acad. de Berlin. 1772, which contains the independent demonstration of Taylor's theorem—in the "Theorie des fonctions Analytiques," wherein he exhibits its application to the various branches of the differential calculus, independently on any consideration of limits, infinitesimals or velocities — and lastly, in the Journal de l' Ecole Polytechnique. Cah. XII. (1802.) and in the "Lecons sur le calcul des Fonctions." The ideas of Arbogast are contained in a Manuscript

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to be found in the Mem. de l' Acad. des Sciences, Paris, 1777; in Lacroix's Calc. Diff. et Int. 2d edit. Art. 107; in the Mecanique Celeste, tom. I. page 172. Adopted (in principle at least) by Lagrange in the "Theorie des Fonct. Analyt. Art. 97, et suiv. And in our own language, in Mr. Woodhouse's Trigonometry, 2nd edition. Arbogast has also demonstrated this theorem. See his Calcul des Derivations, Art. 262, et suiv.

presented to the Academy of Sciences in the year 1789, and of which the outline is given in the Preface to his celebrated work on Derivations. Such is the brief account of the greatest revolution which has yet taken place in Analytical Science.

The operations of the differential calculus once well understood, and rigorously demonstrated, may be employed in improving the theory which gives rise to them. The work of Arbogast just alluded to, has shewn to how vast an extent this application may be carried, and how great is the assistance thus rendered. The peculiar grace of Laplace's Analysis has no where been more beautifully exhibited, than in his improvement and extension of Lagrange's theorem already mentioned. Nor should the labours of Paoli in this field pass unnoticed. By the aid of a very remarkable series derived from reverting that of Taylor, he has been able to assign the developement of any function of a quantity given by any equation whatever, in terms of a function any how composed of the remaining symbols which enter into that equation.

The developement of functions has lately been made, under the name of "Calcul des fonctions generatrices," the foundation of a most elegant theory of finite differences, of which more hereafter.

Soon after the discovery of the integral calculus, on the discussion of some problems, between Leibnitz and the Bernouillis, respecting the variation of the parameters of curves; there occurred certain equations, which, though they satisfied the conditions, were yet not contained in the complete integral of the equation whence they were derived. It is somewhat remarkable, considering the manner in which they first appeared, that their geometrical signification should have remained so long undiscovered. Brook Taylor, according to Lagrange, was the first who arrived at a particular solution by differentiating. Clairaut, in a Memoir presented to the Academy of Sciences at Paris, first remarked, that the equations so found, satisfy the geometrical conditions proposed. Euler styled them Analytical paradoxes, and shewed how in some cases they might be derived from the differential equations. But their theory remained unknown, till the year 1772*; when Laplace explained it in a Memoir of the Academy of Sciences, and pointed out the methods of discovering all the particular solutions of which an equation admits. This subject was still farther pursued in the Berlin Memoirs, by Lagrange, who there developes, with great perspicuity the whole theory both Analytical and

* J. Trembley, in the 5th Vol. of the *Mem. de Turin*, has given a Paper on the derivation of the complete integral, having given a number of particular solutions. His method consists in multiplying the equation by these solutions, each raised to an indeterminate power.

Geometrical *. But the most complete exposition of the subject, which has yet appeared, is to be found in a paper read before the National Institute in the year 1806, by Poisson, in which the theory is extended to partial differential equations, and also to those of finite differences. He observes of certain partial differential equations, that they admit of particular solutions equally general with the complete integral. The Analytical theory, in its present state, is most elegant: still it requires some farther developements when applied to equations of partial differentials, and to those of differences, and might perhaps with advantage be applied to equations of mixed differences. Its Geometrical signification has been beautifully illustrated by Lagrange; but the meaning of particular solutions, when they occur in dynamical problems, which is a question of considerable importance, remains yet undecided. Poisson has shewn a case, in which the particular solution and the complete integral are both required, and has produced others, in which only one is necessary.

As the integration of expressions containing one variable is a matter of considerable importance, and the number of those which are capable of integration, is small, when compared with those which do not admit of it; some attention has been bestowed on the classification of those which are similar, and on the reduction of those which are absolutely different to the least number possible. When this is accomplished, all that remains for the perfection of this branch of Analysis, is to calculate tables which shall afford a value of the integral for any value of the variable. In general, all expressions which do not admit of complete integration, are denominated transcendants. Those which most frequently occur, are logarithmic and circular functions. Tables of these had been long calculated for trigonometrical purposes, and on the discovery of the integral calculus received a vast addition to their utility. It was next proposed to admit as known transcendants, all integrals which could be reduced to the rectification of the conic sections. But, besides the preposterous idea of limiting an Analytical expression by the properties of a curve, no tables had been constructed for them, and of course the determination of their arcs could only be performed by the actual calculation of the integral under consideration: nor, indeed, would it have been possible to form useful tables of any moderate length, without first discussing the properties of the transcendants themselves in the fullest manner.

The theory of the transcendant $\int \frac{P\,dx}{R}$, where P is a rational and integral function of x, and R a quadratic radical of the form $\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4)}$ has at length by the successive labours of Fagnani, Euler, Landen, and Legendre,

* Fontaine first considered a differential equation as the result of the elimination of a constant between an equation and its differential; thus laying the foundation of the theory of equations, both differential, and of differences, and also of their particular solutions.

been brought to great perfection. All the transcendents comprised in this extensive formula, are reducible to three species. Those comprised in the first, are susceptible of multiplication or division, in the same way as the arcs of circles, by algebraic operations only. The transcendants of the second species, are susceptible of a similar multiplication or division, not simply, but when increased or diminished by an algebraic quantity. This algebraic quantity passes in the third species, into a transcendant of the logarithmic or circular kind. Landen has shewn, that an integral of the first species here enumerated may be reduced to two of the second: so that the number of distinct transcendants comprised in the formula $\int \frac{P\,dx}{R}$ is no more than two. A vast variety of integral formulæ have, by dint of indefatigable research on all hands, been reduced to the evaluation of these functions; but to dwell longer on them, would lead us beyond our limits.

