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Algebra, with Arithmetic and Mensuration

The scholar and East India Company administrator Henry Thomas Colebrooke (1765–1837) brought India's rich mathematical heritage to the attention of the wider world with the publication of this book in 1817. Based on Sanskrit texts, it contains English translations of classic works by the Indian mathematicians and astronomers Brahmagupta (598–668) and Bhascara (1114–85), who were instrumental thinkers in the development of algebra. Included here are translations of chapters 12 and 18 of Brahmagupta's best-known work, *Brahmasphutasiddhanta*, focusing on arithmetic and algebra respectively. Also included in this book are translations of two of the greatest works by Bhascara: *Lilavati*, his treatise on arithmetic, and *Bijaganita*, on algebra. Furthermore, Colebrooke's introduction aims to position the Indian advancement of algebra in relation to its development by the Greeks and Arabs.

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Algebra, with Arithmetic and Mensuration

From the Sanscrit of Brahmegupta and Bhascara

TRANSLATED BY H.T. COLEBROOKE

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SANSCRIT.

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BRAHMEGUPTA AND BHÁSCARA.

TRANSLATED BY

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DISSERTATION.

THE history of sciences, if it want the prepossessing attractions of political history and narration of events, is nevertheless not wholly devoid of interest and instruction. A laudable curiosity prompts to inquire the sources of knowledge; and a review of its progress furnishes suggestions tending to promote the same or some kindred study. We would know the people and the names at least of the individuals, to whom we owe particular discoveries and successive steps in the advancement of knowledge. If no more be obtained by the research, still the inquiry has not been wasted, which points aright the gratitude of mankind.

In the history of mathematical science, it has long been a question to whom the invention of Algebraic analysis is due? among what people, in what region, was it devised? by whom was it cultivated and promoted? or by whose labours was it reduced to form and system? and finally from what quarter did the diffusion of its knowledge proceed? No doubt indeed is entertained of the source from which it was received immediately by modern Europe; though the channel have been a matter of question. We are well assured, that the Arabs were mediately or immediately our instructors in this study. But the Arabs themselves scarcely pretend to the discovery of Algebra. They were not in general inventors but scholars, during the short period of their successful culture of the sciences: and the germ at least of the Algebraic analysis is to be found among the Greeks in an age not precisely determined, but more than probably anterior to the earliest dawn of civilization among the Arabs: and this science in a more advanced state subsisted among the Hindus prior to the earliest disclosure of it by the Arabians to modern Europe.

The object of the present publication is to exhibit the science in the state in which the Hindus possessed it, by an exact version of the most approved treatise on it in the ancient language of India, with one of the earlier treatises (the only extant one) from which it was compiled. The design of this preliminary dissertation is to deduce from these and from the evidence which will be here offered, the degree of advancement to which the science had arrived in a remote age. Observations will be added, tending to a comparison of the Indian, with the Arabian, the Grecian, and the modern Algebra: and the subject will be left to the consideration of the learned, for a conclusion to be drawn by them from the internal, no less than the external proof, on the question who can best vindicate a claim to the merit of having originally invented or first improved the methods of computation and analysis, which are the groundwork of both the simple and abstruser parts of Mathematics; that is, Arithmetic and Algebra: so far at least as the ancient inventions are affected; and also in particular points, where recent discoveries are concerned.

In the actual advanced condition of the analytic art, it is not hoped, that this version of ancient *Sanscrit* treatises on Algebra, Arithmetic, and Mensuration, will add to the resources of the art, and throw new light on mathematical science, in any other respect, than as concerns its history. Yet the remark may not seem inapposite, that had an earlier version of these treatises been completed, had they been translated and given to the public, when the notice of mathematicians was first drawn to the attainments of the Hindus in astronomy and in sciences connected with it, some addition would have been then made to the means and resources of Algebra for the general solution of problems by methods which have been re-invented, or have been perfected, in the last age.

The treatises in question, which occupy the present volume, are the *Vîja-gańita* and *Lílávatí* of BHÁSCARA ÁCHÁRYA and the *Gańitád'haya* and *Cuítacád'hyaya* of BRAHMEGUPTA. The two first mentioned constitute the preliminary portion of BHÁSCARA's Course of Astronomy, entitled *Sidd'hánta-śirómańi*. The two last are the twelfth and eighteenth chapters of a similar course of astronomy, by BRAHMEGUPTA, entitled *Brahma-sidd'hánta*.

The questions to be first examined in relation to these works are their authenticity and their age. To the consideration of those points we now proceed.

The period when BHÁSCARA, the latest of the authors now named, flourished, and the time when he wrote, are ascertained with unusual precision.

