DISQUISITIONES ANALYTICÆ

DE

FRACTIONIBUS SIMPLICIBUS

DISSERTATIO INAUGURALIS

QUAM

AMPLISSIMO PHILOSOPHORUM ORDINI

PRO

SUMMIS IN PHILOSOPHIA HONORIBUS

IN

UNIVERSITATE LITTERARIA BEROLINENSI RITE ADIPISCENDIS

EXHIBUIT AUCTOR

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DISQUISITIONES ANALYTICÆ DE FRACTIONIBUS SIMPLICIBUS.

Sectio I.

Demonstratur theorema ab Ill°. Lagrange sine demonstratione propositum.

1.

Mirum videri possit, et fortasse temerarium, si quis in materia inde a primis recentioris Analyseos temporibus a plurimis mathematicis tractata, quam igitur iure optimo decantatam dicere licet, vel novi quid velit afferre, vel ita rem attingere, ut ne acta egisse videatur. Iam vero fractionum simplicium theoriam ita fere decantatam esse vel inde patet, quod mathematici omnes, qui de serierum recurrentium theoria, omnes, qui de calculi integralis elementis egerunt, etiam de illis agere debuerunt. Sane nos quoque ista turba deterruisset, nisi casu in manus incidisset commentatio Illi. Lagrange, quae in Actis Academiae nostrae Berolinensis a. 1792—1793 legitur. Ibi enim, dum ille formulas quasdam in Actis eiusdem Academiae a. 1775 ab ipso exhibitas retractat, curiosam movit quaestionem de eiusmodi fractionum simplicium expressione investiganda, quae etsi denominatorum fractionum simplicium vel duo vel plures inter se aequales evadant, immutata maneret, ita ut ad speciem absurdi, quae istis casibus subnascitur, declinandam, non opus sit ad analyticam confugere transformationem. Ipse eiusmodi expressionem in medium profert, quam ut directa quadam methodo demonstrent, invitat geometras, cum ipse formulae propositae non addiderit demonstrationem. Unde in his quoque non ita omnia absoluta esse videbantur. Quod autem dico, hoc est.

2.

Propositam aliquam fractionem

$$\frac{f(x)}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)},$$

designante $f(x)$ functionem elementi x integram rationalem*) huiusmodi schematis

$$Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\cdots+P,$$

notum est in has resolvi posse fractiones simplices

$$\frac{f(\alpha_1)}{(x-\alpha_1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)}$$
$$+\frac{f(\alpha_2)}{(x-\alpha_2)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\dots(\alpha_2-\alpha_n)}$$
$$+\frac{f(\alpha_3)}{(x-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\dots(\alpha_3-\alpha_n)}$$
$$\vdots$$
$$+\frac{f(\alpha_n)}{(x-\alpha_n)(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\dots(\alpha_n-\alpha_{n-1})}.^{**)}$$

Quia, posito denominatore

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)=\varphi(x),$$

facile patet, fore

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)=\varphi'(\alpha_1)^{***)}$$
$$(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\dots(\alpha_2-\alpha_n)=\varphi'(\alpha_2)$$
$$(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\dots(\alpha_3-\alpha_n)=\varphi'(\alpha_3)$$
$$\vdots$$
$$(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\dots(\alpha_n-\alpha_{n-1})=\varphi'(\alpha_n),$$

fractiones illas simplices ita quoque scribere licet

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)}+\frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)}+\frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)}+\cdots+\frac{f(\alpha_n)}{(x-\alpha_n)\varphi'(\alpha_n)}.$$

3.

Iam ubi quantitatum $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ aliquot aequales fiunt, expressionum $\varphi'(\alpha_1), \varphi'(\alpha_2), \varphi'(\alpha_3), \dots, \varphi'(\alpha_n)$ totidem evanescunt, totidem fractionum simplicium

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)},\quad \frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)},\quad \frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)},\quad \dots,\quad \frac{f(\alpha_n)}{(x-\alpha_n)\varphi'(\alpha_n)}$$

in infinitum abeunt. Scilicet ubi erit e. g.

$$\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_m,$$

*) Euleri Introd. in Anal. Infin. Lib. I. Cap. I. §§ 8, 9.

