ON

CONTINUED PRODUCTS.

ASSUME the equation,

$$\psi x \,.\, \phi x = \chi x \,\ldots\ldots\ldots\ldots \,(a)$$

in which the functions denoted by ψ, ϕ, χ, are only subject to one condition, namely, that the product of the two first is equal to the last. In other respects they are perfectly arbitrary.

Take any other arbitrary function fx. It is evident that the generality of equation (a) is neither increased nor diminished by putting $\phi f x$ for χx.

Let this be done, then,

$$\psi x \,.\, \phi x = \phi f x \,\ldots\ldots\ldots\ldots \,(b)$$

In this equation put for x successively fx, ffx, $fffx$, and $f^n x$; then multiply the resulting equations together,

$$\begin{aligned} \psi f x \,.\, \phi f x &= \phi f^2 x \\ \psi f^2 x \,.\, \phi f^2 x &= \phi f^3 x \\ \psi f^3 x \,.\, \phi f^3 x &= \phi f^4 x \\ \&c. \qquad & \quad \&c. \\ \psi f^n x \,.\, \phi f^n x &= \phi f^{n+1} x \end{aligned}$$

therefore,

$$\frac{\phi f^{n+1} x}{\phi f x} = \psi f x \,.\, \psi f^2 x \,.\, \psi f^3 x \ldots\ldots \psi f^n x \,\ldots\ldots \,(c)$$

In (b) we may without limiting the equation put $(\psi x)^{p-1}$ for ψx, it then becomes

$$(\psi x)^{p-1} \phi x = \phi f x \,\ldots\ldots\ldots\ldots\ldots\ldots \,(d)$$

Dividing by $(\psi x)^p$, we have

$$\frac{\phi x}{\psi x} = \frac{\phi f x}{(\psi x)^p} \ldots\ldots\ldots\ldots (e)$$

Let us assume

$$y = \left\{\frac{\phi f x}{\psi f x}\right.^{\frac{1}{p}} \left\{\frac{\phi f^2 x}{\psi f^2 x}\right.^{\frac{1}{p}} \left\{\frac{\phi f^3 x}{\psi f^3 x}\right.^{\frac{1}{p}} \left\{\right.^{\frac{1}{p}} \text{\&c.} \left\{\frac{\phi f^n x}{\psi f^n x}\right\}^{\frac{1}{p}} *$$

$$y = \left\{\phi f x\right.^{\frac{1}{p}} \left\{\frac{\phi f^2 x}{(\psi f x)^p}\right.^{\frac{1}{p}} \left\{\frac{\phi f^3 x}{(\psi f^2 x)^p}\right.^{\frac{1}{p}} \left\{\right.^{\frac{1}{p}} \text{\&c.} \left\{\frac{\phi f^n x}{(\psi f^{n-1} x)^p}\right.^{\frac{1}{p}} \left\{\frac{1}{(\psi f^n x)^p}\right\}^{\frac{1}{p}}$$

Multiplying by $(\phi f^{n+1} x)^{\frac{1}{p^{n+1}}}$, we have

$$y\,(\phi f^{n+1} x)^{\frac{1}{p^{n+1}}} = \left\{\phi f x\right.^{\frac{1}{p}} \left\{\frac{\phi f^2 x}{(\psi f x)^p}\right.^{\frac{1}{p}} \left\{\frac{\phi f^3 x}{(\psi f^2 x)^p}\right.^{\frac{1}{p}} \left\{\right.^{\frac{1}{p}} \text{\&c.} \left\{\frac{\phi f^n x}{(\psi f^{n-1} x)^p}\right.^{\frac{1}{p}} \left\{\frac{\phi f^{n+1} x}{(\psi f^n x)^p}\right\}^{\frac{1}{p}}$$