He completed his great work, the *Sidd'hánta-sirómańi*, as he himself informs us in a passage of it,[1] in the year 1072 *Saca*. This information receives corroboration, if any be wanted, from the date of another of his works, the *Carańa-cutuhala*, a practical astronomical treatise, the epoch of which is 1105 *Saca*;[2] 33 years subsequent to the completion of the systematic treatise. The date of the *Sidd'hánta-śirómańi*, of which the *Víja-gańita* and *Lílávatí* are parts, is fixt then with the utmost exactness, on the most satisfactory grounds, at the middle of the twelfth century of the Christian era, A.D. 1150.[3]

The genuineness of the text is established with no less certainty by numerous commentaries in *Sanscrit*, besides a Persian version of it. Those commentaries comprise a perpetual gloss, in which every passage of the original is noticed and interpreted: and every word of it is repeated and explained. A comparison of them authenticates the text where they agree; and would serve, where they did not, to detect any alterations of it that might have taken place, or variations, if any had crept in, subsequent to the composition of the earliest of them. A careful collation of several commentaries,[4] and of three copies of the original work, has been made; and it will be seen in the notes to the translation how unimportant are the discrepancies.

From comparison and collation, it appears then, that the work of BHÁSCARA, exhibiting the same uniform text, which the modern transcripts of it do, was in the hands of both Mahommedans and Hindus between two and three centuries ago: and, numerous copies of it having been diffused throughout India, at an earlier period, as of a performance held in high estimation, it was the subject of study and habitual reference in countries and places so remote from each other as the north and west of India and the southern peninsula: or, to speak with the utmost precision, *Jambusara* in the west, *Agra* in North *Hindustan*, and *Párthapúra*, *Gólagráma*, *Amarávatí*, and *Nandigráma*, in the south.

[1] *Góládhyáya*; or lecture on the sphere. c. 11. § 56. As. Res. vol. 12. p. 214.

[2] As. Res. *ibid.*

[3] Though the matter be introductory, the preliminary treatises on arithmetic and algebra may have been added subsequently, as is hinted by one of the commentators of the astronomical part. (*Vártic.*) The order there intimated places them after the computation of planets, but before the treatise on spherics; which contains the date.

[4] Note A.

This, though not marking any extraordinary antiquity, nor approaching to that of the author himself, was a material point to be determined: as there will be in the sequel occasion to show, that modes of analysis, and, in particular, general methods for the solution of indeterminate problems both of the first and second degrees, are taught in the *Vija-gańita*, and those for the first degree repeated in the *Lílávatí*, which were unknown to the mathematicians of the west until invented anew in the last two centuries by algebraists of France and England. It will be also shown, that BHÁSCARA, who himself flourished more than six hundred and fifty years ago, was in this respect a compiler, and took those methods from Indian authors as much more ancient than himself.

That BHÁSCARA's text (meaning the metrical rules and examples, apart from the interspersed gloss;) had continued unaltered from the period of the compilation of his work until the age of the commentaries now current, is apparent from the care with which they have noticed its various readings, and the little actual importance of these variations; joined to the consideration, that earlier commentaries, including the author's own explanatory annotations of his text, were extant, and lay before them for consultation and reference. Those earlier commentaries are occasionally cited by name: particularly the *Gańita-caumudí*, which is repeatedly quoted by more than one of the scholiasts.[1]

No doubt then can be reasonably entertained, that we now possess the arithmetic and algebra of BHÁSCARA, as composed and published by him in the middle of the twelfth century of the Christian era. The age of his precursors cannot be determined with equal precision. Let us proceed, however, to examine the evidence, such as we can at present collect, of their antiquity.

Towards the close of his treatise on Algebra,[2] BHÁSCARA informs us, that it is compiled and abridged from the more diffuse works on the same subject, bearing the names of BRAHME, (meaning no doubt BRAHMEGUPTA,) ŚRÍD'HARA and PADMANÁBHA; and in the body of his treatise, he has cited a passage of ŚRÍD'HARA's algebra,[3] and another of PADMANÁBHA's.[4] He repeatedly adverts to preceding writers, and refers to them in general terms,

[1] For example, by SÚRYADÁSA, under *Lílávatí*, § 74; and still more frequently by RANGANÁT'HA.

[2] *Vija-gańita*, § 218. [3] Ibid. § 131. [4] Ibid. § 142.

where his commentators understand him to allude to ÁRYA-BHAṬṬA, to BRAHMEGUPTA, to the latter's scholiast CHATURVÉDA PRIT'HÚDACA SWÁMÍ,[1] and to the other writers above mentioned.