**) Ubi $f(x)$ ad altiorem quam $(n-1)^{\text{tum}}$ gradum ascendit, quo casu fractio

$$\frac{f(x)}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)}$$

spuria dici solet, i. e. functio rationalis ex integra et fracta conflata: fractiones illae simplices genuinam fractionem exprimunt in spuria illa latitantem.

***) Hic et in sequentibus, duce Ill$^{\text{o}}$. Lagrange, brevitatis causa ponimus $\frac{d\varphi(x)}{dx}=\varphi'(x)$, $\frac{d^2\varphi(x)}{dx^2}=\varphi''(x)$, $\frac{d^3\varphi(x)}{dx^3}=\varphi'''(x)$, et in genere $\frac{d^n\varphi(x)}{dx^n}=\varphi^{(n)}(x)$; ubi tamen melius iudicabitur, veterem quoque designandi modum adhibebimus.

quorum valorem communem ponamus $=\alpha$: denominator $\varphi(x)$ factorem $(x-\alpha)^m$ continebit, quo factore $(x-\alpha)^m$ fractiones nasci constat simplices huiusmodi

$$\frac{a}{(x-\alpha)^m}+\frac{b}{(x-\alpha)^{m-1}}+\frac{c}{(x-\alpha)^{m-2}}+\cdots+\frac{p}{(x-\alpha)},$$

unde anterius fractionum simplicium schema omnino mutatur. Cum vero formulae alicuius schema suppositione quadam prorsus mutatur, per absurdi speciem id plerumque indicatur, sicuti hoc loco fractiones simplices

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)},\quad \frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)},\quad \frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)},\quad \ldots,\quad \frac{f(\alpha_m)}{(x-\alpha_m)\varphi'(\alpha_m)}$$

in infinitum abeunt; ita ut aut quaestio ea suppositione facta de integro retractanda sit, aut ad analyticam transformationem confugere debeamus, qua ista absurdi species declinetur.

Iam illi quidem numeratores $a, b, c, \ldots, p$ facile consideratione sequenti inveniuntur. Sit enim $\varphi(x)=(x-\alpha)^m\psi(x)$, ita ut poni possit

$$\frac{f(x)}{\varphi(x)}=\frac{F(x)}{\psi(x)}+\frac{a}{(x-\alpha)^m}+\frac{b}{(x-\alpha)^{m-1}}+\frac{c}{(x-\alpha)^{m-2}}+\cdots+\frac{p}{x-\alpha}.$$

Iam evoluta fractione proposita $\frac{f(x)}{\varphi(x)}$ ad dignitates ascendentes quantitatis $(x-\alpha)$, negativae, quae in illa evolutione inveniuntur, quantitatis $(x-\alpha)$ dignitates hae ipsae evadunt fractiones simplices, in quas inquirimus:

$$\frac{a}{(x-\alpha)^m}+\frac{b}{(x-\alpha)^{m-1}}+\frac{c}{(x-\alpha)^{m-2}}+\cdots+\frac{p}{x-\alpha}.$$

Quia enim $\psi(x)$ factorem $(x-\alpha)$ non continere supponitur, in fractione $\frac{F(x)}{\psi(x)}$ evoluta ad ascendentes quantitatis $(x-\alpha)$ dignitates, negativae eius dignitates inveniri non possunt.

Ut ipsam indicemus evolutionem, posito

$$\frac{f(x)}{\psi(x)}=\Pi(x),$$

unde fractio proposita $\frac{f(x)}{\varphi(x)}=\frac{\Pi(x)}{(x-\alpha)^m}$, e theoremate Tayloriano fit

$$\Pi(x)=\Pi(\alpha+x-\alpha)=\Pi(\alpha)+\Pi'(\alpha)(x-\alpha)+\frac{\Pi''(\alpha)(x-\alpha)^2}{1.2}+\frac{\Pi'''(\alpha)(x-\alpha)^3}{1.2.3.}+\text{ etc.}$$

Hinc erit

$$\frac{f(x)}{\varphi(x)}=\frac{\Pi(x)}{(x-\alpha)^m}=\frac{\Pi(\alpha)}{(x-\alpha)^m}+\frac{\Pi'(\alpha)}{(x-\alpha)^{m-1}}+\frac{\Pi''(\alpha)}{1.2.(x-\alpha)^{m-2}}+\frac{\Pi'''(\alpha)}{1.2.3.(x-\alpha)^{m-3}}+\text{ etc.},$$

unde statim quaesitas quantitates invenimus

$$a=\Pi(\alpha),\quad b=\Pi'(\alpha),\quad c=\frac{\Pi''(\alpha)}{1.2},\quad \ldots,\quad p=\frac{\Pi^{(m-1)}(\alpha)}{1.2.3\ldots(m-1)}.$$

Etsi non formulam, quam Ill. Lagrange voluit, methodum certe iam tradidimus, quae eadem manet, quicunque sit numerus m, seu quotiescunque denominator $\varphi(x)$ factorem $(x-\alpha)$ contineat.