The only other species of transcendants of any considerable extent which have received much discussion, are those contained in the formulæ $\int \varepsilon^{-x^n}.dx$, and $\int \frac{{}^n\varepsilon^x.dx^n}{1+\varepsilon^x}$. Kramp, at the end of his Analyse des Refractions, has given a table of the values of the first of these, (in the case of $n=2$), the integral being taken between the limits 0·00,....3·00, ∞. The definite integrals dependent on the general form, we shall speak of hereafter. The second formula is (ultimately) that of the logarithmic transcendants, on the various orders of which Mr. Spence, in the year 1809 *, published an Essay, which displays considerable ingenuity, and a depth of reading rarely to be met with among the Mathematical writers of this country. A general property there given of the transcendant ${}^nL(x)$, leads to the summation of some very extraordinary series, which are now in our possession, and which we cannot forbear mentioning.

Their general (or i^{th}) terms are comprised in the formulæ

$$\frac{f\{x.\varepsilon^{i\theta\sqrt{(-1)}}\}+f\{x.\varepsilon^{-i\theta\sqrt{(-1)}}\}}{i^{2n}}, \text{ and } \frac{f\{x.\varepsilon^{i\theta\sqrt{(-1)}}\}-f\{x.\varepsilon^{-i\theta\sqrt{(-1)}}\}}{\sqrt{(-1)}.i^{2n+1}}$$

where f is the characteristic of any function whatever †, developeable in integer powers, either positive or negative, or both.

* Le Gendre published his Exercises de Calcul Integral in 1811. After having given Landen's and Euler's values of particular cases of the function ${}^nL(1+x)$ he adds, "Jusqu'a present, on n'est pas allé plus loin dans la theorie de ces sortes des transcendantes," page 249. It is probable therefore, that, owing to our interrupted intercourse with the continent, he had not seen Mr. Spence's work.

† One singular result of these researches is, the evaluation in terms of the transcendants ϵ, and π, of the function

$$(\tan\theta)^{(\frac{1}{1})^{2n+1}}.(\cot 3\theta)^{(\frac{1}{3})^{2n+1}}.(\tan 5\theta)^{(\frac{1}{5})^{2n+1}}.\&c. \text{ ad infinitum.}$$

Mr. Spence has also given tables (of some extent,) of the successive values of his functions, as we have before remarked of Kramp. Too much praise cannot be bestowed on such examples, which however there is little hope of seeing followed. The ingenious Analyst who has investigated the properties of some curious function, can feel little complaisance in calculating a table of its numerical values; nor is it for the interest of science, that he should *himself* be thus employed, though perfectly familiar with the method of operating on symbols; he may not perform extensive arithmetical operations with equal facility and accuracy; and even should this not be the case, his labours will at all events meet with little remuneration.

It sometimes happens, that the arbitrary constant does not continue the same throughout the whole extent of an integral. An instance of this, in the series

$$C - \frac{\theta}{2} = \frac{\sin\theta}{1} + \frac{\sin 2\theta}{2} + \&c.$$

was remarked by Daniel Bernouilli, in the Act. Petrop. Landen also notices, that this equation is false, when $\theta = 0$, but without explanation. Other instances occur in the "Essay on logarithmic transcendants," pp. 52-3; in Lacroix's Traite de Calcul &c. 4^to^. tom. III. p. 141, as well as some reflections on this difficult subject in page 483 of the same volume. The cause of these anomalies has not been satisfactorily explained. If we may hazard a conjecture, it must be looked for in the evanescent or infinite values of some of the differential coefficients of the function integrated, causing that function for an instant to change its form. Or, it may have some connection with exponentials, since all the instances which have hitherto been adduced depend on that species of function.

The discovery of partial differentials, has been generally attributed to D' Alembert. He certainly was the first who applied them to mechanical problems, and perceived their vast utility in all the more difficult applications of Analysis to physics. But, if he is to be considered as the inventor, who first solved an equation of the kind, and who, when their importance was acknowledged, contributed more than any other to the improvement and progress of this calculus; the glory of their discovery will undoubtedly belong to Euler. In the 7th vol. of the Acta Acad. Petropolitanæ, is a Memoir of his, entitled, "Methodus inveniendi æquationes pro infinitis curvis ejusdem generis." (A. D. 1735.) In this paper, and more particularly in a supplement, are given the solutions of a number of partial differential equations, of which the most general is

$$q = Xz + pR$$

X being a function of x, and R a function of x and y.

In the latter part of the "additamentum ad dissertationem," he proceeds to integrate some equations of the second order. The "Reflexions sur la cause generale des Vents," which contains d' Alembert's first application of partial differential equations, was not published till the year 1747, and it gained the prize of the Academy of Berlin in 1746. It would lead us too far to trace, successively, the various improvements which the new theory underwent in the hands of Euler, Lagrange, Laplace, Monge, Parseval, and a multitude of great men, whom the vast importance of the subject incited to its prosecution. Notwithstanding every exertion, the theory however continues to present a multitude of difficulties. The analogy which was supposed to exist between the arbitrary functions, which enter into the integrals of partial differential equations, and the arbitrary constants in equations of total differentials, is found not to hold beyond the first degree; after which, even the number of these arbitrary functions is unknown. Instances have been adduced, where, besides the arbitrary functions, arbitrary constants also must be introduced to complete the integral*. The application of definite integrals to the integration of these equations, presents a wide field of research, as well as the promise of great discoveries. A very curious Memoir of Laplace is to be found in the Mem. Acad. des Sciences 1779, where this subject, among many others, is discussed with considerable success †.

