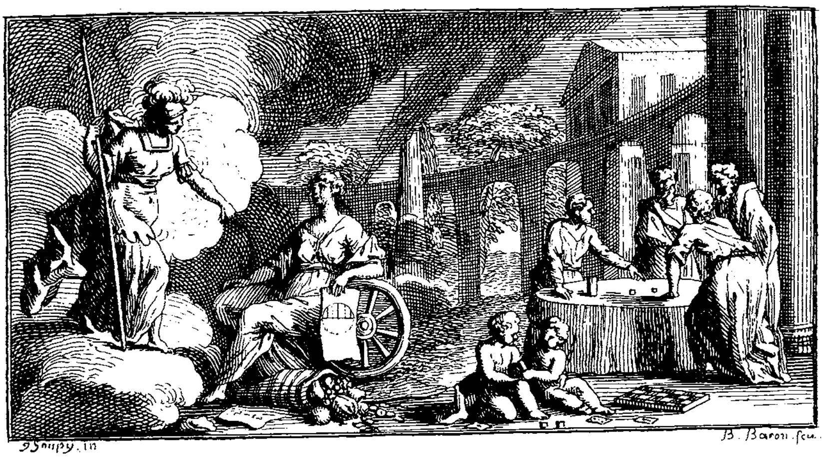

THE DOCTRINE OF CHANCES.

The INTRODUCTION.

1. THE Probability of an Event is greater or leſs, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.

2. Wherefore, if we conſtitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper deſignation of the Probability of happening. Thus

if an Event has 3 Chances to happen, and 2 to fail, the Fraction $\frac{3}{5}$ will fitly repreſent the Probability of its happening, and may be taken to be the meaſure of it.

The ſame thing may be ſaid of the Probability of failing, which will likewiſe be meaſured by a Fraction whoſe Numerator is the number of Chances whereby it may fail, and the Denominator the whole number of Chances, both for its happening and failing; thus the Probability of the failing of that Event which has 2 Chances to fail and 3 to happen will be meaſur'd by the Fraction $\frac{2}{5}$.

3. The Fractions which repreſent the Probabilities of happening and failing, being added together, their Sum will always be equal to Unity, ſince the Sum of their Numerators will be equal to their common Denominator; now it being a certainty that an Event will either happen or fail, it follows that Certainty which may be conceived under the notion of an infinitely great degree of Probability, is fitly repreſented by Unity.

Theſe things will eaſily be apprehended, if it be conſider'd, that the word Probability includes a double Idea; firſt, of the number of Chances whereby an Event may happen; ſecondly, of the number of Chances whereby it may either happen or fail.

If I ſay that I have three Chances to win any Sum of Money, it is impoſſible from that bare aſſertion to judge whether I am like to obtain it; but if I add that the number of Chances either to obtain it, or to miſs it, is five in all, from hence will enſue a compariſon between the Chances that favour me, and the whole number of Chances that are for or againſt me, whereby a true judgment will be formed of my Probability of ſucceſs: from whence it neceſſarily follows that it is the comparative magnitude of the number of Chances to happen, in reſpect to the whole number of Chances either to happen or to fail, which is the true meaſure of Probability.

4. If upon the happening of an Event, I be intitled to a Sum of Money, my Expectation of obtaining that Sum has a determinate value before the happening of the Event.

Thus, if I am to have 10 *L.* in caſe of the happening of an Event which has an equal Probability of happening and failing, my Expectation before the happening of the Event is worth 5 *L.* for I am preciſely in the ſame circumſtances as he who at an equal Play ventures 5 *L.* either to have 10, or to loſe his 5. Now he who ventures 5 *L.* in an equal Play, is poſſeſſor of 5 *L.* before the deciſion of the Play;

Play; therefore my Expectation in the cafe above-mentioned muft alfo be worth 5 *L.*

5. In all cafes, the Expectation of obtaining any Sum is eftimated by multiplying the value of the Sum expected by the Fraction which reprefents the Probability of obtaining it.