Which becomes, by applying equation (e),

$$(\phi f^{n+1} x)^{\frac{1}{p^{n+1}}} y = \left\{\phi f x\right.^{\frac{1}{p}} \left\{\frac{\phi f x}{\psi f x}\right.^{\frac{1}{p}} \left\{\frac{\phi f^2 x}{\psi f^2 x}\right.^{\frac{1}{p}} \left\{\right.^{\frac{1}{p}} \text{\&c.} \left\{\frac{\phi f^n x}{\psi f^n x}\right\}^{\frac{1}{p}} = \left\{\phi f x \,.\, y\right\}^{\frac{1}{p}}$$

Hence,

$$y^p\,(\phi f^{n+1} x)^{\frac{1}{p^n}} = y\,\phi f x$$

$$y^{p-1} = \frac{\phi f x}{(\phi f^{n+1} x)^{\frac{1}{p^n}}}$$

$$y = \left\{\frac{\phi f x}{(\phi f^{n+1} x)^{\frac{1}{p^n}}}\right\}^{\frac{1}{p-1}}$$

$$\left\{\frac{\phi f x}{(\phi f^{n+1} x)^{\frac{1}{p^n}}}\right\}^{\frac{1}{p-1}} = \left\{\frac{\phi f x}{\psi f x}\right.^{\frac{1}{p}} \left\{\frac{\phi f^2 x}{\psi f^2 x}\right.^{\frac{1}{p}} \left\{\right.^{\frac{1}{p}} \text{\&c.} \left\{\frac{\phi f^n x}{\psi f^n x}\right\}^{\frac{1}{p}} \ldots\ldots\ldots\ldots (f)$$

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* Braces with an index as $\left(\frac{1}{p}\right)$ over them signify the $\frac{1}{p}$ power of all the following part of the expression.

Putting p for $\frac{1}{p}$, we find

$$\left\{\frac{(\phi f^{n+1}x)^{p^n}}{\phi f x}\right\}^{\frac{p}{p-1}} = \left\{\frac{\phi f x}{\psi f x}\right\}^{p} \left\{\frac{\phi f^2 x}{\psi f^2 x}\right\}^{p} \left\{ \&\text{c.} \right\}^{p} \left\{\frac{\phi f^n x}{\psi f^n x}\right\}^{p} \dots\dots\dots\dots (g)$$

And the equation determining the functions becomes,

$$\{\phi x\}^p \{\psi x\}^{p-1} = \{\phi f x\}^p$$

These equations in their present general form are sufficiently concise, but as they will become rather complex by the substitution of particular cases, I shall make use of the following notation to abbreviate them. P with an index above it placed before any function of n and other quantities signifies the continued product of that function, n being successively equal to 1. 2. and n; thus,

$$\overset{n}{P} \{\psi f^n x\} = \psi f x \,.\, \psi f^2 x \,.\, \psi f^3 x \dots\dots \psi f^n x$$

If the product is to be continued to infinity, I shall make use of Euler's method of denoting the limits of integrals.

In (b) assume

$$\psi x = 1 + x + x^2 + \&\text{c.} + x^a \dots\dots f x = x^{a+1}$$

$$\{1 + x + x^2 + \&\text{c.} + x^a\} \ \phi x = \phi x^{a+1}$$

or, $$\frac{x^{a+1} - 1}{x - 1} \phi x = \phi x^{a+1}$$

Let, $y_z = x$ and $y_{z+1} = x^{a+1}$ then,

$$y_z^{a+1} = y_{z+1}$$

By integrating $$y_z = y_0^{\overline{a+1}^{\,z}} = c^{\overline{a+1}^{\,z}} = x$$

Let $$\phi x = \phi y_z = u_z$$

$$\phi x^{a+1} = \phi y_{z+1} = u_{z+1}$$

then, $$\frac{c^{\overline{a+1}^{\,z+1}} - 1}{c^{\overline{a+1}^{\,z}} - 1} u_z = z_{z+1}$$

$$(c^{\overline{a+1}^{\,z+1}} - 1) \, u_z = (c^{\overline{a+1}^{\,z}} - 1) \, u_{z+1}$$

