NOVA METHODUS, AEQUATIONES DIFFERENTIALES PARTIALES PRIMI ORDINIS INTER NUMERUM VARIABILIUM QUEMCUNQUE PROPOSITAS INTEGRANDI

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(Ex Ill. C. G. J. Jacobi manuscriptis posthumis in medium protulit A. Clebsch.)

Reductio problematis generalis in formam simpliciorem*).

1.

Sit V functio quaesita, sint $q_1, q_2, \ldots, q_m$ variabiles independentes atque $p_1, p_2, \ldots, p_m$ differentialia partialia ipsius V secundum $q_1, q_2, \ldots, q_m$. Problema de integratione aequationum differentialium partialium primi ordinis inter numerum quemcunque variabilium hoc est:

Data aequatione inter quantitates $V, q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m$, *ipsam* V *ut functionem ipsarum* $q_1, q_2, \ldots, q_m$ *determinare.*

Supponam aequationem propositam ipsam functionem quaesitam V non continere. Quoties enim continet, problema ad aliud revocari potest, in quo numerus variabilium independentium unitate auctus est, sed functio ipsa incognita ex aequatione differentiali evasit. Introducta enim nova variabili t, sit

$$W = t.V,$$

erit

$$V = \frac{\partial W}{\partial t}, \quad p_1 = \frac{\partial V}{\partial q_1} = \frac{1}{t}\frac{\partial W}{\partial q_1}, \quad p_2 = \frac{\partial V}{\partial q_2} = \frac{1}{t}\frac{\partial W}{\partial q_2}, \quad \ldots\text{**)}.$$

Quibus valoribus substitutis in aequatione inter V et quantitates $q_1, q_2, \ldots, q_m$, $p_1, p_2, \ldots, p_m$ proposita, prodibit aequatio inter variabiles independentes t, q_1, $q_2, \ldots, q_m$ atque differentialia partialia functionis W secundum variabiles illas sumta, ipsam functionem W non continens. Hinc, quia numerum variabilium independentium *m quemcunque* assumpsimus, concessa est suppositio, aequationem differentialem propositam functionem incognitam non continere.

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*) Epitomae paragraphorum in ipso manuscripto praeter paragraphos 66, 67 non inveniuntur. Quae tamen in usum lectoris, ut longioris commentationis decursus facilius perspiceretur, adjiciendae videbantur. *C.*

**) Significandis differentialibus partialibus signum characteristicum — ∂ —, significandis completis signum — d — adhibebo. Quod bene tenendum est.

Problema sub ea, qua in sequentibus utatur, forma proponitur.

2.

Si functio incognita ipsa aequationem differentialem partialem propositam non ingreditur, problema maxima generalitate sic enuntiari potest:

Proposita expressione

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m,$$

si data est aequatio inter quantitates $q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m$, *invenire* $m-1$ *alias aequationes inter easdem quantitates, e quibus quantitates* $p_1, p_2, \ldots, p_m$ *tales prodeant functiones ipsarum* $q_1, q_2, \ldots, q_m$, *ut expressio proposita*

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

evadat differentiale completum dV.

Ut expressio

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

sit differentiale completum, satisfieri debet $\frac{m(m-1)}{2}$ aequationibus conditionalibus hoc schemate contentis:

$$\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right),$$

in qua aequatione indicibus i et k valores $1, 2, 3, \ldots, m$ tribui possunt, vel ut aequationes tantum inter se diversae obtineantur, indici i tribuantur valores $1, 2, 3, \ldots, m-1$ et pro singulis ipsius i valoribus tribuantur indici k valores tantum ipso i maiores.

In aequationibus praecedentibus quantitates $p_1, p_2, \ldots, p_m$ ut functiones ipsarum $q_1, q_2, \ldots, q_m$ consideratae sunt. Quod quoties fit, differentialia partialia illarum quantitatum uncis includam, sicuti antecedentibus factum est.

Conditionum integrabilitatis forma prima exhibetur.

3.