Most, if not all, of the treatises, to which he thus alludes, must have been extant, and in the hands of his commentators, when they wrote; as appears from their quotations of them; more especially those of BRAHMEGUPTA and ÁRYA-BHAṬṬA, who are cited, and particularly the first mentioned, in several instances[2] A long and diligent research in various parts of India, has, however, failed of recovering any part of the *Padmanábha víja*, (or Algebra of PADMANÁBHA,) and of the Algebraic and other works of ÁRYA-BHAṬṬA.[3] But the translator has been more fortunate in regard to the works of ŚRÍD'HARA and BRAHMEGUPTA, having in his collection ŚRÍD'HARA's compendium of arithmetic, and a copy, incomplete however, of the text and scholia of BRAHMEGUPTA's *Brahma-sidd'hánta*, comprising among other no less interesting matter, a chapter treating of arithmetic and mensuration; and another, the subject of which is algebra: both of them fortunately complete.[4]

The commentary is a perpetual one; successively quoting at length each verse of the text; proceeding to the interpretation of it, word by word; and subjoining elucidations and remarks: and its colophon, at the close of each chapter, gives the title of the work and name of the author.[5] Now the name, which is there given, CHATURVÉDA PRIT'HÚDACA SWÁMÍ, is that of a celebrated scholiast of BRAHMEGUPTA, frequently cited as such by the commentators of BHÁSCARA and by other astronomical writers: and the title of the work, *Bráhma-sidd'hánta*, or sometimes *Bráhma sphuṭa-sidd'hánta*, corresponds, in the shorter form, to the known title of BRAHMEGUPTA's treatise in the usual references to it by BHÁSCARA's commentators;[6] and answers, in the longer form, to the designation of it, as indicated in an introductory couplet which is quoted from BRAHMEGUPTA by LACSHMÍDÁSA, a scholiast of BHÁSCARA.[7]

Remarking this coincidence, the translator proceeded to collate, with the

[1] *Víj.-gań.* Ch. 5. note of SÚRYADÁSA. Also *Víj.-gań.* § 174; and *Líl.* § 246 ad finem.

[2] For example, under *Líl.* Ch. 11. [3] Note G. [4] Note B.

[5] *Vásaná-bháshya* by CHATURVÉDA PRIT'HÚDACA SWÁMÍ, son of MAD'HUSÚDANA, on the *Brahma-sidd'hánta*; (or sometimes *Brahma*-sphuṭa-*sidd'hánta*.)

[6] They often quote from the *Brahma-sidd'hánta* after premising a reference to BRAHMEGUPTA.

[7] Note C.

text and commentary, numerous quotations from both, which he found in BHÁSCARA's writings or in those of his expositors. The result confirmed the indication, and established the identity of both text and scholia as BRAHMEGUPTA's treatise, and the gloss of PRIT'HÚDACA. The authenticity of this *Brahma-sidd'hánta* is further confirmed by numerous quotations in the commentary of BHAṬṬÓTPALA on the *sanhitá* of VARÁHA MIHIRA: as the quotations from the *Brahma-sidd'hánta* in that commentary, (which is the work of an author who flourished eight hundred and fifty years ago,) are verified in the copy under consideration. A few instances of both will suffice; and cannot fail to produce conviction.[1]

It is confidently concluded, that the Chapters on Arithmetic and Algebra, fortunately entire in a copy, in many parts imperfect, of BRAHMEGUPTA's celebrated work, as here described, are genuine and authentic. It remains to investigate the age of the author.

Mr. DAVIS, who first opened to the public a correct view of the astronomical computations of the Hindus,[2] is of opinion, that BRAHMEGUPTA lived in the 7th century of the Christian era.[3] Dr. WILLIAM HUNTER, who resided for some time with a British Embassy at *Ujjayaní*, and made diligent researches into the remains of Indian science, at that ancient seat of Hindu astronomical knowledge, was there furnished by the learned astronomers whom he consulted, with the ages of the principal ancient authorities. They assigned to BRAHMEGUPTA the date of 550 *Śaca;* which answers to A. D. 628. The grounds, on which they proceeded, are unfortunately not specified: but, as they gave BHÁSCARA's age correctly, as well as several other dates right, which admit of being verified; it is presumed, that they had grounds, though unexplained, for the information which they communicated.[4]

Mr. BENTLEY, who is little disposed to favour the antiquity of an Indian astronomer, has given his reasons for considering the astronomical system which BRAHMEGUPTA teaches, to be between twelve and thirteen hundred years old ($1263\frac{2}{3}$ years in A. D. 1799).[5] Now, as the system taught by this author is professedly one corrected and adapted by him to conform with the observed positions of the celestial objects when he wrote,[6] the age, when their positions would be conformable with the results of computations made as by him directed, is precisely the age of the author himself: and so far as

[1] Note D.
[2] As. Res. 2. 225.
[3] Ibid. 9. 242.
[4] Note E.
[5] As. Res. 6. 586.
[6] Supra.