But this equation is of the form

$$Q_{z+1} u_z = Q_z u_{z+1}$$

whose integral is

$$u_z = b Q_z$$

therefore,

$$u_z = \phi x = b\left(c^{\overline{a+1}^{z}} - 1\right) = (x - 1)\, b \ldots\ldots b = 1$$

Substituting these values of ψ f and ϕ in (c), we have

$$\frac{x^{\overline{a+1}^{n+1}} - 1}{x^{a+1} - 1} = \overset{n}{P}\left\{1 + x^{1\cdot\overline{a+1}^{n}} + x^{2\cdot\overline{a+1}^{n}} + \&c. + x^{a\cdot\overline{a+1}^{n}}\right\} \ldots\ldots (1)$$

Let $a = 1$

$$\frac{x^{2^{n+1}} - 1}{x^2 - 1} = \overset{n}{P}\left\{1 + x^{2^n}\right\} = (1 + x^2)(1 + x^4)(1 + x^8)\ldots\ldots(1 + x^{2^n})\ldots\ldots (2)$$

If $a = 2$

$$\frac{x^{3^{n+1}} - 1}{x^3 - 1} = \overset{n}{P}\left\{1 + x^{3^n} + x^{2\cdot 3^n}\right\} = (1 + x^3 + x^6)(1 + x^9 + x^{18})..(1 + x^{3^n} + x^{2\cdot 3^n})\ldots (3)$$

In (1) for a put any even number as $2a$, and make x negative,

$$\frac{1 + x^{\overline{2a+1}^{n+1}}}{1 + x^{2a+1}} = \overset{n}{P}\left\{1 - x^{1\cdot\overline{2a+1}^{n}} + x^{2\cdot\overline{2a+1}^{n}} - \&c. + x^{2a\cdot\overline{2a+1}^{n}}\right\} \ldots\ldots\ldots\ldots (4)$$

Let $a = 1$

$$\frac{1 + x^{3^{n+1}}}{1 + x^3} = \overset{n}{P}\left\{1 - x^{3^n} + x^{2\cdot 3^n}\right\} = (1 - x^3 + x^6)(1 - x^9 + x^{18})\cdot\cdot(1 - x^{3^n} + x^{2\cdot 3^n})\ldots (5)$$

Assume $fx = x^2$, and

$$\psi x = 1 - x + x^2 - \&c. + x^{2a} = \frac{1 + x^{2a+1}}{1 + x}$$

$$x = y_z \qquad x^2 = y_{z+1}$$

$$y_z = c^{2^z}$$

Let $\phi y_z = u_z \qquad \phi y_{z+1} = u_{z+1}$

Equation (b) becomes

$$\frac{1+c^{\overline{2a+1}\cdot 2^{n}}}{1+c^{2^{z}}}\,u_z = u_{z+1}$$

$$u_z\,(1+c^{\overline{2a+1}\cdot 2^{z}}) = u_{z+1}\,(1+c^{2^{z}})$$

In order to reduce this to the form of

$$Q_{z+1}u_z = Q_z u_{z+1}$$

multiply each side by $(1-c^{\overline{2a+1}\cdot 2^{z}}).(1-c^{2^{z}})$

$$u_z\,(1+c^{\overline{2a+1}\cdot 2^{z}}).(1-c^{\overline{2a+1}\cdot 2^{z}}).(1-c^{2^{z}}) = u_{z+1}(1-c^{\overline{2a+1}\cdot 2^{z}}).(1-c^{2^{z}}).(1+c^{2^{z}})$$

$$u_z\,(1-c^{\overline{2a+1}\cdot 2^{z+1}}).(1-c^{2^{z}}) = u_{z+1}\,(1-c^{\overline{2a+1}\cdot 2^{z}}).(1-c^{2^{z+1}})$$

$$u_z \times \frac{1-c^{\overline{2a+1}\cdot 2^{z+1}}}{1-c^{2^{z+1}}} = u_{z+1} \times \frac{1-c^{\overline{2a+1}\cdot 2^{z}}}{1-c^{2^{z}}}$$

Hence,

$$u_z = \phi x = \frac{1-c^{\overline{2a+1}\cdot 2^{z}}}{1-c^{2^{z}}} = \frac{1-x^{\overline{2a+1}}}{1-x}$$