4.

Operae tamen pretium esse videbatur Analystis inquirere, quomodo hae formulae ex ipsa expressione §. 2 exhibita

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)}+\frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)}+\frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)}+\cdots+\frac{f(\alpha_n)}{(x-\alpha_n)\varphi'(\alpha_n)},$$

eo casu quo erit $\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_m=\alpha$, analytica transformatione deducerentur. Quam quaestionem inter alios video suscepisse Illm. Malfatti in commentatione doctissima inscripta: *delle serie ricorrenti.* V. *Memorie di Matematica e Fisica della Societa Italiana*, Tom. III. pag. 571—663. Ibi ille, quas Illm. Lagrange in Actis Academiae nostrae a. 1775 sine demonstratione ea de re tradiderat formulas, falsas esse demonstravit, correctasque adstruxit, per calculos tamen valde prolixos et taediosos incedens. (Plus XL illi paginas occupant.) Rem postea retractavit ipse Lagrange, iam a me citatus, in Actis Academiae nostrae Berolinensis a. 1792—93. Uterque eo artificio alibi etiam saepissime adhibito usus est, quod quantitates

$$\alpha_1,\quad \alpha_2,\quad \alpha_3,\quad \ldots,\quad \alpha_m$$

non quidem aequales ab initio, sed quantitate infinite parva diversas statuerent. Quam denuo aggredi quaestionem operae pretium videbatur; quem ad finem duo antea proponamus lemmata, quae generaliori usui inservire possunt.

5.

Lemma I.

Posito $F(x)=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)$, fractionem

$$\frac{1}{F(x)}=\frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)}$$

in simplices resolutam vidimus fieri (§. 2)

$$\frac{1}{(x-\alpha_1)F'(\alpha_1)}+\frac{1}{(x-\alpha_2)F'(\alpha_2)}+\frac{1}{(x-\alpha_3)F'(\alpha_3)}+\cdots+\frac{1}{(x-\alpha_m)F'(\alpha_m)}.$$

Iam evoluta fractione

$$\frac{1}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)}$$

in seriem secundum descendentes elementi x dignitates procedentem, fit

$$\frac{1}{F(x)} = \frac{1}{x^m} + \frac{'\overset{1}{C}}{x^{m+1}} + \frac{'\overset{2}{C}}{x^{m+2}} + \frac{'\overset{3}{C}}{x^{m+3}} + \text{etc.}$$

$$[\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots\ \alpha_m],$$

ubi per characteres

$$'\overset{1}{C},\quad '\overset{2}{C},\quad '\overset{3}{C},\quad ,\quad \text{etc.}$$

$$[\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots,\ \alpha_m]$$

more inter Analystas Germanos recepto combinationes designantur singulorum (i. summa), binorum, ternorum, etc. ex elementis

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots,\ \alpha_m,$$

quae indice subscripto indicantur, ipsis elementorum admissis repetitionibus. (V. *Euleri Introd. in Anal. Infinit. l. I. cap. XV. §. 270.*)

Evolutis igitur etiam fractionibus simplicibus, in quas fractionem $\frac{1}{F(x)}$ resolvimus, in seriem secundum descendentes elementi x dignitates procedentem, singularum dignitatum coëfficientes in utraque evolutione inter se comparando, sequentes eruimus aequationes satis memorabiles:

$$\frac{1}{F'(\alpha_1)} + \frac{1}{F'(\alpha_2)} + \frac{1}{F'(\alpha_3)} + \cdots + \frac{1}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1}{F'(\alpha_1)} + \frac{\alpha_2}{F'(\alpha_2)} + \frac{\alpha_3}{F'(\alpha_3)} + \cdots + \frac{\alpha_m}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^2}{F'(\alpha_1)} + \frac{\alpha_2^2}{F'(\alpha_2)} + \frac{\alpha_3^2}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^2}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^3}{F'(\alpha_1)} + \frac{\alpha_2^3}{F'(\alpha_2)} + \frac{\alpha_3^3}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^3}{F'(\alpha_m)} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\alpha_1^{m-2}}{F'(\alpha_1)} + \frac{\alpha_2^{m-2}}{F'(\alpha_2)} + \frac{\alpha_3^{m-2}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m-2}}{F'(\alpha_m)} = 0$$

$$\frac{\alpha_1^{m-1}}{F'(\alpha_1)} + \frac{\alpha_2^{m-1}}{F'(\alpha_2)} + \frac{\alpha_3^{m-1}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m-1}}{F'(\alpha_m)} = 1$$

$$\frac{\alpha_1^{m}}{F'(\alpha_1)} + \frac{\alpha_2^{m}}{F'(\alpha_2)} + \frac{\alpha_3^{m}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m}}{F'(\alpha_m)} = '\overset{1}{C}$$

$$\frac{\alpha_1^{m+1}}{F'(\alpha_1)} + \frac{\alpha_2^{m+1}}{F'(\alpha_2)} + \frac{\alpha_3^{m+1}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m+1}}{F'(\alpha_m)} = '\overset{2}{C}$$

$$\frac{\alpha_1^{m+2}}{F'(\alpha_1)} + \frac{\alpha_2^{m+2}}{F'(\alpha_2)} + \frac{\alpha_3^{m+2}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m+2}}{F'(\alpha_m)} = '\overset{3}{C}$$

etc. etc.,

sive in universum

$$\frac{\alpha_1^{m+p}}{F'(\alpha_1)} + \frac{\alpha_2^{m+p}}{F'(\alpha_2)} + \frac{\alpha_3^{m+p}}{F'(\alpha_3)} + \cdots + \frac{\alpha_m^{m+p}}{F'(\alpha_m)} = '\overset{p+1}{C},$$

in quibus formulis characteres $\overset{1}{'C}$, $\overset{2}{'C}$, $\overset{3}{'C}$, et in universum $\overset{p+1}{'C}$ ad indicem communem

$$[\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m]$$

referendi sunt.

6.

Lemma II.

Iam designante $\chi(x)$ functionem aliquam elementi x, investigetur, quaenam evadat expressio

$$\frac{\chi(\alpha)}{F'(\alpha_1)}+\frac{\chi(\alpha_2)}{F'(\alpha_2)}+\frac{\chi(\alpha_3)}{F'(\alpha_3)}+\cdots+\frac{\chi(\alpha_m)}{F'(\alpha_m)}$$

posito $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m$, quo casu singulas has fractiones in infinitum abire videmus.

Ponatur $\alpha_1 = \alpha + h_1$, $\alpha_2 = \alpha + h_2$, $\alpha_3 = \alpha + h_3$, $\ldots$, $\alpha_m = \alpha + h_m$; atque $\Phi(x) = (x-h_1)(x-h_2)(x-h_3)\ldots(x-h_m)$; unde erit

$$\Phi(x-\alpha) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m) = F(x).$$

Iam ex aequatione $F(x) = \Phi(x-\alpha)$ sequitur $F'(x) = \Phi'(x-\alpha)$, unde

$$\begin{aligned}
F'(\alpha_1) &= \Phi'(\alpha_1-\alpha) = \Phi'(h_1)\\
F'(\alpha_2) &= \Phi'(\alpha_2-\alpha) = \Phi'(h_2)\\
F'(\alpha_3) &= \Phi'(\alpha_3-\alpha) = \Phi'(h_3)\\
&\ldots\ldots\ldots\ldots\\
F'(\alpha_m) &= \Phi'(\alpha_m-\alpha) = \Phi'(h_m).
\end{aligned}$$

Hinc expressio proposita

$$\frac{\chi(\alpha_1)}{F'(\alpha_1)}+\frac{\chi(\alpha_2)}{F'(\alpha_2)}+\frac{\chi(\alpha_3)}{F'(\alpha_3)}+\cdots+\frac{\chi(\alpha_m)}{F'(\alpha_m)}$$

in hanc abit

$$\frac{\chi(\alpha+h_1)}{\Phi'(h_1)}+\frac{\chi(\alpha+h_2)}{\Phi'(h_2)}+\frac{\chi(\alpha+h_3)}{\Phi'(h_3)}+\cdots+\frac{\chi(\alpha+h_m)}{\Phi'(h_m)}.$$