Mr. BENTLEY's calculations may be considered to approximate to the truth, the date of BRAHMEGUPTA's performance is determined with like approach to exactness, within a certain latitude however of uncertainty for allowance to be made on account of the inaccuracy of Hindu observations.

The translator has assigned on former occasions[1] the grounds upon which he sees reason to place the author's age, soon after the period, when the vernal equinox coincided with the beginning of the lunar mansion and zodiacal asterism *Aświní*, where the Hindu ecliptic now commences. He is supported in it by the sentiments of BHÁSCARA and other Indian astronomers, who infer from BRAHMEGUPTA's doctrine concerning the solstitial points, of which he does not admit a periodical motion, that he lived when the equinoxes did not, sensibly to him, deviate from the beginning of *Aświní* and middle of *Chitrá* on the Hindu sphere.[2] On these grounds it is maintained, that BRAHMEGUPTA is rightly placed in the sixth or beginning of the seventh century of the Christian era; as the subjoined calculations will more particularly show.[3] The age when BRAHMEGUPTA flourished, seems then, from the concurrence of all these arguments, to be satisfactorily settled as antecedent to the earliest dawn of the culture of sciences among the Arabs; and consequently establishes the fact, that the Hindus were in possession of algebra before it was known to the Arabians.

BRAHMEGUPTA's treatise, however, is not the earliest work known to have been written on the same subject by an Indian author. The most eminent scholiast of BHÁSCARA[4] quotes a passage of ÁRYA-BHAṬṬA specifying algebra under the designation of *Víja*, and making separate mention of *Cuṭṭaca*, which more particularly intends a problem subservient to the general method of resolution of indeterminate problems of the first degree: he is understood by another of BHÁSCARA's commentators[5] to be at the head of the elder writers, to whom the text then under consideration adverts, as having designated by the name of *Mad'hyamáharańa* the resolution of affected quadratic equations by means of the completion of the square. It is to be presumed, therefore, that the treatise of ÁRYA-BHAṬṬA then extant, did extend to quadratic equations in the determinate analysis; and to indeterminate problems of the first degree; if not to those of the second likewise, as most probably it did.

This ancient astronomer and algebraist was anterior to both VARÁHA-MIHIRA

[1] As. Res. 9. 329. [2] Ibid. 12. p. 215. [3] Note F.
[4] GAŃÉŚA, a distiguished mathematician and astronomer. [5] SÚR. on *Víj.-gań.* § 128.

and BRAHMEGUPTA; being repeatedly named by the latter; and the determination of the age when he flourished is particularly interesting, as his astronomical system, though on some points agreeing, essentially disagreed on others, with that which those authors have followed, and which the Hindu astronomers still maintain.[1]

He is considered by the commentators of the *Súrya sidd'hánta* and *Śirómańi*,[2] as the earliest of uninspired and mere human writers on the science of astronomy; as having introduced requisite corrections into the system of PARÁŚARA, from whom he took the numbers for the planetary mean motions; as having been followed in the tract of emendation, after a sufficient interval to make further correction requisite, by DURGASINHA and MIHIRA; who were again succeeded after a further interval by BRAHMEGUPTA son of JISHNU.[3]

In short, ÁRYA-BHAŤŤA was founder of one of the sects of Indian astronomers, as PULÍŚA, an author likewise anterior to both VARÁHAMIHIRA and BRAHMEGUPTA, was of another: which were distinguished by names derived from the discriminative tenets respecting the commencement of planetary motions at sun-rise according to the first, but at midnight according to the latter,[4] on the meridian of *Lancá*, at the beginning of the great astronomical cycle. A third sect began the astronomical day, as well as the great period, at noon.

His name accompanied the intimation which the Arab astronomers (under the Abbasside Khalifs, as it would appear,) received, that three distinct astronomical systems were current among the Hindus of those days: and it is but slightly corrupted, certainly not at all disguised, in the Arabic representation of it *Arjabahar*, or rather *Arjabhar*.[5] The two other systems were, first, BRAHMEGUPTA's *Sidd'hánta*, which was the one they became best acquainted with, and to which they apply the denomination of the *sind-hind;* and second, that

[1] Note G. [2] *Nrĭsinha* on *Súr*. GAŃÉŚA pref. to *Grah. lágh.*

[3] As. Res. 2. 235, 242, and 244; and Note H.

[4] BRAHMEGUPTA, ch. 11. The names are *Audayaca* from *Udaya* rising; and *Árdharátrica* from *Ardharátri*, midnight. The third school is noticed by BHAŤŤÓTPALA the scholiast of VARÁHA MIHIRA, under the denomination of *Mádhyandinas*, as alleging the commencement of the astronomical period at noon: (from *Madhyandina*, mid-day.)