Putting for f, ψ, and ϕ, their values in (b), we have

$$\frac{1-x^{\overline{2a+1}\cdot 2^{n+1}}}{1-x^{2\cdot\overline{2a+1}}} \times \frac{1-x^{2}}{1-x^{2^{n+1}}} = \overset{n}{P}\left\{1-x^{2^{n}}+x^{2\cdot 2^{n}}-\&c.+x^{2a\cdot 2^{n}}\right\}\ldots\ldots(6)$$

Let $a=1$,

$$\frac{1-x^{3\cdot 2^{n+1}}}{1-x^{2\cdot 3}} \times \frac{1-x^{2}}{1-x^{2^{n+1}}} = \frac{1+x^{2^{n+1}}+x^{2^{n+2}}}{1+x^{2}+x^{4}} =$$

$$= \overset{n}{P}\left\{1-x^{2^{n}}+x^{2^{n+1}}\right\} = (1-x^{2}+x^{4})\,(1-x^{4}+x^{8})\,..\,(1-x^{2^{n}}+x^{2^{n+1}})\ldots\ldots(7)$$

Divide (5) by (3)

$$\frac{1+x^{3^{n+1}}}{1-x^{3^{n+1}}}\times\frac{1-x^{3}}{1+x^{3}} = P\left\{\frac{1-x^{3^{n}}+x^{2\cdot 3^{n}}}{1+x^{3^{n}}+x^{2\cdot 3^{n}}}\right\} = \left(\frac{1-x^{3}+x^{6}}{1+x^{3}+x^{6}}\right)..\left(\frac{1-x^{3^{n}}+x^{2\cdot 3^{n}}}{1+x^{3^{n}}+x^{2\cdot 3^{n}}}\right)\ldots.(7)$$

In equation (1), for x put x^2, and let a be any even number as $2a$, then,

$$\frac{x^{\overline{2a+1}^{\,n+1}.2}-1}{x^{\overline{2a+1}.2}-1}=\overset{n}{P}\left\{1+x^{1.\overline{2a+1}^{\,n}.2}+x^{2.\overline{2a+1}^{\,n}.2}+\&c.+x^{2a.\overline{2a+1}^{\,n}.2}\right\}$$

Divide both sides by

$$\overset{n}{P}\left\{x^{2a.\overline{2a+1}^{\,n}}\right\}=x^{2a\frac{\left\{\overline{2a+1}+\overline{2a+1}^{\,2}+\&c.+\overline{2a+1}^{\,n}\right\}}{\cdots\cdots\cdots\cdots}}=x^{\overline{2a+1}^{\,n+1}-\overline{2a+1}}$$

By properly arranging the right side of the equation, it becomes,

$$\frac{x^{\overline{2a+1}^{\,n+1}}-x^{-\overline{2a+1}^{\,n+1}}}{x^{\overline{2a+1}}-x^{-\overline{2a+1}}}=$$

$$=\overset{n}{P}\left\{1+(x^{2.\overline{2a+1}^{\,n}}+x^{-2.\overline{2a+1}^{\,n}})+(x^{4.\overline{2a+1}^{\,n}}+x^{-4.\overline{2a+1}^{\,n}})+\&c.+(x^{2a.\overline{2a+1}^{\,n}}+x^{-2a.\overline{2a+1}^{\,n}})\right\}$$

For $x^{+1}+x^{-1}$ put $2\cos.\theta$

This substitution gives, $$\frac{\sin.\overline{2a+1}^{\,n+1}.\theta}{\sin.\overline{2a+1}.\theta}=$$

$$=\overset{n}{P}\left\{1+2\cos.2.\overline{2a+1}^{\,n}\theta+2\cos.4.\overline{2a+1}^{\,n}\theta+\&c.+2\cos.2a.\overline{2a+1}^{\,n}\theta\right\}\ldots\ldots(8)$$