Thus, if I have 3 Chances in 5 to obtain 100 *L.* I fay that the prefent value of my Expectation is the product of 100 *L.* by the fraction $\frac{3}{5}$, and confequently that my Expectation is worth 60 *L.*

For fuppofing that an Event may equally happen to any one of 5 different Perfons, and that the Perfon to whom it happens fhould in confequence of it obtain the Sum of 100 *L.* it is plain that the right which each of them in particular has upon the Sum expected is $\frac{1}{5}$ of 100 *L.* which right is founded in this, that if the five Perfons concerned in the happening of the Event, fhould agree not to ftand the Chance of it, but to divide the Sum expected among themfelves, then each of them muft have $\frac{1}{5}$ of 100 *L.* for his pretenfion. Now whether they agree to divide that fum equally among themfelves, or rather chufe to ftand the Chance of the Event, no one has thereby any advantage or difadvantage, fince they are all upon an equal foot, and confequently each Perfon's expectation is worth $\frac{1}{5}$ of 100 *L.* Let us fuppofe farther, that two of the five Perfons concerned in the happening of the Event, fhould be willing to refign their Chance to one of the other three, then the Perfon to whom thofe two Chances are thus refigned has now three Chances that favour him, and confequently has now a right triple of that which he had before, and therefore his expectation is now worth $\frac{3}{5}$ of 100 *L.*

Now if we confider that the fraction $\frac{3}{5}$ expreffes the Probability of obtaining the Sum of 100 *L.*, and that $\frac{3}{5}$ of 100, is the fame thing as $\frac{3}{5}$ multiply'd by 100, we muft naturally fall into this conclufion, which has been laid down as a principle, that the value of the Expectation of any Sum, is determined by multiplying the Sum expected by the Probability of obtaining it.

This manner of reafoning, tho' deduced from a particular cafe will eafily be perceived to be general, and applicable to any other cafe.

COROLLARY.

From what precedes, it neceſſarily follows that if the Value of an Expectation be given, as alſo the Value of the thing expected, then dividing the firſt value by the ſecond, the quotient will expreſs the Probability of obtaining the Sum expected; thus if I have an Expectation worth 60 $^{L.}$ and that the Sum which I may obtain be worth 100 $^{L.}$ the Probability of obtaining it will be expreſt by the quotient of 60 divided by 100, that is by the fraction $\frac{60}{100}$ or $\frac{3}{5}$.

6. The Riſk of loſing any Sum is the reverſe of Expectation, and the true meaſure of it, is the product of the Sum adventured, multiply'd by the Probability of the Loſs.

7. Advantage or Diſadvantage in Play, reſults from the combination of the ſeveral Expectations of the Gameſters, and of their ſeveral Riſks.

Thus ſuppoſing that *A* and *B* play together, that *A* has depoſited 5 $^{L.}$ and *B* 3 $^{L.}$ that the number of Chances which *A* has to win is 4, and the number of Chances which *B* has to win is 2, and that it were required in this circumſtance to determine the advantage or diſadvantage of the Adventurers, we may reaſon in this manner: Since the whole Sum depoſited is 8, and that the Probability which *A* has of getting it is $\frac{4}{6}$, it follows that the Expectation of *A* upon the whole Sum depoſited is $8 \times \frac{4}{6} = 5\frac{1}{3}$, and for the ſame reaſon the Expectation of *B* upon that whole Sum depoſited is $8 \times \frac{2}{6} = 2\frac{2}{3}$.

Now, if from the reſpective Expectations which the Adventurers have upon the whole ſum depoſited, be ſubtracted the particular Sums which they depoſit, that is their own Stakes, there will remain the Advantage or Diſadvantage of either, according as the difference is poſitive or negative.

And therefore, if from $5\frac{1}{3}$, which is the Expectation of *A* upon the whole Sum depoſited, 5 which is his own Stake, be ſubtracted, there will remain $\frac{1}{3}$ for his advantage; likewiſe if from $2\frac{2}{3}$ which is the Expectation of *B*, 3 which is his own Stake be ſubtracted, there will remain $-\frac{1}{3}$, which being negative ſhews that his Diſadvantage is $\frac{1}{3}$.

Theſe concluſions may alſo be derived from another conſideration; for if from the Expectation which either Adventurer has upon the Sum

Sum depofited by his Adverfary, be fubtracted the Rifk of what he himfelf depofits, there will likewife remain his Advantage or Difadvantage, according as the difference is pofitive or negative.