[5] The *Sanscrĭt ť*, it is to be remembered, is the character of a peculiar sound often mistaken for *r*, and which the Arabs were likely so to write, rather than with a *te* or with a *tau*. The *Hindi ť* is generally written by the English in India with an *r*. Example: *Ber (vaťa)*, the Indian fig. vulg. Banian tree.

of *Árca* the sun, which they write *Árcand*, a corruption still prevalent in the vulgar *Hindi.*[1]

Áryabhaťťa appears to have had more correct notions of the true explanation of celestial phenomena than Brahmegupta himself; who, in a few instances, correcting errors of his predecessor, but oftener deviating from that predecessor's juster views, has been followed by the herd of modern Hindu astronomers, in a system not improved, but deteriorated, since the time of the more ancient author.

Considering the proficiency of Áryabhaťťa in astronomical science, and adverting to the fact of his having written upon Algebra, as well as to the circumstance of his being named by numerous writers as the founder of a sect, or author of a system in astronomy, and being quoted at the head of algebraists, when the commentators of extant treatises have occasion to mention early and original[2] writers on this branch of science, it is not necessary to seek further for a mathematician qualified to have been the great improver of the analytic art, and likely to have been the person, by whom it was carried to the pitch to which it is found to have attained among the Hindus, and at which it is observed to be nearly stationary through the long lapse of ages which have since passed: the later additions being few and unessential in the writings of Brahmegupta, of Bháscara, and of Jnyána rája, though they lived at intervals of centuries from each other.

Áryabhaťťa then being the earliest author known to have treated of Algebra among the Hindus, and being likely to be, if not the inventor, the improver, of that analysis, by whom too it was pushed nearly to the whole degree of excellence which it is found to have attained among them; it becomes in an especial manner interesting to investigate any discoverable trace in the absence of better and more direct evidence, which may tend to fix the date of his labours, or to indicate the time which elapsed between him and Brahmegupta, whose age is more accurately determined.[3]

Taking Áryabhaťťa, for reasons given in the notes,[3] to have preceded Brahmegupta and Varáhamihira by several centuries; and Brahmegupta to have flourished about twelve hundred years ago;[4] and Varáha mihira, concerning whose works and age some further notices will be found in a sub-

[1] See notes I, K, and N.

[2] Súrya-dása on *Víja-gańita*, ch. 5.

[3] Note I.

[4] See before and note F.

joined note,[1] to have lived at the beginning of the sixth century after Christ,[2] it appears probable that this earliest of known Hindu algebraists wrote as far back as the fifth century of the Christian era; and, perhaps, in an earlier age. Hence it is concluded, that he is nearly as ancient as the Grecian algebraist DIOPHANTUS, supposed, on the authority of ABULFARAJ,[3] to have flourished in the time of the Emperor JULIAN, or about A. D. 360.

Admitting the Hindu and Alexandrian authors to be nearly equally ancient, it must be conceded in favour of the Indian algebraist, that he was more advanced in the science; since he appears to have been in possession of the resolution of equations involving several unknown, which it is not clear, nor fairly presumable, that DIOPHANTUS knew; and a general method for indeterminate problems of at least the first degree, to a knowledge of which the Grecian algebraist had certainly not attained; though he displays infinite sagacity and ingenuity in particular solutions; and though a certain routine is discernible in them.

A comparison of the Grecian, Hindu, and Arabian algebras, will more distinctly show, which of them had made the greatest progress at the earliest age of each, that can be now traced.

The notation or algorithm of Algebra is so essential to this art, as to deserve the first notice in a review of the Indian method of analysis, and a comparison of it with the Grecian and Arabian algebras. The Hindu algebraists use abbreviations and initials for symbols: they distinguish negative quantities by a dot;[4] but have not any mark, besides the absence of the negative sign, to discriminate a positive quantity. No marks or symbols indicating operations of addition, or multiplication, &c. are employed by them: nor any announcing equality[5] or relative magnitude (greater or less).[6] But a factum is denoted by the initial syllable of a word of that import,[7] subjoined to the terms which compose it, between which a dot is sometimes interposed. A fraction is indicated by placing the divisor under the dividend,[8] but without a line of separation. The two sides of an equation are ordered in the same

[1] Note K. [2] See before and note E. [3] Pococke's edition and translation, p. 89.

[4] *Víj.-gaṅ.* § 4.

[5] The sign of equality was first used by Robert Recorde, because, as he says, no two things can be more equal than a pair of parallels, or *gemowe* lines of one length. *Hutton.*

[6] The signs of relative magnitude were first introduced into European algebra by Harriot.