In applying the test of integrability to differential equations, some were found, which could not be made to satisfy the equations of condition. These were for a long time deemed absurd, until about the year 1784, when Monge perceived their connection with the theory of curve surfaces, and demonstrated that these equations admit of solutions, corresponding to the curves of double curvature, formed by the successive intersections of a curve surface, whose parameter varies; and discovered methods of transforming any equation of this kind into an equation of partial differentials, and also of solving the converse problem. From this, he proposed obtaining solutions of equations of partial differentials, which are not integrable by other methods. But the example he gives, shows, that to expect success in such an enquiry, we must be familiar with space, considered as of three dimensions, and also with a numerous collection of curves and curve surfaces situated in it; such considerations would but add intricacy to a subject already difficult: hints for the advancement of Analysis may be derived from foreign sources, but must always be improved and cultivated by its own powers. La Croix has given an excellent Analytical theory of this kind of equations.

To the practice which prevailed in the infancy of Analytics, of proposing to

* Monge; Savans Etrangers, vol. VII. p. 322.

† The great desideratum in the integration of an equation by definite integrals is, that whenever it is susceptible of actual resolution, these integrals should give it—a condition which does not always hold good. See Laplace. Mémoire sur divers points d' Analyse. Journ. de l' Ecole Polytechnique, N°. 15.

Geometers the solution of difficult problems, we owe many of its improvements. The method of variations, among others, is much indebted to this source. Obliged however to consult brevity, we must refer to the late treatise on "Isoperimetrical Problems," for a full account of its history: a work which being undoubtedly in the hands of the majority of our readers, must render superfluous all we could say on that subject. The chief difficulty now consists in distinguishing the maxima from the minima, which depends in general on the solution of difficult differential equations. The inverse of the method of variations does not appear to have received any attention; it would consist in solving such problems as the following: "Given a curve, to find what properties of maxima and minima it possesses."

The method of variations was applied by its inventor to differential equations, and also to those of finite differences. Cases may occur, in which it would be necessary to apply it to equations of mixed differences; these relate to a number of difficult problems, such, for instance, as this: "What must be the nature of a curve, such, that drawing to any point, an ordinate and also a normal, and at the foot of this normal another ordinate; the curvilinear area intercepted between the first and last ordinate, may be a maximum or a minimum."

There are but few instances in the history of Science, in which the path of the inventor has been the shortest and most direct. Thus it occurs, that the method of finite differences, which would most naturally have preceded that of the differential calculus, was not discovered until many years after. Its inventor, Brook Taylor, published it in a work, entitled, "Methodus Incrementorum directa et inversa," a book noted for its obscurity. Montmort, Stirling, and Emerson, made several improvements, which may be found in the Philosophical Transactions, and also in their respective works. Moivre first investigated the nature of recurring series, on which, Laplace remarks, "Sa theorie est une des choses les plus curieuses, et les plus utiles que l'on ait trouvees sur les suites." It has been well observed, by the same author, that the first who summed a Geometrical or Arithmetical series, had really integrated an equation of finite differences. The same remark, as is well known, applies to any recurrent series. It was not, however, till Lagrange in the Melanges de Turin, vol. I. applied Alembert's method of indeterminate coefficients, to an equation of differences of the first degree, that this truth was perceived. In the fourth volume of the same work, Laplace published a Memoir, in which the two celebrated theorems of Lagrange, respecting equations of common differentials, are extended to those of differences, and to those of partial differentials of a similar description, with constant coefficients. Returning to the subject, in a Memoir communicated to the Academy of Paris, he integrates a very extensive class of equations of partial differences, involving any number of variable indices,—and also a singular species of equations, frequent in the theory of chances, called by him, *equations rentrantes*.

Of equations beyond the first degree, very few have been solved, if we consider their amazing variety and importance. Monge has given a short paper on the subject, in the Memoirs of the Academy of Sciences, for 1783. Laplace also in the 15th number of the Journal de l' Ecole Polytechnique, has by a most happy combination of the equations of differences, with the discovery of Euler, respecting elliptic transcendants, integrated a few very difficult ones. The nature of the integrals obtained by Charles, requires a fuller investigation. They might perhaps receive considerable extension. When the variables are mixed in the indices, thus, u_{x+y}, u_{x-y+1}, u_{xy}, &c.; the subject seems to have passed altogether unnoticed. Many equations containing such expressions, are impossible or contradictory.

Euler first remarked, that the constant introduced by integrating an equation between u_x, u_{x+1}, &c. may be an arbitrary function of $\cos 2\pi x$, a remark which afterwards in the hands of Laplace, (Savans Etrangers, 1773,) became the foundation of a very general theory of determining functions from given conditions. To notice all the applications of the theory of finite differences, or all the profound researches which have enriched it, would occupy volumes. We cannot, however, pass over the theorems relating to the analogy of differences to powers, given first by Lagrange without demonstration, in Mem. de Berlin. 1772. A demonstration by Laplace, appeared in the Mem. des Savans Etrangers for the following year, and another in the Mem. de l' Acad. for 1777. In 1779, appeared that noted Memoir, in which the same author exhibited the principles of his powerful and elegant "Calcul des fonctions generatrices." In this *, he gives a far more systematic proof of the theorems, and extends them to any number of variables. Since that period, they have been demonstrated by Arbogast, in the 6th article of his, "Calcul des derivations," where, by a peculiarly elegant mode of separating the symbols of operation from those of quantity, and operating *upon them* as upon analytical symbols; he derives not only these, but many other much more general theorems with unparalleled conciseness. Brinkley has given a demonstration of the theorem

$$\Delta^n u_x = \left\{ \varepsilon^{\frac{\Delta x}{dx} \cdot d \cdot} - 1 \right\}^n \times u_x$$

of considerable elegance, and a simplicity truly elementary †.