Negotium, quod suscipiam, primum est transformatio aequationum conditionalium. Quippe quas ita exhibeamus, quales fiunt, si non ut antea omnes $p_1, p_2, \ldots, p_m$ ut ipsarum $q_1, q_2, \ldots, q_m$ functiones considerantur, sed

$$\begin{array}{lll}
p_1 & \text{ut ipsarum} & p_2,\ p_3,\ p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m, \\
p_2 & \text{ut ipsarum} & p_3,\ p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m, \\
p_3 & \text{ut ipsarum} & p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m, \\
\cdots & \cdots & \cdots \\
p_{m-1} & \text{ut ipsarum} & p_m,\ q_1,\ q_2,\ \ldots,\ q_m, \\
p_m & \text{ut ipsarum} & q_1,\ q_2,\ \ldots,\ q_m
\end{array}$$

functiones. Ad quam suppositionem referam sequentibus differentiationes per partes instituendas, nisi aliud disertis verbis statutum sit, aut differentialia uncis inclusa reperias, quo facto semper innuitur, omnes $p_1, p_2, \ldots, p_m$ tamquam ipsarum $q_1, q_2, \ldots, q_m$ functiones spectari.

Systema *primum* aequationum conditionalium, quod respondet valori $i = 1$, hoc est:

$$\left(\frac{\partial p_1}{\partial q_2}\right) = \left(\frac{\partial p_2}{\partial q_1}\right),\quad \left(\frac{\partial p_1}{\partial q_3}\right) = \left(\frac{\partial p_3}{\partial q_1}\right),\quad \ldots,\quad \left(\frac{\partial p_1}{\partial q_m}\right) = \left(\frac{\partial p_m}{\partial q_1}\right).$$

Quod e supra statutis sic repraesentari potest:

$$\begin{aligned}
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_2}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_2}\right) + \frac{\partial p_1}{\partial q_2} = \left(\frac{\partial p_2}{\partial q_1}\right), \\
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_3}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_3}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_3}\right) + \frac{\partial p_1}{\partial q_3} = \left(\frac{\partial p_3}{\partial q_1}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_m}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_m}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_m} = \left(\frac{\partial p_m}{\partial q_1}\right).
\end{aligned}$$

Quae aequationes per aequationes conditionales in has transformari possunt:

$$\begin{aligned}
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_2}{\partial q_3}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_2}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_2} = \left(\frac{\partial p_2}{\partial q_1}\right), \\
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_3}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_3}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_3}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_3} = \left(\frac{\partial p_3}{\partial q_1}\right), \\
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_4}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_4}{\partial q_3}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_4}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_4} = \left(\frac{\partial p_4}{\partial q_1}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_m}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_m}{\partial q_3}\right) + \cdots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_m} = \left(\frac{\partial p_m}{\partial q_1}\right).
\end{aligned}$$

Multiplicemus aequationem $2^{\text{am}}, 3^{\text{am}}, \ldots, (m-1)^{\text{tam}}$ per $\frac{\partial p_2}{\partial p_3}, \frac{\partial p_2}{\partial p_4}, \ldots, \frac{\partial p_2}{\partial p_m}$ et productarum summam deducamus a prima; multiplicemus aequationem 3^{am},

$4^{\text{tam}}, \ldots (m-1)^{\text{tam}}$ per $\frac{\partial p_3}{\partial p_4}, \frac{\partial p_3}{\partial p_5}, \ldots, \frac{\partial p_3}{\partial p_m}$ et productarum summam subducamus de secunda: multiplicemus aequationem 4^{tam}, $5^{\text{tam}}, \ldots (m-1)^{\text{tam}}$ per $\frac{\partial p_4}{\partial p_5}, \frac{\partial p_4}{\partial p_6}, \ldots \frac{\partial p_4}{\partial p_m}$ et productarum summam deducamus de tertia; et ita porro. Quibus patratis aliud eruimus systema aequationum, systemati primo aequivalens, hoc:

$$
(\mathrm{A})\left\{
\begin{aligned}
&(1)\quad \frac{\partial p_1}{\partial p_2}\frac{\partial p_2}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_2}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_2}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_2}{\partial q_5}+\cdots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_2}{\partial q_m}\\
&\qquad +\frac{\partial p_1}{\partial q_2}-\frac{\partial p_2}{\partial p_3}\frac{\partial p_1}{\partial q_3}-\frac{\partial p_2}{\partial p_4}\frac{\partial p_1}{\partial q_4}-\frac{\partial p_2}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\cdots-\frac{\partial p_2}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_2}{\partial q_1}\\
&(2)\quad \frac{\partial p_1}{\partial p_2}\frac{\partial p_3}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_3}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_3}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_3}{\partial q_5}+\cdots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_3}{\partial q_m}\\
&\qquad +\frac{\partial p_1}{\partial q_3}-\frac{\partial p_3}{\partial p_4}\frac{\partial p_1}{\partial q_4}-\frac{\partial p_3}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\cdots-\frac{\partial p_3}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_3}{\partial q_1}\\
&(3)\quad \frac{\partial p_1}{\partial p_2}\frac{\partial p_4}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_4}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_4}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_4}{\partial q_5}+\cdots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_4}{\partial q_m}\\
&\qquad +\frac{\partial p_1}{\partial q_4}-\frac{\partial p_4}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\cdots-\frac{\partial p_4}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_4}{\partial q_1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(m-2)\quad \frac{\partial p_1}{\partial p_2}\frac{\partial p_{m-1}}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_{m-1}}{\partial q_3}+\cdots+\frac{\partial p_1}{\partial p_{m-1}}\frac{\partial p_{m-1}}{\partial q_{m-1}}+\frac{\partial p_1}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m}\\
&\qquad +\frac{\partial p_1}{\partial q_{m-1}}-\frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_{m-1}}{\partial q_1}\\
&(m-1)\quad \frac{\partial p_1}{\partial p_2}\frac{\partial p_m}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_m}{\partial q_3}+\cdots+\frac{\partial p_1}{\partial p_{m-1}}\frac{\partial p_m}{\partial q_{m-1}}+\frac{\partial p_1}{\partial p_m}\frac{\partial p_m}{\partial q_m}\\
&\qquad +\frac{\partial p_1}{\partial q_m}=\frac{\partial p_m}{\partial q_1}.
\end{aligned}
\right.
$$

E quibus aequationibus differentialia partialia uncis inclusa evaserunt.

4.

Systema *secundum* aequationum conditionalium, quod respondet valori $i=2$, hoc est:

$$\left(\frac{\partial p_2}{\partial q_3}\right)=\left(\frac{\partial p_3}{\partial q_2}\right),\quad \left(\frac{\partial p_2}{\partial q_4}\right)=\left(\frac{\partial p_4}{\partial q_2}\right),\quad \ldots,\quad \left(\frac{\partial p_2}{\partial q_m}\right)=\left(\frac{\partial p_m}{\partial q_2}\right).$$

Designante k quemlibet e numeris $3, 4, \ldots, m$, aequatio

$$\left(\frac{\partial p_2}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_2}\right)$$

sic etiam exhiberi potest:

$$\frac{\partial p_2}{\partial p_3}\left(\frac{\partial p_3}{\partial q_k}\right)+\frac{\partial p_2}{\partial p_4}\left(\frac{\partial p_4}{\partial q_k}\right)+\cdots+\frac{\partial p_2}{\partial p_m}\left(\frac{\partial p_m}{\partial q_k}\right)+\frac{\partial p_2}{\partial q_k} = \left(\frac{\partial p_k}{\partial q_2}\right);$$

quae adhibendo aequationes

$$\left(\frac{\partial p_{k'}}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_{k'}}\right)$$

in sequentem abit:

$$\frac{\partial p_2}{\partial p_3}\left(\frac{\partial p_k}{\partial q_3}\right)+\frac{\partial p_2}{\partial p_4}\left(\frac{\partial p_k}{\partial q_4}\right)+\cdots+\frac{\partial p_2}{\partial p_m}\left(\frac{\partial p_k}{\partial q_m}\right)+\frac{\partial p_2}{\partial q_k} = \left(\frac{\partial p_k}{\partial q_2}\right).$$