[7] *Víj.-gan.* § 21. [8] *Líl.* § 33.

manner, one under the other:[1] and, this method of placing terms under each other being likewise practised upon other occasions,[2] the intent is in the instance to be collected from the recital of the steps of the process in words at length, which always accompanies the algebraic process. That recital is also requisite to ascertain the precise intent of vertical lines interposed between the terms of a geometric progression, but used also upon other occasions to separate and discriminate quantities. The symbols of unknown quantity are not confined to a single one: but extend to ever so great a variety of denominations: and the characters used are initial syllables of the names of colours,[3] excepting the first, which is the initial of *yávat-távat*, as much as; words of the same import with BOMBELLI's *tanto;* used by him for the same purpose. Colour therefore means unknown quantity, or the symbol of it: and the same *Sanscrit* word, *varńa*, also signifying a literal character, letters are accordingly employed likewise as symbols; either taken from the alphabet;[4] or else initial syllables of words signifying the subjects of the problem; whether of a general nature,[5] or specially the names of geometric lines in algebraic demonstrations of geometric propositions or solution of geometric problems.[6] Symbols too are employed, not only for unknown quantities, of which the value is sought; but for variable quantities of which the value may be arbitrarily put, (*Víj.* Ch. 6, note on commencement of § 153—156,) and, especially in demonstrations, for both given and sought quantities. Initials of the terms for square and solid respectively denote those powers; and combined they indicate the higher. These are reckoned not by the sums of the powers; but by their products.[7] An initial syllable is in like manner used to mark a surd root.[8] The terms of a compound quantity are ordered according to the powers; and the absolute number invariably comes last. It also is distinguished by an initial syllable, as a discriminative token of known quantity.[9] Numeral coefficients are employed, inclusive of unity which is always noted, and comprehending fractions;[10] for the numeral divisor is generally so placed, rather than under the symbol of the unknown: and in like manner the negative dot is set over the numeral coefficient: and not over the literal character. The coefficients are placed after the symbol of the

[1] *Víj.-gań.* and *Brahm.* 18, passim.
[2] *Víj.-gań.* § 55.
[3] *Víj.-gań.* § 17. *Brahm.* c. 18, § 2.
[4] *Víj.-gań.* ch. 6.
[5] *Víj.-gań.* § 111.
[6] *Víj.-gań.* § 146.
[7] *Líl.* § 26.
[8] *Víj.-gań.* § 29.
[9] *Víj.-gań.* § 17.
[10] Stevinus in like manner included fractions in coefficients.

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unknown quantity.[1] Equations are not ordered so as to put all the quantities positive; nor to give precedence to a positive term in a compound quantity: for the negative terms are retained, and even preferably put in the first place. In stating the two sides of an equation, the general, though not invariable, practice is, at least in the first instance, to repeat every term, which occurs in the one side, on the other: annexing nought for the coefficient, if a term of that particular denomination be there wanting.

If reference be made to the writings of DIOPHANTUS, and of the Arabian algebraists, and their early disciples in Europe, it will be found, that the notation, which has been here described, is essentially different from all theirs; much as they vary. DIOPHANTUS employs the inverted medial of ἔλλειψις, defect or want (opposed to υπαρξις, substance or abundance[2]) to indicate a negative quantity. He prefixes that mark ⫛ to the quantity in question. He calls the unknown, αριθμος; representing it by the final ς, which he doubles for the plural; while the Arabian algebraists apply the equivalent word for number to the constant or known term; and the *Hindus*, on the other hand, refer the word for numerical character to the coefficient. He denotes the monad, or unit absolute, by μ^{o}; and the linear quantity is called by him *arithmos;* and designated, like the unknown, by the final *sigma.* He marks the further powers by initials of words signifying them: δ^{υ}, κ^{υ}, $\delta\delta^{\upsilon}$, $\delta\kappa^{\upsilon}$, $\kappa\kappa^{\upsilon}$, &c. for *dynamis*, power (meaning the square); *cubos*, cube; *dynamo-dynamis*, biquadrate, &c. But he reckons the higher by the sums, not the products, of the lower. Thus the sixth power is with him the *cubo-cubos*, which the *Hindus* designate as the quadrate-cube, (cube of the square, or square of the cube).

The Arabian Algebraists are still more sparing of symbols, or rather entirely destitute of them.[4] They have none, whether arbitrary or abbreviated, either for quantities known or unknown, positive or negative, or for the steps and operations of an algebraic process: but express every thing by words, and phrases, at full length. Their European scholars introduced a few and very few abbreviations of names: c^{o}, c^{e}, c^{u}, for the three first powers; c^{o}, q^{a}, for the first and second unknown quantities; p, m, for plus and minus; and

[1] VIETA did so likewise.

[2] A word of nearly the same import with the *Sanscrït d'hana*, wealth, used by *Hindu* algebraists for the same signification.

[3] Def. 9. [4] As. Res. 12. 183.