Thus in the preceding cafe, the Stake of *B* being 3, and the Probability which *A* has of winning it, being $\frac{4}{6}$, the Expectation of *A* upon that Stake is $3 \times \frac{4}{6} = 2$; moveover the Stake of *A* being 5, and the Probability of lofing it, being $\frac{2}{6}$, his Rifk ought to be eftimated by $5 \times \frac{2}{6} = 1\frac{2}{3}$; wherefore, if from the Expectation 2, the Rifk $1\frac{2}{3}$ be fubtracted, there will remain $\frac{1}{3}$ as before for the Advantage of *A*; and by the fame way of proceeding, the Difadvantage of *B* will be found to be $\frac{1}{3}$.

It is very carefully to be obferved, that what is here call'd Advantage or Difadvantage, and which may properly be call'd Gain or Lofs, is always eftimated before the Event is come to pafs; and altho' it be not cuftomary to call that Gain or Lofs which is to be derived from an Event not yet determined, neverthelefs in the Doctrine of Chances, that appellation is equivalent to what in common difcourfe is call'd Gain or Lofs.

For in the fame manner as that he who ventures a Guinea in an equal Game may, before the determination of the Play, be faid to be poffeffor of that Guinea, and may, in confideration of that Sum, refign his place to another; fo he may be faid to be a Gainer or Lofer, who would get fome Profit, or fuffer fome Lofs, if he would fell his Expectation upon equitable terms, and fecure his own Stake for a Sum equal to the Rifk of lofing it.

8. If the obtaining of any Sum requires the happening of feveral Events that are independent on each other, then the Value of the Expectation of that Sum is found by multiplying together the feveral Probabilities of happening, and again multiplying the product by the Value of the Sum expected.

Thus fuppofing that in order to obtain 90*L.* two Events muft happen; the firft whereof has 3 Chances to happen, and 2 to fail, the fecond has 4 Chances to happen, and 5 to fail, and I would know the value of that Expectation; I fay,

The Probability of the firft's happening is $\frac{3}{5}$, the Probability of the fecond's happening is $\frac{4}{9}$; now multiplying thefe two Probabilities together, the product will be $\frac{12}{45}$ or $\frac{4}{15}$; and this product being again

again multiply'd by 90, the new product will be $\frac{360}{15}$ or 24, therefore that Expectation is worth 24 *L.*

The Demonſtration of this will be very eaſy, if it be conſider'd, that ſuppoſing the firſt Event had happen'd, then that Expectation depending now intirely upon the ſecond, would before the determination of the ſecond, be found to be exactly worth $\frac{4}{9} \times 90$ *L.* or 40 *L.* (by Art. 5th) We may therefore look upon the happening of the firſt, as a condition of obtaining an Expectation worth 40 *L.* but the Probability of the firſt's happening has been ſuppoſed $\frac{3}{5}$, wherefore the Expectation ſought for, is to be eſtimated by $\frac{3}{5} \times 40$, or by $\frac{3}{5} \times \frac{4}{9} \times 90$, that is by the product of the two Probabilities of happening multiply'd by the Sum expected.

And likewiſe, if an Expectation depends on the happening of one Event, and the failing of another, then its Value will be the product of the Probability of the firſt's happening by the Probability of the ſecond's failing, and of that again by the Value of the Sum expected.

And again, if an Expectation depends on the failing of two Events, the Rule will be the ſame, for that Expectation will be found by multiplying together the two Probabilities of failing, and multiplying that again by the Value of the Sum expected.

And the ſame Rule is applicable to the happening or failing of as many Events as may be aſſigned.

COROLLARY.

If we make abſtraction of the Value of the Sum to be obtain'd, the bare Probability of obtaining it, will be the product of the ſeveral Probabilities of happening, which evidently appears from this 8th Art. and from the Corollary to the 4th.

Hitherto, I have confin'd myſelf to the conſideration of Events independent; but for fear that in what is to be ſaid afterwards, the terms independent or dependent might occaſion ſome obſcurity, it will be neceſſary, before I proceed any farther, to ſettle intirely the notion of thoſe terms.

Two Events are independent, when they have no connexion one with the other, and that the happening of one neither forwards nor obſtructs the happening of the other.