For a in (1), put any odd number as $2a-1$, and substituting x^2 for x,

$$\frac{x^{(2a)^{n+1}.2}-1}{x^{(2a).2}-1}=\overset{n}{P}\left\{1+x^{1.(2a)^{n}.2}+x^{2.(2a)^{n}.2}+\&c.+x^{\overline{2a-1}.(2a)^{n}.2}\right\}$$

Divide each side by

$$\overset{n}{P}\left\{x^{\overline{2a-1}.(2a)^{n}}\right\}=x^{\overline{2a-1}\frac{\left\{(2a)^{1}+(2a)^{2}+\&c.(2a)^{n}\right\}}{\cdots\cdots\cdots\cdots}}=x^{(2a)^{n+1}-2a}$$

then,

$$\frac{x^{(2a)^{n+1}}-x^{-(2a)^{n+1}}}{x^{2a}-x^{-2a}}=$$

$$=\overset{n}{P}\left\{(x^{1.(2a)^{n}}+x^{-1.(2a)^{n}})+(x^{3.(2a)^{n}}+x^{-3.(2a)^{n}})+\&c.+(x^{\overline{2a-1}.(2a)^{n}}+x^{-\overline{2a+1}.(2a)^{n}}\right\}$$

Using the substitution

$$x^{+1} + x^{-1} = 2\cos.\,\theta$$

$$\frac{\sin.\,(2a)^{n+1}\theta}{\sin.\,2a\,\theta} = P\left\{\cos.\,1\,(2a)^n\theta+\cos.\,3\,(2a)^n\theta+\&c.+\cos.\,\overline{2a-1}\,(2a)^n\theta\right\}\times\overset{n}{P}\{2\}$$

But

$$\overset{n}{P}\{2\} = 2^n$$

$$\frac{1}{2^n}\cdot\frac{\sin.\,(2a)^{n+1}\theta}{\sin.\,2a\,\theta} = P\left\{\cos.\,1\,(2a)^n\theta+\cos.\,3\,(2a)^n\theta+\&c.+\cos.\,\overline{2a-1}\,(2a)^n\theta.\right\}\,..\,(9)$$

If $a = 1$, it becomes the well known expression of Euler

$$\frac{1}{2^n}\cdot\frac{\sin.\,2^{n+1}\theta}{\sin.\,2\,\theta} = \cos.\,2\theta\,.\,\cos.\,2^2\theta\ldots\ldots\cos.\,2^n\theta\ldots\ldots\ldots\ldots(1,\,1)$$

Let $a = 2$

$$\frac{1}{2^n}\cdot\frac{\sin.\,4^{n+1}\theta}{\sin.\,4\,\theta} = P\left\{\cos.\,1.\,4^n\theta+\cos.\,3.\,4^n\theta\right\}\ldots\ldots\ldots\ldots(1,\,2)$$

If the same operations be performed on (4) and (6) that have been made use of on (1), the results will be

$$\frac{\cos.\,\overline{2a+1}^{n+1}\theta}{\cos.\,\overline{2a+1}\,\theta} = \overset{n}{P}\left\{\pm(1-2\cos.\,2.\,\overline{2a+1}^n\theta+\&c.+2\cos.\,2a.\,\overline{2a+1}^n\theta)\right\}$$

and

$$\frac{\sin.\,2\,.\,\overline{2a+1}^{n+1}.\,\theta}{\sin.\,2\,\overline{2a+1}\,\theta}\cdot\frac{\sin.\,2\,\theta}{\sin.\,2^n\theta} =$$

$$= \overset{n}{P}\left\{\pm(1-2\cos.\,2.\,2^n\theta+2\cos.\,4.\,2^n\theta-\&c.+2\cos.\,2a.\,2^n\theta)\right\}$$

plus or minus being used as a is an even or odd number.