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* This Memoir forms the greater part of the first Chapter of the "Theorie Analytique des Probabilites." Its first principles, and the demonstration of the theorems, on the analogy of differences with powers, are given briefly in the 15th N°. of the "Journal de l' Ecole Polytechnique." A slight sketch of the method alluded to, is also to be found in the 9th book of the Mécan. Cél. tom. IV. p. 204.

† Philos. Transactions, 1807. Part I. He has extended his researches to the actual expansion of the series themselves, to which these theorems lead; such, for instance, as

$$\Sigma u_x = \frac{1}{\Delta x} \cdot \int u_x \, dx - \frac{u_x}{2} + \Delta x \cdot {}^1A_x + (\Delta x)^2 \cdot {}^2A_x + \&c.$$

But in point of clearness and elegance, by no means with equal success: partly owing to an unfortunate notation, and partly to the perpetual employment of the theory of combinations. In his results, he has for the most part been anticipated by Laplace, and others.

We are now naturally led to say a few words on the "Calcul des fonctions generatrices." Its object may be best stated, in the words of its inventor *: "C'est de ramener au simple developpement des fonctions, toutes les operations relatives aux differences et specialement l' integration des equations aux differences ordinaires et partielles," and from the extreme facility with which all the known theorems flow from it, and its fecundity in affording new ones, it is, perhaps, in the present state of science, the best adapted of any, for explaining the general theory of differences, and the developement and transformation of series. At the same time, it must be confessed, that owing to its extreme generality, and the consequent complexity of many of its operations, particularly in what regards the transformation of series, it is an instrument to be placed only in the hands of an experienced Analyst. If we except the calculus of variations, it is the only method, perhaps, of any considerable importance, which has received its first and last touches from the same hand; and which first appeared in a state of perfection, very little short of what it at present possesses. The latest work which treats of this subject, is the "Theorie Analytique des Probabilités;" the first part of which is dedicated to a very full exposition of the method.

After the solution of differential equations, and those of finite differences, it was natural to consider those, in which the difference and differential of a quantity both occurred. These have been called equations of mixed differences. They were not attempted until about the year 1779, when Condorcet and Laplace obtained the integrals of some few particular cases. But problems which required their application had been proposed and resolved by several Geometers in the "Acta eruditorum," and in the "Acta Acad. Petrop." long before this time; their solutions were obtained by certain insulated artifices, dependent on the peculiar nature of the problems.

On this subject, a wide field is extended for investigation, and one which abounds with difficulties. The little that has yet been discovered, is chiefly contained in two papers, one by Biot, in the *Memoires de l' Institut*, and the other by Poisson, in the *Journal de l' Ecole Polytechnique;* the former treats chiefly of that kind of equations, called *equations successives;* the latter integrates a few particular equations, by employing a substitution used by Laplace, in integrating some equations of partial differentials.

There is no branch of mathematical science which has not received improvements, from the profound and original genius of Euler. Several are indebted to him for their existence; of this latter class is the knowledge of the nature, and use of definite integrals, a subject, to which the greatest Geometers of the present age look as the most probable source of future discoveries and improvements.

* Journal de l' Ecole Polyt. Mém. sur divers points d' Analyse.

Legendre, in a work lately published, entitled " Exercises de Calcul Integral," has collected all that has been discovered on this subject, and demonstrated the results with peculiar elegance; the greater part is extracted from the various works of Euler; and also a considerable portion from some Memoirs of Laplace. Its application to the solution of differential equations, from which so much is expected, does not enter into the plan of his work; this however has been well treated, by Lacroix, in the third volume of his *Traite de Calcul, &c.*

But the most elegant part of the theory of definite integrals is, their application to such problems in finite differences, as involve functions of very high numbers. In many cases (particularly in the theory of chances,) it has been well remarked, that the mere impracticability of the arithmetical operations requisite to obtain a result, (however simple its analytical expression,) must for ever preclude our advancement; were it not for some mode of approximation, which, grasping the prominent terms of an expression, in a formula easily reduced to numbers, should throw the minor ones into the back-ground, to be valued by a series converging the more rapidly the higher the numbers employed become. A more appropriate instance cannot be adduced, than the equation

$$1 . 2 . 3 \ldots\ldots s = s^{s+\frac{1}{2}} . \varepsilon^{-s} . \sqrt{(2\pi)} \left\{ 1 + \frac{1}{12 s} + \frac{1}{288 s^2} + \&c. \right\}$$

in page 129 of the " Theorie Anal. des Probabilites," or the method in which the author, in page 259 of the same work, computing the probability of a primitive cause influencing the inclinations of the cometary orbits, employs a definite integral to effect with conciseness, a calculation surpassing, without that assistance, the utmost limits of human patience and industry.

In this part of his career, Laplace stands unrivalled. Stirling indeed, and Moivre had seen, and in some cases obviated the difficulties, arising from the immensity of the numbers under consideration. The discoveries of Euler, gave a connection and unity to their results. But it was not till the labours of Lagrange, Condorcet, and Laplace had brought the theory of finite differences to considerable perfection, that the definite integrals were applied by the latter, to the solution of these equations, and a clear and strong light thrown over this most obscure part of the Mathematics. The whole of this interesting theory, has been digested into one work, (" Theor. de Prob." above cited) which for comprehensive views, for depth of investigation, and the purity of its analysis, may justly be looked up to, as marking the highest point to which the science of abstract number has yet attained.