Two Events are dependent, when they are ſo connected together as that the Probability of either's happening is alter'd by the happening of the other.

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In order to illuſtrate this, it will not be amiſs to propoſe the two following eaſy Problems.

1°. Suppoſe there is a heap of 13 Cards of one colour, and another heap of 13 Cards of another colour, what is the Probability that taking two Cards at a venture out of each heap, I ſhall take the two Aces?

The Probability of taking the Ace out of the firſt heap is $\frac{1}{13}$; now it being very plain that the taking or not taking the Ace out of the firſt heap has no influence in the taking or not taking the Ace out of the ſecond, it follows, that ſuppoſing that Ace taken out, the Probability of taking the Ace out of the ſecond will alſo be $\frac{1}{13}$; and therefore, thoſe two Events being independent, the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

2°. Suppoſe that out of one ſingle heap of 13 Cards of one colour, it ſhould be undertaken to take out the Ace in the firſt place, and then the Deux; and that it were required to aſſign the Probability of doing it, we are to conſider that altho' the Probability of the Ace's being in the firſt place be $\frac{1}{13}$, and that the Probability of the Deux's being in the ſecond place, would alſo be $\frac{1}{13}$, if that ſecond Event were conſider'd in itſelf without any relation to the firſt; yet that the Ace being ſuppoſed as taken out at firſt, there will remain but 12 Cards in the heap, and therefore that upon the ſuppoſition of the Ace being taken out at firſt, the Probability of the Deux's being next taken will be alter'd, and become $\frac{1}{12}$; and therefore, we may conclude that thoſe two Events are dependent, and that the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{12} = \frac{1}{156}$.

From whence it may be infer'd that the Probability of the happening of two Events dependent, is the product of the Probability of the happening of one of them, by the Probability which the other will have of happening, when the firſt ſhall have been conſider'd as having happen'd; and the ſame Rule will extend to the happening of as many Events as may be aſſign'd.

9. But to determine in the eaſieſt manner poſſible, the Probability of the happening of ſeveral Events dependent, it will be convenient to diſtinguiſh by thought the order of thoſe Events, and to ſuppoſe one of them to be the firſt, another to be the ſecond, and ſo on: which being done, the Probability of the happening of the firſt may be look'd upon as independent, the Probability of the happening of the

the ſecond, is to be determin'd from the ſuppoſition of the firſt's having happen'd, the Probability of the third's happening, is to be determin'd from the ſuppoſition of the firſt's and ſecond's having happened, and ſo on: then the Probability of the happening of them all will be the product of the Multiplication of the ſeveral Probabilities which have been determined in the manner preſcrib'd.

We had ſeen before how to determine the Probability of the happening or failing of as many Events independent as may be aſſign'd; we have ſeen likewiſe in the preceding Article how to determine the Probability of the happening of as many Events dependent as may be aſſign'd; but in the caſe of Events dependent, how to determine the Probability of the happening of ſome of them, and at the ſame time the Probability of the failing of ſome others, is a diſquiſition of a greater degree of difficulty, which for that reaſon will be more conveniently transfer'd to another place.

10. If I have ſeveral Expectations upon ſeveral Sums, it is very evident that my Expectation upon the whole is the Sum of the Expectations I have upon the particulars.

Thus ſuppoſe two Events ſuch, that the firſt may have 3 Chances to happen and 2 to fail, and the ſecond 4 Chances to happen and 5 to fail, and that I be intitled to 90$^{L.}$ in caſe the firſt happens, and to another like Sum of 90$^{L.}$ in caſe the ſecond happens alſo, and that I would know the Value of my Expectation upon the whole: I ſay,

The Sum expected in the firſt caſe being 90$^{L.}$ and the Probability of obtaining it being $\frac{3}{5}$, it follows that my Expectation on that account, is worth $90 \times \frac{3}{5} = 54$; and again the Sum expected in the ſecond caſe being 90, and the Probability of obtaining it being $\frac{4}{9}$, it follows that my Expectation of that ſecond Sum is worth $90 \times \frac{4}{9} = 40$; and therefore my Expectation upon the whole is worth $54^{L} + 40^{L} = 94^{L.}$