To obviate the ambiguity of the signs, it will be better to put for a, $2a$, and $2a-1$, then,

$$\frac{\cos.\,\overline{4a+1}^{n+1}.\,\theta}{\cos.\,\overline{4a+1}\,\theta} =$$

$$= \overset{n}{P}\left\{1-2\cos.\,2.\,\overline{4a+1}^n.\,\theta+2\cos.\,4.\,\overline{4a+1}^n\theta-\&c.+2\cos.\,4a.\,\overline{4a+1}^n\theta\right\}\ldots(1,\,3)$$

$$\frac{\cos.\,\overline{4a-1}^{\,n+1}\,\theta}{\cos.\,\overline{4a-1}\,\theta} =$$

$$= \overset{n}{P}\left\{2\cos.\,\overline{4a-2}.\overline{4a-1}^{\,n}\theta - 2\cos.\,\overline{4a-4}.\overline{4a-1}^{\,n}\theta + \&c. + 2\cos.\,2\overline{4a-1}^{\,n}\theta - 1\right\} .\,.\,(1,4)$$

$$\frac{\sin.\,2\,\overline{4a+1}^{\,n+1}\,\theta}{\sin.\,2\,\overline{4a+1}\,\theta}\cdot\frac{\sin.\,2\,\theta}{\sin.\,2^{n+1}\theta} =$$

$$= \overset{n}{P}\left\{1 - 2\cos.\,2\,.\,2^{n}\theta + 2\cos.\,4\,.\,2^{n}\theta - \&c. + 2\cos.\,4a\,.\,2^{n}\theta\right\} .\,.\,.\,.\,.\,.\,.\,.\,(1,5)$$

$$\frac{\sin.\,2\,\overline{4a-1}^{\,n+1}\,\theta}{\sin.\,2\,.\,\overline{4a-1}\,\theta}\cdot\frac{\sin.\,2\,\theta}{\sin.\,2^{n+1}\theta} =$$

$$= \overset{n}{P}\left\{2\cos.\,\overline{4a-2}\,.\,2^{n}\theta - 2\cos.\,\overline{4a-4}\,.\,2^{n}\theta + \&c. + 2\cos.\,2\,.\,2^{n}\theta - 1\right\} .\,.\,.\,.\,(1,6)$$

In (8) make $a = 1$, and also make $a = 1$ in (1, 4) Dividing the last result by the first.

$$\frac{\cos.\,3^{n+1}\,\theta}{\cos.\,3\,\theta}\cdot\frac{\sin.\,3\,\theta}{\sin.\,3^{n+1}\theta} = \frac{\tan.\,3\,\theta}{\tan.\,3^{n+1}\theta} = \overset{n}{P}\left\{\frac{2\cos.\,2.3^{n}\,\theta - 1}{2\cos.\,2.3^{n}\theta + 1}\right\} = \overset{n}{P}\left\{\frac{2 - \sec.\,2.3^{n}\,\theta}{2 + \sec.\,2.3^{n}\theta}\right\} .\,.\,(1,7)$$

Or,

$$\frac{\tan.\,\theta}{\tan.\,3^{n+1}\theta} = \left\{\frac{2 - \sec.\,2\,.\,3\,\theta}{2 + \sec.\,2\,.\,3\,\theta}\right\}\cdot\left\{\frac{2 - \sec.\,2\,.\,3^{2}\theta}{2 + \sec.\,2\,.\,3^{2}\theta}\right\} .\,.\,.\,.\left\{\frac{2 - \sec.\,2\,.\,3^{n}\theta}{2 + \sec.\,2\,.\,3^{n}\theta}\right\}$$

In (1, 6) make $a = 1$, and in (1, 1), put 2θ for θ; then,

$$\frac{\sin.\,3\,.\,2^{n+1}\theta}{\sin.\,2^{n+1}\theta}\cdot\frac{\sin.\,2\,\theta}{\sin.\,3\,.\,2\theta} = \overset{n}{P}\left\{2\cos.\,2\,.\,2^{n}\theta - 1\right\} .\,.\,.\,.\,.\,.\,.\,.\,(1,8)$$

And,
$$\frac{1}{2^{n}}\cdot\frac{\sin.\,2\,.\,2^{n+1}\theta}{\sin.\,2\,.\,2\,\theta} = \overset{n}{P}\left\{\cos.\,2\,.\,2^{n}\theta\right\}$$