In analytical investigations, we frequently meet with a series of quantities connected together by multiplication, whose differences are constant. These have received various names from different Geometers; Vandermonde called them powers of the second order; Kramp adopted the appellation of Facultes numeriques; and Arbogast named them Factorials. Each of these writers treats them in a different manner, and employs a peculiar notation for them: that of Vandermonde is perhaps the best adapted to the subject, as it has a considerable resemblance to that of exponents, and possesses also an advantage, which is by no means inconsiderable, that of being capable of a ready extension to powers of all orders. Lacroix has adopted it, and by its help demonstrated many properties of powers of the second order. Those of the superior orders have not as yet been examined. Kramp has deduced from his "Theorie des facultes numeriques," some contradictions which require examination. One of his most useful theorems affords a method of transforming any power of the second order, into a series which converges *ad libitum.* It appears probable, that the theory of powers of different orders may afford a useful method of classing transcendents, as they can frequently be reduced to definite integrals, and by this means their value be obtained, when their index is fractional.

Interpolations were at first considered, as a branch of the method of finite differences, and as such they were usually treated of together. Wallis appears first to have applied this name, to the determination of the intermediate term of a series, whose law of formation is known. The extraction of roots is an interpolation of powers, and may be considered as an extension of the meaning of exponents, from whole numbers to fractions. Perhaps it might not be unworthy of consideration, whether the meaning of the indices of differentiation, could not be considerably extended. Euler seems to have had the first idea* of interpolating the series

$$dy, \quad d^2 y, \quad d^3 y, \quad \&c.$$

Laplace has extended his researches on this subject to considerable length, and has given the value of such expressions as the following, by converging series and definite integrals

$$\frac{d^n . x^m}{d x^n}, \qquad \Delta^n . x^m$$

so as to allow of evaluation for fractional values of n. But the indices themselves might be supposed to vary *continuously*, and such expressions as these

$$\left\{\frac{d^n . \left(\frac{d^p y}{d x^p}\right)}{d p^n}\right\}, \qquad \left(\frac{d \Delta^n y_x}{d n}\right)$$

become the subject of Analytical investigation. Or the index of a function might vary, as in the following instance:

* Leibnitz, it is true, in a letter to J. Bernouilli, mentions fractional indices of differentiation. Euler, however, first determined the value of such an expression as $d^{\frac{1}{2}} y$, y being a certain function of x.

Let $f(x) = \sqrt{\left(\frac{2x}{1+x}\right)}$, $f^2(x) = \sqrt{\left(\frac{2f(x)}{1+f(x)}\right)}$, and so on,

then shall we have,

$$\frac{d f^n(\sec v)}{dn} = -\frac{v \cdot \log 2}{2^n} \cdot \tan\left(\frac{v}{2^n}\right) \cdot \sec\left(\frac{v}{2^n}\right)$$

and again, (making use of Arbogast's notation,) if $y = a^x$

$$\overset{p_n}{\underset{p_{n-1}}{\mathrm{D}}} \cdot \overset{p_{n-1}}{\underset{p_{n-2}}{\mathrm{D}}} \cdot \ldots \overset{p_2}{\underset{p_1}{\mathrm{D}}} \cdot \overset{p_1}{\mathrm{D}_x} \cdot y = a^x \cdot (\log a)^{p_1} \cdot (\log^2 a)^{p_2} \cdot \ldots (\log^n a)^{p_n}$$

where $\log^2 a = \log\log a$, $\log^3 a = \log\log\log a$, and so on.

It has been observed by Charles, that the equation

$$\Delta y_n = b \cdot \frac{d y_n}{dx}$$

may be transformed into an integral, in which the index of integration is variable. Its solution then is

$$y_{-n} = b^{-n} \varepsilon^{-\frac{x}{b}} \cdot \int^n \varepsilon^{\frac{x}{b}} \cdot y_0 \cdot dx^n$$

or, which comes to the same,

$$y_n = b^n \varepsilon^{-\frac{x}{b}} \cdot \mathrm{D}^n_x \left\{ y_0 \cdot \varepsilon^{\frac{x}{b}} \right\}$$

y_0 being any function of x. Laplace, in the "Theorie Anal. des Prob." gives other instances of the same kind *.

That such expressions are not merely analytical curiosities, but relate to the most difficult and important theories, is confirmed by the opinions of the most

* The integral of the equation of mixed partial differences

$$u_x = \mathrm{A} \cdot \left(\frac{d^{\alpha} u_{x-1}}{da^{\alpha}}\right) + \mathrm{B} \cdot \left(\frac{d^{\beta} u_{x-2}}{db^{\beta}}\right) + \mathrm{C} \cdot \left(\frac{d^{\gamma} u_{x-3}}{dc^{\gamma}}\right) + \&c.$$

may be easily shown to be

$$u_x = \mathrm{A}^{\frac{x}{1}} \cdot \left(\frac{d^{\frac{\alpha x}{1}} \cdot \psi_1(a)}{da^{\frac{\alpha x}{1}}}\right) + \mathrm{B}^{\frac{x}{2}} \cdot \left(\frac{d^{\frac{\beta x}{2}} \psi_2(b)}{db^{\frac{\beta x}{2}}}\right) + \mathrm{C}^{\frac{x}{3}} \cdot \left(\frac{d^{\frac{\gamma x}{3}} \psi_3(c)}{dc^{\frac{\gamma x}{3}}}\right) + \&c.$$

ψ_1, ψ_2, &c. being the characteristics of arbitrary functions: an expression which except α, β, γ,.... are respectively multiples of 1, 2, 3,.... must necessarily involve fractional indices of differentiation for some values of x.

eminent Analysts; and though on mathematical subjects when proof can be produced, no weight must be allowed to authority; yet when the former is deficient, our judgement may surely be influenced by the latter.