But if I am to have 90$^{L.}$ once for all for the happening of one or the other of the two afore-mentioned Events, the method of proceſs in determining the value of my Expectation will be ſomewhat alter'd; for altho' my Expectation of the firſt Event be worth 54$^{L.}$ as it was in the preceding Example, yet I conſider that my Expectation of the ſecond will ceaſe upon the happening of the firſt, and that therefore this Expectation takes place only in caſe the firſt does happen to fail. Now the Probability of the firſt's failing is $\frac{2}{5}$; and ſuppoſing it has fail'd, then my Expectation will be 40; wherefore $\frac{2}{5}$ being the meaſure of the Probability of my obtaining an Expectation worth 40 *L.*, it follows that this Expectation (to eſtimate it before the time of the firſt's being determin'd) will be worth $40 \times \frac{2}{5} = 16$, and therefore my Expectation upon the whole is worth $54^{L.} + 16^{L.} = 70^{L.}$

If that which was call'd the ſecond Event be now conſider'd as the firſt, and that which was call'd the firſt be now conſider'd as the ſecond, the concluſion will be the ſame as before.

In order to make the preceding Rules familiar, it will be convenient to apply them to the Solution of ſome eaſy caſes, ſuch as are the following.

CASE I[ſt.]

To find the Probability of throwing an Ace in two throws.

SOLUTION.

The Probability of throwing an Ace the firſt time is $\frac{1}{6}$; wherefore $\frac{1}{6}$ is the firſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill it may be thrown on the ſecond, but the Probability of miſſing it the firſt time is $\frac{5}{6}$, and the Probability of throwing it the ſecond time is $\frac{1}{6}$; wherefore the Probability of miſſing it the firſt time and throwing it the ſecond, is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$: and this is the ſecond part of the Probability required, and therefore the Probability required is in all $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$.

To this caſe is analogous a queſtion commonly propoſed about throwing with two Dice either ſix or ſeven in two throws, which will be eaſily ſolv'd, provided it be known that Seven has 6 Chances to come up, and Six 5 Chances, and that the whole number of Chances in two Dice is 36; for the number of Chances for throwing ſix or ſeven being 11, it follows that the Probability of throwing either Chance the firſt time is $\frac{11}{36}$, but if both are miſſed the firſt time, ſtill either may be thrown the ſecond time, but the Probability of miſſing both the firſt time is $\frac{25}{36}$, and the Probability of throwing either of them on the ſecond is $\frac{11}{36}$; wherefore the Probability of miſſing both of them the firſt time, and throw-

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ing either of them the ſecond time is $\frac{25}{36} \times \frac{11}{36} = \frac{275}{1296}$, and therefore the Probability required is $\frac{11}{36} + \frac{275}{1296} = \frac{671}{1296}$, and the Probability of the contrary is $\frac{625}{1296}$.

CASE II^{d}.

To find the Probability of throwing an Ace in three throws.

SOLUTION.

The Probabiliy of throwing an Ace the firſt time is $\frac{1}{6}$, which is the firſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill it may be thrown in the two remaining throws; but the Probability of miſſing it the firſt time is $\frac{5}{6}$, and the Probability of throwing it in the two remaining times is (by Caſe 1st) $= \frac{11}{36}$. And therefore the Probability of miſſing it the firſt time, and throwing it in the two remaining times is $\frac{5}{6} \times \frac{11}{36} = \frac{55}{216}$, which is the ſecond part of the Probability required; wherefore the Probability required will be $\frac{1}{6} + \frac{55}{216} = \frac{91}{216}$.

CASE III^{d}.

To find the Probability of throwing an Ace in four throws.

SOLUTION.

The Probability of throwing an Ace the firſt time is $\frac{1}{6}$, which is the firſt part of the Probability required.

If the Ace be miſs'd the firſt time, of which the Probability is $\frac{5}{6}$, there remains the Probability of throwing it in three times, which (by Caſe 2^{d}) is $\frac{91}{216}$; wherefore the Probability of miſſing the Ace the firſt time, and throwing it in the three remaining times is $= \frac{5}{6} \times \frac{91}{216} = \frac{455}{1296}$, which is the ſecond part of the Probability required, and therefore the Probability required is in the whole $\frac{1}{6} + \frac{455}{1296} = \frac{671}{1296}$, and the Probability of the contrary $\frac{625}{1296}$.