℞ for the note of radicality; occur in the first printed work which is that of PACIOLO.[1] LEONARDO BONACCI of *Pisa*, the earliest scholar of the Arabians,[2] is said by TARGIONI TOZZETTI to have used the small letters of the alphabet to denote quantities.[3] But LEONARDO only does so because he represents quantities by straight lines, and designates those lines by letters, in elucidation of his Algebraic solutions of problems.[4]

The Arabians termed the unknown (and they wrought but on one) *shai* thing. It is translated by LEONARDO of *Pisa* and his disciples, by the correspondent Latin word *res* and Italian *cosa;* whence *Regola de la Cosa,* and Rule of *Coss*, with *Cossike* practise and *Cossike* number of our older authors,[5] for Algebra or Speculative practice, as PACIOLO[6] denominates the analytic art; and *Cossic* number, in writers of a somewhat later date, for the root of an equation.

The *Arabs* termed the square of the unknown *mál,* possession or wealth; translated by the Latin *census* and Italian *censo;* as terms of the same import: for it is in the acceptation of amount of property or estate[7] that *census* was here used by LEONARDO.

The cube was by the Arabs termed *Cáb,* a die or cube; and they combined these terms *mál* and *cáb* for compound names of the more elevated powers; in the manner of DIOPHANTUS by the sums of the powers; and not like the *Hindus* by their products. Such indeed, is their method in the modern elementary works: but it is not clear, that the same mode was observed by their earlier writers; for their Italian scholars denominated the biquadrate and higher powers Relato primo, secundo, tertio, &c.

Positive they call *záïd* additional; and negative *nákis* deficient: and, as before observed, they have no discriminative marks for either of them.

The operation of *restoring* negative quantities, if any there be, to the positive form, which is an essential step with them, is termed *jebr*, or with the article *Aljebr*, the mending or restoration. That of *comparing* the terms and taking like from like, which is the next material step in the process of resolu-

[1] Or PACIOLI, PACIUOLO,—LI, &c. For the name is variously written by Italian authors.
[2] See Note L.
[3] *Viaggi*, 2d Edit. vol. 2, p. 62.
[4] COSSALI, Origine dell'Algebra, i.
[5] Robert Recorde's Whetstone of Witte.
[6] Secondo noi detta Pratica Speculativa. *Summa* 8. 1.
[7] *Census,* quicquid fortunarum quis habet. *Steph. Thes.*

tion, is called by them *mukábalah* comparison. Hence the name of *Tarík aljebr wa almukabala*, 'the method of restoration and comparison,' which obtained among the Arabs for this branch of the Analytic art; and hence our name of Algebra, from LEONARDO of *Pisa's* exact version of the Arabic title. *Fi istakhráju'l majhulát ba tarik aljebr wa almukábalah,*[1] De solutione quarundam quæstionum secundum modum *Algebræ* et *Almuchabalæ.*[2]

The two steps or operations, which have thus given name to the method of analysis, are precisely what is enjoined without distinctive appellations of them, in the introduction of the arithmetics of DIOPHANTUS, where he directs, that, if the quantities be positive on both sides, like are to be taken from like until one species be equal to one species; but, if on either side or on both, any species be negative, the negative species must be added to both sides, so that they become positive on both sides of the equation: after which like are again to be taken from like, until one species remain on each side.[3]

The *Hindu* Algebra, not requiring the terms of the equation to be all exhibited in the form of positive quantity, does not direct the preliminary step of *restoring* negative quantity to the affirmative state: but proceeds at once to the operation of equal subtraction *(samasódhana)* for the difference of like terms which is the process denominated by the Arabian Algebraists comparison *(mukábalah)*. On that point, therefore, the Arabian Algebra has more affinity to the Grecian than to the Indian analysis.

As to the progress which the Hindus had made in the analytic art, it will be seen, that they possessed well the arithmetic of surd roots;[4] that they were aware of the infinite quotient resulting from the division of finite quantity by cipher;[5] that they knew the general resolution of equations of the second degree; and had touched upon those of higher denomination; resolving them in the simplest cases, and in those in which the solution happens to be practicable by the method which serves for quadratics:[6] that they had attained a general solution of indeterminate problems of the first degree:[7] that they had arrived at a method for deriving a multitude of solutions of answers to pro-

[1] *Khulásatúl hisáb.* c. 8. *Calcutta.*
[2] *Liber abbaci,* 9. 15. 3. M.S. in Magliab. Libr.
[3] Def. 11.
[4] *Brahm.* 18. § 27—29. *Víj.-gań.* § 29—52.
[5] *Líl.* § 45. *Víj.-gań.* § 15—16 and § 135.
[6] *Víj.-gań.* § 129. and § 137—138.
[7] *Brahm.* 18. § 3—18. *Víj.-gań.* 53—73. *Líl.* § 248—265.

blems of the second degree from a single answer found tentatively;[1] which is as near an approach to a general solution of such problems, as was made until the days of LAGRANGE, who first demonstrated, that the problem, on which the solutions of all questions of this nature depend, is always resolvable in whole numbers.[2] The *Hindus* had likewise attempted problems of this higher order by the application of the method which suffices for those of the first degree;[3] with indeed very scanty success, as might be expected.