The importance of adopting a clear and comprehensive notation did not, in the early period of analytical science, meet with sufficient attention; nor were the advantages resulting from it, duly appreciated. In proportion as science advanced, and calculations became more complex, the evil corrected itself, and each improvement in one, produced a corresponding change in the other. Perhaps no single instance of the improvement or extension of notation, better illustrates this opinion, than the happy idea of defining the result of every operation, that can be performed on quantity, by the general term of function, and expressing this generalization by a characteristic letter. It had the effect of introducing into investigations, two qualities once deemed incompatible, generality and simplicity. It now points out a calculus* perhaps more general than any hitherto discovered, and which should be called the calculus of functions, a name that more naturally belongs to it, than to that which Lagrange has so classically treated in the work which bears this name, although this latter is a branch of it.

Its object would be in general, the determination of functions from given conditions of whatever nature, whether depending on the successive terms of their developements, or on a series of indices differing by unity; or lastly, on a species of equations depending on the successive *orders* of the same function, of which the first mention we believe is made in one of the papers which compose the present volume. The necessity of this calculus was perceived soon after the discovery of equations of partial differentials, when it became requisite to determine the arbitrary functions which enter into their integrals, so as to satisfy given conditions. Euler and Alembert determined a few particular cases. Lagrange, in a Memoir entitled, "Solution de differens Problemes de calcul integral," resolved the equation

$$T = \alpha . \phi \{t + a . (h + kt)\} + \beta . \phi \{t + b (h + kt)\} + \&c.$$

Monge also has given several papers upon the subject, in the Memoirs of the Societies of Turin and Paris. The method of Laplace for reducing an equation of the first order, where the difference of the independent variable is any function of the variable itself, to one wherein that difference is constant, is well known. It had not, however, hitherto been shewn to be possible to reduce *every* equation of the former kind, to one of the latter. This object is, however, accomplished in the following pages. Still, it appears by no means natural, to resolve these equations, by

* Quamobrem non solum in hoc negotio, sed in plurimis aliis casibus, maximé utile foret, si functionum doctrina magis perficeretur et excoleretur. Euler, Act. Acad. Petropol. tom. VII.

means of finite differences, and it were much to be wished, that some independent method could be discovered, by which they might be treated.

One of the most striking advantages of the theory *of functions*, is, that it seems equally adapted to the proof of the most elementary truths, and to that of the most complicated and abstruse theorems. The latter part of this assertion, no one will be inclined to deny. An example of the former may be found in Laplace's proof of the decomposition of forces in the " Mecanique Celeste" *.

There are still many problems in the theory of functions, which analysis seems

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* This demonstration consists of two parts. In the first he proves, that the diagonal of a rectangle whose sides represent the separate forces, will on the same scale represent the *quantity* of the resultant. The second is devoted to shew, that it represents also its direction. This part seems to be generally considered, as deficient in clearness and simplicity. What we have here to remark, is, that it is *redundant*. In fact, by the combination of Laplace's three equations,

$$x = z \,.\, \phi(\theta); \qquad y = z \,.\, \phi\left(\frac{\pi}{2} - \theta\right), \qquad x^2 + y^2 = z^2$$

we obtain

$$\left\{\phi(\theta)\right\}^2 + \left\{\phi\left(\frac{\pi}{2} - \theta\right)\right\}^2 = 1$$

an equation which suffices for determining the nature of the function ϕ, and from which, by known processes, combined with the conditions of the question; it is easy to obtain $\phi(\theta) = \cos\theta$, and $x = z \,.\, \cos\theta$, which is the equation to be deduced.

Another very remarkable instance of the use of the theory of functions, in demonstrating elementary truths, may be found in the following demonstration of Euclid's 47th, which has generally been thought to admit of none, but a geometrical proof: Call a, b, c, the sides, A, B, C, the opposite angles of a right-angled triangle, $C = \frac{\pi}{2}$. It is easy to see, that the following equations hold good:

$$b = c \,.\, \phi(A); \qquad a = c \,.\, \phi(B); \ldots\ldots\ldots\ldots(1)$$

drop a perpendicular p, dividing c into two parts x, y, and we shall have, in the same manner,

$$x = b \,.\, \phi(A); \qquad y = a \,.\, \phi(B)$$

and of course, $$x + y = c = b \,.\, \phi(A) + a \,.\, \phi(B)$$

from which, eliminating $\phi(A)$, and $\phi(B)$, by equations (1)

$$c^2 = a^2 + b^2, \qquad \text{Q E D.}$$

Having proved from other principles, that $A + B + C = \pi$, we shall have the very same equation

$$1 = \left\{\phi(A)\right\} + \left\{\phi\left(\frac{\pi}{2} - A\right)\right\}^2$$

and thus we obtain $b = c \,.\, \cos A$, $a = c \,.\, \cos B$. It is only by this way of proceeding, or some analogous one, that we can ever hope to see the elementary principles of Trigonometry, brought under the dominion of Analysis. But this is not the place to proceed farther with the subject. It may suffice to have thrown out a hint, which may be followed up at some future opportunity.