Quod evolutum secundum theorema Taylorianum ponatur

$$\begin{aligned}
&= A_0\chi(\alpha)+A_1\chi'(\alpha)+\frac{A_2\chi''(\alpha)}{1.2}+\frac{A_3\chi'''(\alpha)}{1.2.3}+\cdots+\frac{A_{m-2}\chi^{(m-2)}(\alpha)}{1.2\ldots(m-2)}\\
&+\frac{A_{m-1}\chi^{(m-1)}(\alpha)}{1.2\ldots(m-1)}+\frac{A_m\chi^{(m)}(\alpha)}{1.2\ldots m}+\frac{A_{m+1}\chi^{(m+1)}(\alpha)}{1.2\ldots(m+1)}+\frac{A_{m+2}\chi^{(m+2)}(\alpha)}{1.2\ldots(m+2)}+\text{ etc.};
\end{aligned}$$

unde erit

$$\begin{aligned}
A_0 &= \frac{1}{\Phi'(h_1)} + \frac{1}{\Phi'(h_2)} + \frac{1}{\Phi'(h_3)} + \cdots + \frac{1}{\Phi'(h_m)} \\
A_1 &= \frac{h_1}{\Phi'(h_1)} + \frac{h_2}{\Phi'(h_2)} + \frac{h_3}{\Phi'(h_3)} + \cdots + \frac{h_m}{\Phi'(h_m)} \\
A_2 &= \frac{h_1^2}{\Phi'(h_1)} + \frac{h_2^2}{\Phi'(h_2)} + \frac{h_3^2}{\Phi'(h_3)} + \cdots + \frac{h_m^2}{\Phi'(h_m)} \\
A_3 &= \frac{h_1^3}{\Phi'(h_1)} + \frac{h_2^3}{\Phi'(h_2)} + \frac{h_3^3}{\Phi'(h_3)} + \cdots + \frac{h_m^3}{\Phi'(h_m)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_{m-2} &= \frac{h_1^{m-2}}{\Phi'(h_1)} + \frac{h_2^{m-2}}{\Phi'(h_2)} + \frac{h_3^{m-2}}{\Phi'(h_3)} + \cdots + \frac{h_m^{m-2}}{\Phi'(h_m)} \\
A_{m-1} &= \frac{h_1^{m-1}}{\Phi'(h_1)} + \frac{h_2^{m-1}}{\Phi'(h_2)} + \frac{h_3^{m-1}}{\Phi'(h_3)} + \cdots + \frac{h_m^{m-1}}{\Phi'(h_m)} \\
A_m &= \frac{h_1^m}{\Phi'(h_1)} + \frac{h_2^m}{\Phi'(h_2)} + \frac{h_3^m}{\Phi'(h_3)} + \cdots + \frac{h_m^m}{\Phi'(h_m)} \\
A_{m+1} &= \frac{h_1^{m+1}}{\Phi'(h_1)} + \frac{h_2^{m+1}}{\Phi'(h_2)} + \frac{h_3^{m+1}}{\Phi'(h_3)} + \cdots + \frac{h_m^{m+1}}{\Phi'(h_m)} \\
A_{m+2} &= \frac{h_1^{m+2}}{\Phi'(h_1)} + \frac{h_2^{m+2}}{\Phi'(h_2)} + \frac{h_3^{m+2}}{\Phi'(h_3)} + \cdots + \frac{h_m^{m+2}}{\Phi'(h_m)} \\
&\text{etc. etc.}
\end{aligned}$$

Iam e lemmate I. (§. 5) sequitur, ubi loco $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$ ponitur $h_1, h_2, h_3, \ldots, h_m$, atque $\Phi(x)$ loco $F(x)$,

$$\begin{gathered}
A_0 = 0, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \quad \ldots, \quad A_{m-2} = 0, \\
A_{m-1} = 1, \quad A_m = '\overset{1}{C}, \quad A_{m+1} = '\overset{2}{C}, \quad A_{m-2} = '\overset{3}{C}, \quad \text{etc. etc.,}
\end{gathered}$$

characteribus $'\overset{1}{C}$, $'\overset{2}{C}$, $'\overset{3}{C}$, etc. relatis ad indicem communem:

$$[h_1, h_2, h_3, \ldots, h_m];$$

ita ut sit expressio nostra proposita

$$\frac{\chi(\alpha_1)}{F'(\alpha_1)} + \frac{\chi(\alpha_2)}{F'(\alpha_2)} + \frac{\chi(\alpha_3)}{F'(\alpha_3)} + \cdots + \frac{\chi(\alpha_m)}{F'(\alpha_m)}$$

$$= \frac{\chi^{(m-1)}(\alpha)}{1.2\ldots(m-1)} + \frac{'\overset{1}{C}\chi^{(m)}(\alpha)}{1.2\ldots m} + \frac{'\overset{2}{C}\chi^{(m+1)}(\alpha)}{1.2\ldots(m+1)} + \frac{'\overset{3}{C}\chi^{(m+2)}(\alpha)}{1.2\ldots(m+2)} + \text{etc.}$$

$$[h_1, h_2, h_3, \ldots, h_m]$$

Posito

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m = \alpha,$$

exsistit

$$h_1 = h_2 = h_3 = \cdots = h_m = 0,$$

ita ut etiam, ea suppositione facta,

$$'\overset{1}{C}=0,\quad '\overset{2}{C}=0,\quad '\overset{3}{C}=0,\quad \text{etc.},$$
$$[h_1,\ h_2,\ h_3,\ \ldots,\ h_m]$$

unde expressio proposita fit

$$\frac{\chi^{(m-1)}(\alpha)}{1.2\ldots(m-1)}=\frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}\chi(\alpha)}{d\alpha^{m-1}}.$$

7.

Iam ad propositam quaestionem redeamus. Quaesivimus enim, resoluta fractione

$$\frac{f(x)}{\varphi(x)}=\frac{f(x)}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)}$$

in simplices hasce

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)}+\frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)}+\frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)}+\cdots+\frac{f(\alpha_n)}{(x-\alpha_n)\varphi'(\alpha_n)},$$

quaenam evadant fractiones simplices, quae e denominatoris $\varphi(x)$ factoribus $(x-\alpha_1)$, $(x-\alpha_2)$, $(x-\alpha_3)$, $\ldots$, $(x-\alpha_m)$ ortum ducunt, videlicet

$$\frac{f(\alpha_1)}{(x-\alpha_1)\varphi'(\alpha_1)}+\frac{f(\alpha_2)}{(x-\alpha_2)\varphi'(\alpha_2)}+\frac{f(\alpha_3)}{(x-\alpha_3)\varphi'(\alpha_3)}+\cdots+\frac{f(\alpha_m)}{(x-\alpha_m)\varphi'(\alpha_m)},$$

casu quo erit $\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_m=\alpha$.

Posito $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_m)=F(x)$, atque

$$\frac{f(x)}{(x-\alpha_{m+1})(x-\alpha_{m+2})(x-\alpha_{m+3})\ldots(x-\alpha_n)}=\Pi(x),$$

fit fractio proposita

$$\frac{f(x)}{\varphi(x)}=\frac{\Pi(x)}{F(x)},$$

unde etiam

$$\frac{\varphi(x)}{f(x)}=\frac{F(x)}{\Pi(x)},$$

qua differentiata aequatione prodit

$$\frac{\varphi'(x)}{f(x)}+\varphi(x)\left(\frac{1}{f(x)}\right)'=\frac{F'(x)}{\Pi(x)}+F(x)\left(\frac{1}{\Pi(x)}\right)',$$

unde in locum elementi x substitutis α_1, α_2, α_3, $\ldots$, α_m, quia ea substitutione facta evanescunt $\varphi(x)$ atque $F(x)$, fit

$$\frac{\varphi'(\alpha_1)}{f(\alpha_1)}=\frac{F'(\alpha_1)}{\Pi(\alpha_1)},\quad \frac{\varphi'(\alpha_2)}{f(\alpha_2)}=\frac{F'(\alpha_2)}{\Pi(\alpha_2)},\quad \frac{\varphi'(\alpha_3)}{f(\alpha_3)}=\frac{F'(\alpha_3)}{\Pi(\alpha_3)},\quad \ldots,\quad \frac{\varphi'(\alpha_m)}{f(\alpha_m)}=\frac{F'(\alpha_m)}{\Pi(\alpha_m)}$$