The first divided by the second produces

$$\frac{2^{n}\sin.\,3\,.\,2^{n+1}\theta}{\sin.\,3\,.\,2\,\theta}\cdot\frac{\sin.\,2\,\theta\,\sin.\,2\,.\,2\,\theta}{\sin.\,2^{n+1}\theta\,\sin.\,2\,.\,2^{n+1}\theta} = \overset{n}{P}\left\{2 - \sec.\,2\,.\,2^{n}\theta\right\} .\,.\,.\,.\,(1,9)$$

If in theorems (8) (9) (1, 3) (1, 4) (1, 5) (1, 6) for θ be substituted in each respectively,

$$\frac{\theta}{(2a+1)^{n+1}},\quad \frac{\theta}{(2a)^{n+1}},\quad \frac{\theta}{(4a+1)^{n+1}},\quad \frac{\theta}{(4a-1)^{n+1}},\quad \frac{\theta}{2^{n+1}},\quad \frac{\theta}{2^{n+1}}$$

The following are the results:

$$\frac{\sin.\theta}{\sin.\frac{\theta}{(2a+1)^n}} = \overset{n}{P}\left\{1+2\cos.\frac{2\theta}{(2a+1)^n}+2\cos.\frac{4\theta}{(2a+1)^n}+\&c.+2\cos.\frac{2a\theta}{(2a+1)^n}\right\} \ldots\ldots (2,1)$$

$$\frac{1}{2^n}\cdot\frac{\sin.\theta}{\sin.\frac{\theta}{(2a)^n}} = \overset{n}{P}\left\{\cos.\frac{1.\theta}{(2a)^n}+\cos.\frac{3.\theta}{(2a)^n}+\&c.+\cos.\frac{\overline{2a-1}\,\theta}{(2a)^n}\right\}\ldots\ldots\ldots (2,2)$$

$$\frac{\cos.\theta}{\cos.\frac{\theta}{(4a+1)^n}} = \overset{n}{P}\left\{1-2\cos.\frac{2\theta}{(4a+1)^n}+2\cos.\frac{4\theta}{(4a+1)^n}-\&c.+2\cos.\frac{4a\theta}{(4a+1)^n}\right\} \ldots\ldots (2,3)$$

$$\frac{\cos.\theta}{\cos.\frac{\theta}{(4a-1)^n}} = \overset{n}{P}\left\{2\cos.\frac{\overline{4a-2}\,\theta}{(4a-1)^n}-2\cos.\frac{\overline{4a-4}\,\theta}{(4a-1)^n}+\&c.+2\cos.\frac{2\theta}{(4a-1)^n}-1\right\} \ldots\ldots (2,4)$$

$$\frac{\sin.\overline{4a+1}\,\theta}{\sin.\frac{4a+1}{2^n}\theta}\cdot\frac{\sin.\frac{\theta}{2^n}}{\sin.\theta} = \overset{n}{P}\left\{1-2\cos.\frac{2\theta}{2^n}+2\cos.\frac{4\theta}{2^n}-\&c.+2\cos.\frac{4a\theta}{2^n}\right\}\ldots\ldots (2,5)$$

$$\frac{\sin.\overline{4a-1}\,\theta}{\sin.\frac{4a-1}{2^n}\theta}\cdot\frac{\sin.\frac{\theta}{2^n}}{\sin.\theta} = \overset{n}{P}\left\{2\cos.\frac{\overline{4a-2}\,\theta}{2^n}-2\cos.\frac{\overline{4a-4}\,\theta}{2^n}+\&c.+2\cos.\frac{2\theta}{2^n}-1\right\} \ldots\ldots (2,6)$$