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to afford no means of attacking directly. Among which, may be enumerated the greater part of those which lead to an equation, containing a definite integral, where the unknown function enters under the integral sign. For instance, suppose it were required to find the form of a function $\phi(x)$, such that the integral

$$\int dx . F\{a, x, \phi(f(x))\}$$

taken between the limits $x = 0$, $x = a$, should equal any assigned function of a. F and f being given characteristics.

The examination of the properties and relations of numbers, constitutes a distinct branch of mathematical enquiry, almost entirely of modern origin; so abstract, and apparently so far removed from the confines of utility, that it seems to have attracted little attention from the generality of those, who have dedicated themselves to the pursuits of science. To the few, however, who have thought it worth their while to explore its more profound recesses, it has proved a mine, fertile in the most brilliant produce. Euclid, in his 7th book, has given the elements of transcendental Arithmetic, (a name appropriated to it by Professor Gauss). In the work of Diophantus, notwithstanding the ingenuity of the author, we discover the infancy of science in the absence of that generalization so happily adopted by modern writers. It consists of a variety of insulated problems, relating to the solution of certain indeterminate equations, rather than to the properties of numbers. Indeed, the indeterminate analysis, and the theory of numbers, form two branches of enquiry, which, (however nearly connected), ought to be carefully distinguished, in any systematic arrangement of our knowledge. The former must be considered as a province of the pure Analytics. Its attainment is indeed necessary for the perfection of the latter, (which should rather be regarded in the light of an *application* of analysis,) but is by no means limited to this one object. Lacroix * has introduced it with a very elegant effect, in that part of the theory of curves, which relates to their construction by points.

The resolution of the indeterminate equation of the first degree, is said to be due to Bachet. Euler, in the Petersburg Commentaries, exhibited a method of obtaining any number of solutions of that of the second, provided one particular one be known; but it has been remarked, that his methods do not afford *all* the possible solutions. This, however, has been effected at length by Lagrange, in the Mem. Acad. de Berlin (1767 and 1768.) His method consists in reducing successively by a series of operations, the coefficients of the equation

$$a x^2 + b y^2 = z^2$$

(to which form, every equation of the second degree may be reduced,) till one of

* Traité de Calc. &c. 2d edit. vol. I. Note to page 417.

them becomes unity; in which case, the resolution is easy. Gauss also, in his "Disquisitiones Arithmeticæ," N° 216. has, by a method entirely different, shewn how to obtain all the solutions, of which an equation of the second degree admits. It is extremely remarkable, that the Hindu Algebraist, Bhaskara Acharya, who flourished about the year 1188, had also succeeded so far in his attempts on this difficult problem, as to derive any number of solutions from one, previously known, in the case of $a x^2 + b = y^2$.

The theorems given by Fermat without demonstration, form, without doubt, the most remarkable era in the theory of numbers. Too general, and too extraordinary in their nature to escape notice, they seem to have been the principal cause of the advances, which have since been made in this theory, by drawing the attention of Mathematicians to their demonstration. Euler, than whom none ever entered with greater ardency into this career, has proved some of the principal. Lagrange has supplied the demonstrations to others: still, however, many remain, of which no proof has been offered. It has been suggested, that Fermat was indebted to the method of induction, for the discovery of many of his theorems; an opinion rendered probable by the observation of Euler, that one of them relating to prime numbers is not true. This method is perhaps more applicable to researches in the theory of numbers, than to any other branch of abstract investigation; but it is of dangerous use, and should be supported by a large number of instances *.

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* The substitution of 0, 1, 2, ... for x, in the expression $x^2 + x + 41$, gives a series of numbers, of which the 40 first terms are primes, as Euler has remarked: yet it is easy to shew, that no algebraic function of x can in all cases represent a prime. The reason of this singular circumstance, and of a variety of similar coincidences, has since been satisfactorily explained, and the property demonstrated *a priori*. Fermat, deceived by a similar induction, asserted that all the numbers contained in the formula $2^{2^n} + 1$, are primes which Eulér has since (as above alluded to) shewn to fail, in the case of $n = 5$.

The following theorems are derived solely by induction:

1mo. An indefinite number of integer values of x may be found, which render $\frac{7^x - 1}{10^a}$ an integer, to which we may add, that the formula $\frac{7^{8.10^x} - 1}{10^{x+2}}$ is always an integer, as are also the formulæ $\frac{3^{2^x} - 1}{2^x}$, $\frac{5^{2^x} - 1}{2^x}$ and, in general, $\frac{(2n-1)^{2^x} - 1}{2^x}$, which last may be easily shewn, *a priori*, as well as a variety of expressions of the same description.

2do. The expression $\frac{3^{10^x} - 1}{10^{x+1}}$ is an integer, and, it is somewhat remarkable, that this integer is always of the form $100n + 22$.

3°. If A be such, that 5^n is congruous to A, (modul. 10^k) then will also $5^{n+2^{k-2}.i}$ be congruous to the same A, to the same modulus, and consequently, $5^{n+2^{k-2}.i} \equiv 5^n$. (modul. 10^k) i being any integer. In other words, the formula

$$\frac{5^{n+2^x.i} - 5^n}{10^{x+2}}$$

is always an integer, x and i being integers.

Among the later discoveries in the theory of numbers, we have to enumerate two of the most surprising, which perhaps are to be found in the whole circle of Analytics. The first is a formula of Lagrange, obtained by induction, for determining the number of primes contained between given limits. It has been demonstrated by Legendre, in his "Essai sur la Theorie des Nombres," although not very rigorously, it must be confessed; such is the difficulty of the subject. Indeed, the author has given it only as an attempt. The other is that celebrated theorem of Gauss, given in his "Disquisitiones Arithmeticæ," on the resolution of the equation $x^n - 1 = 0$, n being any prime, viz. that this equation may be reduced to α equations of the degree a, β of the degree b, and so on, where $n-1=a^\alpha . b^\beta \ldots$ Of course, the resolution of the equation $x^n-1=0$, where n is a prime of the form 2^m+1, requires only the application of quadratics. Thus the division of the circle into 17, 257, 65537 parts, may be accomplished by the description only of circles and straight lines.