Si aequationem 1^{am}, 2^{am}, ..., $(m-2)^{\text{tam}}$ vocamus, quae prodeunt ex aequatione praecedente loco k respective ponendo valores $3, 4, \ldots, m$, multiplicemus aequationem 2^{am}, 3^{am}, ..., $(m-2)^{\text{tam}}$ per $\frac{\partial p_3}{\partial p_4}$, $\frac{\partial p_3}{\partial p_5}$, ..., $\frac{\partial p_3}{\partial p_m}$ et productarum summam deducamus de prima; multiplicemus 3^{am}, 4^{tam}, ..., $(m-2)^{\text{tam}}$ per $\frac{\partial p_4}{\partial p_5}$, $\frac{\partial p_4}{\partial p_6}$, ..., $\frac{\partial p_4}{\partial p_m}$ et productarum summam deducamus de secunda; et ita porro. Eruetur his transactis systema aequationum hoc:

$$(\text{B})\begin{cases}
(1)\quad \frac{\partial p_2}{\partial p_3}\frac{\partial p_3}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_3}{\partial q_4}+\frac{\partial p_2}{\partial p_5}\frac{\partial p_3}{\partial q_5}+\cdots+\frac{\partial p_2}{\partial p_m}\frac{\partial p_3}{\partial q_m} \\
\qquad +\frac{\partial p_2}{\partial q_3}-\frac{\partial p_3}{\partial p_4}\frac{\partial p_2}{\partial q_4}-\frac{\partial p_3}{\partial p_5}\frac{\partial p_2}{\partial q_5}-\cdots-\frac{\partial p_3}{\partial p_m}\frac{\partial p_2}{\partial q_m} = \frac{\partial p_3}{\partial q_2} \\
(2)\quad \frac{\partial p_2}{\partial p_3}\frac{\partial p_4}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_4}{\partial q_4}+\frac{\partial p_2}{\partial p_5}\frac{\partial p_4}{\partial q_5}+\cdots+\frac{\partial p_2}{\partial p_m}\frac{\partial p_4}{\partial q_m} \\
\qquad +\frac{\partial p_2}{\partial q_4}-\frac{\partial p_4}{\partial p_5}\frac{\partial p_2}{\partial q_5}-\cdots-\frac{\partial p_4}{\partial p_m}\frac{\partial p_2}{\partial q_m} = \frac{\partial p_4}{\partial q_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(m-3)\quad \frac{\partial p_2}{\partial p_3}\frac{\partial p_{m-1}}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_{m-1}}{\partial q_4}+\cdots+\frac{\partial p_2}{\partial p_{m-1}}\frac{\partial p_{m-1}}{\partial q_{m-1}}+\frac{\partial p_2}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m} \\
\qquad +\frac{\partial p_2}{\partial q_{m-1}}-\frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_2}{\partial q_m} = \frac{\partial p_{m-1}}{\partial q_2} \\
(m-2)\quad \frac{\partial p_2}{\partial p_3}\frac{\partial p_m}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_m}{\partial q_4}+\cdots+\frac{\partial p_2}{\partial p_{m-1}}\frac{\partial p_m}{\partial q_{m-1}}+\frac{\partial p_2}{\partial p_m}\frac{\partial p_m}{\partial q_m} \\
\qquad +\frac{\partial p_2}{\partial q_m} = \frac{\partial p_m}{\partial q_2}.
\end{cases}$$

Quod aequationum systema e praecedente (A) eruitur, si indices omnes unitate augentur, quantum fieri per limites indicum potest.

5.

Prorsus eadem ratione demonstratur *generalis* aequatio:

$$(a)\quad \begin{cases} \frac{\partial p_i}{\partial p_{i+1}}\frac{\partial p_k}{\partial q_{i+1}}+\frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_k}{\partial q_{i+2}}+\frac{\partial p_i}{\partial p_{i+3}}\frac{\partial p_k}{\partial q_{i+3}}+\cdots+\frac{\partial p_i}{\partial p_m}\frac{\partial p_k}{\partial q_m} \\ \quad +\frac{\partial p_i}{\partial q_k}-\frac{\partial p_k}{\partial p_{k+1}}\frac{\partial p}{\partial q_{k+1}}-\frac{\partial p_k}{\partial p_{k+2}}\frac{\partial p_i}{\partial q_{k+2}}-\cdots-\frac{\partial p_k}{\partial p_m}\frac{\partial p_i}{\partial q_m}=\frac{\partial p_k}{\partial q_i}. \end{cases}$$

in qua i designare potest unumquemque e numeris $1, 2, 3, \ldots, m-1$ atque pro singulis ipsius i valoribus designare potest k numerum unumquemque ipso i majorem usque ad valorem $k=m$. Quae igitur aequatio generalis amplectitur numerum $\frac{m(m-1)}{2}$ aequationum inter se diversarum, quae e totidem aequationibus

$$\left(\frac{\partial p_i}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_i}\right)$$

derivatae sunt.