Other theorems nearly similar may be thus derived. In (1), for x put vx, and $\frac{x}{v}$; multiply together the results; and in the left side of the equation put for $v^{+1}+v^{-1}$ its value $2\cos.\theta$; then,

$$\frac{x^{2(a+1)^{n+1}} - 2x^{(a+1)^{n+1}}\cos.\overline{a+1}^{\,n+1}\theta+1}{x^{2(a+1)} - 2x^{a+1}\cos.\overline{a+1}\,\theta+1} =$$

$$= P\left\{1+(xv)^{1.\overline{a+1}^n}+(xv)^{2.\overline{a+1}^n}+\&c.+(xv)^{a.\overline{a+1}^n}\right\}$$

$$\times\ P\left\{1+\left(\frac{x}{v}\right)^{1.\overline{a+1}^n}+\left(\frac{x}{v}\right)^{2.\overline{a+1}^n}+\&c.+\left(\frac{x}{v}\right)^{a.\overline{a+1}^n}\right\}$$

In order to reduce the right side of the equation to a series of multiple arcs, let us consider the product of two functions of this form;

$$f(xv) = 1+(xv)+(xv)^2+\&c.+(xv)^a$$

$$f\left(\frac{x}{v}\right) = 1+\left(\frac{x}{v}\right)+\left(\frac{x}{v}\right)^2+\&c.+\left(\frac{x}{v}\right)^a$$

Their product is of the form

$$f(xv)\times f\left(\frac{x}{v}\right) =$$

$$(1+x^2+\&c.x^{2a})+x(1+x^2+\&c.x^{2a-2})(v^{+1}+v^{-1})+x^2(1+x^2+\&c.+x^{2a-4})(v^{+2}+v^{-2})+\&c.+x^a(v^{+a}+v^{-a})$$

$$=\left\{1-x^2\right\}^{-1}.\left\{(1-x^{2a+2})+x(1-x^{2a})(v^{+1}+v^{-1})+x^2(1-x^{2a-2})(v^{+2}+v^{-2})+\&c.x^a(v^{+a}+v^{-a})\right\}$$

Putting $v^{+1}+v^{-1}=2\cos.\theta$, and applying this to the preceding expression,

$$\frac{x^{2.\overline{a+1}^{\,n+1}}-2x^{\overline{a+1}^{\,n+1}}\cos.\overline{a+1}^{\,n+1}\theta+1}{x^{2.a+1}-2x^{a+1}\cos.a+1\,\theta+1}=$$

$$=(1-x^2)^{-n}\overset{n}{P}\left\{(1-x^{\overline{2a+2.a+1}^{\,n}})+2(1-x^{\overline{2a.a+1}^{\,n}})x^{\overline{1.a+1}^{\,n}}\cos.1.\overline{a+1}^{\,n}\theta+\&c.+2x^{\overline{a.a+1}^{\,n}}\cos.a.\overline{a+1}^{\,n}\theta\right\}$$

In the case of x equal to unity, this expression becomes

$$\frac{1-\cos.\overline{a+1}^{\,n+1}\theta}{1-\cos.a+1\,\theta}=$$

$$P\left\{(a+1)+2(a)\cos.1.\overline{a+1}^{\,n}\theta+2(a-1)\cos.2\overline{a+1}^{\,n}\theta+\&c.+2(1)\cos.a.\overline{a+1}^{\,n}\theta\right\}..(2,6)$$

If x do not equal unity, and if $a=1$;

$$\frac{x^{2.2^{n+1}}-2x^{2^{n+1}}\cos.2^{n+1}\theta+1}{x^{2.2}-2x^2\cos.2\theta+1}=P\left\{1+2x^{2^n}\cos.2^n\theta+x^{2^{n+1}}\right\}......(2,7)$$

If the same operations are performed on (4) and (6), we shall obtain the following results;

$$\frac{1+\cos.\overline{2a+1}^{\,n+1}.\theta}{1+\cos.\overline{2a+1}.\theta}=$$

$$=\overset{n}{P}\left\{(2a+1)-2(2a)\cos.1.\overline{2a+1}^{\,n}\theta+2(2a-1)\cos.2\overline{2a+1}^{\,n}\theta-\&c.+2(1)\cos.2a.\overline{2a+1}^{\,n}\theta\right\}..(2,8)$$