To enter into any account of the advances made in the mixed Analytics, would far exceed our limits. There is one point, however, which we cannot forbear cursorily touching upon, on account of the great difficulty of reducing its conditions into symbolic language. We allude to the geometry of situation. Like the theory of numbers, at the first glance it seems barren and useless, but on a nearer examination is found abounding with interesting relations. Like that theory too, its cultivators have hitherto been few, but eminent, distinguished for that restless spirit of enquiry, which is ever upon the wing in search of new truths, and that invention which knows how to extract them, from the most unpromising hints. Leibnitz, appears to have found its first application, in considering the game of solitaire. A similar case (the problem of the knight's move at chess,) occupied the attention of Euler, and afterwards of Vandermonde, who adapted to it a notation analogous to that, by which the position of a point in space is determined, by three rectangular co-ordinates. In a more advanced state, it might, perhaps, embrace problems of a much higher order of difficulty, such as the following: "Given n points in space; to find the course to be pursued, so that setting off from any one, and passing at least once through all the rest, on returning to the original position, the least possible space shall have been described." Such is the brief account of a theory yet in its first infancy. On its basis some future LEIBNITZ may perhaps hereafter lay the foundation of a name great as that of its original inventor *.

* In the "Journal de l' Ecole Polytechnique, An. 10." is a Memoir on polygons and polyhedrons, by Poinsot, in which he shews, that there exist other regular polygons besides the equilateral triangle, the sum of whose angles is equal to two right angles, and also, that there are more than five regular polyhedrons. The whole Memoir relates to geometry of situation, and forms the introduction to some more considerable researches, which the author promises in a future paper.

The preceding pages have been devoted to a slight account of the history and present state of Analytical Science, that branch of human knowledge, of which Laplace has justly observed " C'est le guide le plus sur qui peut nous conduire dans la recherche de la verite." But some account will naturally be expected of the source itself, from which the present work emanates. Of this however, very little need be said, but, that it consists of a few individuals, perhaps too sanguine in their hopes of promoting their favourite science, and of adding at least some trifling aid to that spirit of enquiry, which seems lately to have been awakened in the minds of our countrymen, and which will no longer suffer them to receive discoveries in science at second hand, or to be thrown behind in that career, whose first impulse they so eminently partook. The time perhaps is not far distant, when such an attempt will be regarded in an honourable light, whatever may be its success.

Meanwhile the view we have taken of the subject, appears by no means to lead to the mortifying conclusion, deduced by a foreign Geometer of considerable eminence ; " que la puissance de notre analyse est a-peupres epuisee." The golden age of mathematical literature is undoubtedly past. Another, " less fine in carat," may however yet succeed. The motive which could draw forth so severe a sentence on the success of future exertions we will forbear to enquire, but it must surely be looked for elsewhere, than in the real interest of science which can never be promoted by repressing the ardour of research, or extinguishing the hope of reward. The foundations of a vast edifice have been laid; some of its apartments have been finished ; others yet remain incomplete : but the strength and solidity of the basis will justify the expectation of large additions to the superstructure.

Attentively to observe the operations of the mind in the discovery of new truths, and to retain at the same time those fleeting links, which furnish a momentary connection with distant ideas, the knowledge of whose existence we derive from reason rather than perception, are objects in whose pursuit nothing but the most patient assiduity can expect success. Powerful indeed, must be that mind, which can simultaneously carry on two processes, each of which requires the most concentrated attention. Yet these obstacles must be surmounted, before we can hope for the discovery of a philosophical theory of invention; a science which Lord Bacon reported to be wholly deficient two centuries ago, and which has made since that time but slight advances. Probably, the era which shall produce this discovery is yet far distant. The capital of science, however, from its very nature, must continue to increase by gradual yet permanent additions; at the same time that all such additions to the common stock yield an interest in the power they afford of multiplying our combinations, and examining old difficulties in new points of view. It is this connection with fresher sources, which can restore fertility to subjects appa-

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rently the most exhausted, and which cannot be too earnestly recommended to those who wish to enlarge the limits of analysis. The fire of improvement, however dormant, and seemingly extinct, may yet break forth at the contact of some external flame. The history of Mathematics affords too many instances of the most distant principles coming into play on the most unexpected occasions, to allow of our ever despairing of success in such enquiries.

One inconvenience however, results as a necessary consequence from the continued accumulation of indestructible knowledge. The beaten field of analysis, limited as it is when compared with the almost boundless extent which remains to be explored, is yet so considerable with respect to the powers of human reason, and (if we may be allowed to pursue the metaphor a little farther,) so intersected with the tracks of those who have traversed it in every direction, as to become bewildering and oppressive to the last degree. The labour of one life would be more than occupied in perusing those works on the subject which the labour of so many has been spent in composing. The multitude of different methods and artifices, which for the most part lead only to the same results, and whose power is limited by the same points of difficulty, is at length grown into a very serious evil. Our continental neighbours seem sensible of this, if we may judge from the number of works which have appeared within these few years, digesting various points into a systematic form. But there is still much to be done in this line. That man would render a most invaluable service to science, who would undertake the labour of reducing into a reasonable compass the whole essential part of analysis, with its applications, curtailing its superfluous luxuriance, rejecting its artificial difficulties, and giving connection and unity to its scattered members.