De forma usitata conditionum integrabilitatis ex ea, quae proponitur, derivanda.

6.

Vice versa ex aequationibus (a) deduci possunt aequationes conditionales initio propositae

$$\left(\frac{\partial p_i}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_i}\right),$$

sive demonstrari potest Theorema sequens:

Theorema I.

Supponatur:

p_1	*functio*	*quantitatum*	$p_2,\ p_3,\ p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m,$
p_2	-	-	$p_3,\ p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m,$
p_3	-	-	$p_4,\ p_5,\ \ldots,\ p_m,\ q_1,\ q_2,\ \ldots,\ q_m,$
. . .	. . .	. . .	. . . .
p_{m-1}	-	-	$p_m,\ q_1,\ q_2,\ \ldots,\ q_m,$
p_m	-	-	$q_1,\ q_2,\ \ldots,\ q_m,$

quae tales sint functiones, ut habeatur identice:

$$(a)\quad \begin{cases} 0 = -\dfrac{\partial p_k}{\partial q_\iota}+\dfrac{\partial p_\iota}{\partial p_{\iota+1}}\dfrac{\partial p_k}{\partial q_{\iota+1}}+\dfrac{\partial p_\iota}{\partial p_{\iota+2}}\dfrac{\partial p_k}{\partial q_{i+2}}+\cdots+\dfrac{\partial p_\iota}{\partial p_m}\dfrac{\partial p_k}{\partial q_m} \\ \qquad +\dfrac{\partial p_\iota}{\partial q_k}-\dfrac{\partial p_k}{\partial p_{k+1}}\dfrac{\partial p_\iota}{\partial q_{k+1}}-\dfrac{\partial p_k}{\partial p_{k+2}}\dfrac{\partial p_\iota}{\partial q_{k+2}}-\cdots-\dfrac{\partial p_k}{\partial p_m}\dfrac{\partial p_\iota}{\partial q_m}, \end{cases}$$

designante i unumquemque e numeris 1, 2, 3, ..., $m-1$ et pro singulis ipsius i valoribus designante k unumquemque e numeris $i+1$, $i+2$, ..., m, unde numerus totus aequationum est $\frac{m(m-1)}{2}$; erunt aequationes illae numero $\frac{m(m-1)}{2}$ conditiones quum necessariae tum sufficientes, ut, expressis omnibus p_1, p_2, ..., p_m per quantitates q_1, q_2, ..., q_m, expressio

$$p_1 dq_1+p_2 dq_2+\cdots+p_m dq_m$$

evadat differentiale completum.

Forma secunda conditionum integrabilitatis.

7.

Conditiones illas esse necessarias antecedentibus comprobavi, quippe quas locum habere demonstravi, quoties expressio

$$p_1 dq_1+p_2 dq_2+\cdots+p_m dq_m$$

differentiale completum sit. Iam demonstrabo, easdem conditiones esse *sufficientes*, sive, quoties aequationes illae numero $\frac{m(m-1)}{2}$ locum habeant, expressionem

$$p_1 dq_1+p_2 dq_2+\cdots+p_m dq_m$$

esse differentiale completum.

Posito $k = m$, aequatio proposita fit:

$$(1)\quad 0 = -\left(\frac{\partial p_m}{\partial q_\iota}\right)+\frac{\partial p_\iota}{\partial p_{\iota+1}}\left(\frac{\partial p_m}{\partial q_{\iota+1}}\right)+\frac{\partial p_\iota}{\partial p_{\iota+2}}\left(\frac{\partial p_m}{\partial q_{\iota+2}}\right)+\cdots+\frac{\partial p_\iota}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right)+\frac{\partial p_\iota}{\partial q_m}.$$

Uncis rursus utimur, si p_1, p_2, ..., p_m ut solarum q_1, q_2, ..., q_m functiones spectamus, unde pro ipsa p_m perinde scribi potest $\frac{\partial p_m}{\partial q_\iota}$ sive $\left(\frac{\partial p_m}{\partial q_\iota}\right)$.

Posito $k = m-1$, fit:

$$0=-\frac{\partial p_{m-1}}{\partial q_\iota}+\frac{\partial p_\iota}{\partial p_{\iota+1}}\frac{\partial p_{m-1}}{\partial q_{\iota+1}}+\frac{\partial p_\iota}{\partial p_{\iota+2}}\frac{\partial p_{m-1}}{\partial q_{\iota+2}}+\cdots+\frac{\partial p_\iota}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m}$$
$$+\frac{\partial p_\iota}{\partial q_{m-1}}-\frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_\iota}{\partial q_m}.$$

Cui aequationi si addimus aequationem (1) multiplicatam per $\frac{\partial p_{m-1}}{\partial p_m}$, prodit:

$$(2)\quad 0=-\left(\frac{\partial p_{m-1}}{\partial q_\iota}\right)+\frac{\partial p_\iota}{\partial p_{\iota+1}}\left(\frac{\partial p_{m-1}}{\partial q_{\iota+1}}\right)+\frac{\partial p_\iota}{\partial p_{\iota+2}}\left(\frac{\partial p_{m-1}}{\partial q_{\iota+2}}\right)+\cdots+\frac{\partial p_\iota}{\partial p_m}\left(\frac{\partial p_{m-1}}{\partial q_m}\right)$$
$$+\frac{\partial p_\iota}{\partial q_{m-1}}.$$

Posito $k=m-2$, fit:

$$0=-\frac{\partial p_{m-2}}{\partial q_\iota}+\frac{\partial p_\iota}{\partial p_{\iota+1}}\frac{\partial p_{m-2}}{\partial q_{\iota+1}}+\frac{\partial p_i}{\partial p_{\iota+2}}\frac{\partial p_{m-2}}{\partial q_{\iota+2}}+\cdots+\frac{\partial p_\iota}{\partial p_m}\frac{\partial p_{m-2}}{\partial q_m}$$
$$+\frac{\partial p_i}{\partial q_{m-2}}-\frac{\partial p_{m-2}}{\partial p_{m-1}}\frac{\partial p_\iota}{\partial q_{m-1}}-\frac{\partial p_{m-2}}{\partial p_m}\frac{\partial p_\iota}{\partial q_m}.$$

Cui aequationi addo aequationem (1) per $\frac{\partial p_{m-2}}{\partial p_m}$ et aequationem (2) per $\frac{\partial p_{m-2}}{\partial p_{m-1}}$ multiplicatam, prodit:

$$(3)\quad 0=-\left(\frac{\partial p_{m-2}}{\partial q_\iota}\right)+\frac{\partial p_\iota}{\partial p_{\iota+1}}\left(\frac{\partial p_{m-2}}{\partial q_{\iota+1}}\right)+\frac{\partial p_\iota}{\partial p_{\iota+2}}\left(\frac{\partial p_{m-2}}{\partial q_{\iota+2}}\right)+\cdots+\frac{\partial p_\iota}{\partial p_m}\left(\frac{\partial p_{m-2}}{\partial q_m}\right)$$
$$+\frac{\partial p_\iota}{\partial q_{m-2}}.$$

Et ita continuando demonstras aequationem *generalem:*

$$(b)\quad 0=-\left(\frac{\partial p_k}{\partial q_\iota}\right)+\frac{\partial p_\iota}{\partial p_{\iota+1}}\left(\frac{\partial p_k}{\partial q_{\iota+1}}\right)+\frac{\partial p_\iota}{\partial p_{\iota+2}}\left(\frac{\partial p_k}{\partial q_{\iota+2}}\right)+\cdots+\frac{\partial p_\iota}{\partial p_m}\left(\frac{\partial p_k}{\partial q_m}\right)$$
$$+\frac{\partial p_\iota}{\partial q_k},$$

in qua k valores omnes induere potest m, $m-1$, $m-2$, ... usque ad $i+1$. Unde, si ipsi i rursus valores 1, 2, 3, ..., $m-1$ tribuuntur, numerus aequationum (b) fit $\frac{m(m-1)}{2}$.