They not only applied algebra both to astronomy[4] and to geometry;[5] but conversely applied geometry likewise to the demonstration of Algebraic rules.[6] In short, they cultivated Algebra much more, and with greater success, than geometry; as is evident from the comparatively low state of their knowledge in the one,[7] and the high pitch of their attainments in the other: and they cultivated it for the sake of astronomy, as they did this chiefly for astrological purposes. The examples in the earliest algebraic treatise extant (BRAHMEGUPTA's) are mostly astronomical: and here the solution of indeterminate problems is sometimes of real and practical use. The instances in the later treatise of Algebra by BHÁSCARA are more various: many of them geometric; but one astronomical; the rest numeral: among which a great number of indeterminate; and of these some, though not the greatest part, resembling the questions which chiefly engage the attention of DIOPHANTUS. But the general character of the Diophantine problems and of the Hindu unlimited ones is by no means alike: and several in the style of Diophantine are noticed by BHASCARA in his arithmetical, instead of his algebraic, treatise.[8]

To pursue this summary comparison further, DIOPHANTUS appears to have been acquainted with the direct resolution of affected quadratic equations; but less familiar with the management of them, he seldom touches on it. Chiefly busied with indeterminate problems of the first degree, he yet seems to have possessed no general rule for their solution. His elementary instructions for the preparation of equations are succinct.[9] His notation, as

[1] *Brahm.* 18. § 29—49. *Víj.-gań.* § 75—99.

[2] Mem. of Acad. of Turin: and of Berlin.

[3] *Víj.-gań.* § 206—207.

[4] *Brahm.* 18. passim. *Víj.-gań.*

[5] *Víj.-gań.* § 117—127. § 146—152.

[6] *Víj.-gań.* § 212—214.

[7] *Brahm.* 12. § 21; corrected however in *Líl.* § 169—170.

[8] *Líl.* § 59—61, where it appears, however, that preceding writers had treated the question algebraically. See likewise § 139—146.

[9] Def. 11.

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before observed, scanty and inconvenient. In the whole science, he is very far behind the Hindu writers: notwithstanding the infinite ingenuity, by which he makes up for the want of rule: and although presented to us under the disadvantage of mutilation; if it be, indeed, certain that the text of only six, or at most seven, of thirteen books which his introduction announces, has been preserved.[1] It is sufficiently clear from what does remain, that the lost part could not have exhibited a much higher degree of attainment in the art. It is presumable, that so much as we possess of his work, is a fair specimen of the progress which he and the Greeks before him (for he is hardly to be considered as the inventor, since he seems to treat the art as already known;) had made in his time.

The points, in which the Hindu Algebra appears particularly distinguished from the Greek, are, besides a better and more comprehensive algorithm,—1st, The management of equations involving more than one unknown term. (This adds to the two classes noticed by the Arabs, namely simple and compound, two, or rather three, other classes of equation.) 2d, The resolution of equations of a higher order, in which, if they achieved little, they had, at least, the merit of the attempt, and anticipated a modern discovery in the solution of biquadratics. 3d, General methods for the solution of indeterminate problems of 1st and 2d degrees, in which they went far, indeed, beyond DIOPHANTUS, and anticipated discoveries of modern Algebraists. 4th, Application of Algebra to astronomical investigation and geometrical demonstration: in which also they hit upon some matters which have been reinvented in later times.

This brings us to the examination of some of their anticipations of modern discoveries. The reader's notice will be here drawn to three instances in particular.

The first is the demonstration of the noted proposition of PYTHAGORAS, concerning the square of the base of a rectangular triangle, equal to the squares of the two legs containing a right angle. The demonstration is given two ways in BHÁSCARA's Algebra, (*Víj.-gańi.* § 146.) The first of them is the same which is delivered by WALLIS in his treatise on angular sections, (Ch. 6.) and, as far as appears, then given for the first time.[2]

[1] Note M.

[2] He designates the sides C. D. Base B. Segments $\varkappa$, δ. Then

$$\left.\begin{matrix} B : C :: C : \varkappa \\ B : D :: D : \delta \end{matrix}\right\} \text{ and therefore } \left\{\begin{matrix} C^2 = B\varkappa \\ D^2 = B\delta \end{matrix}\right.$$

On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by BHA'SCARA himself, towards the close of his Algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities. The rule, which he demonstrates, is of great antiquity in Hindu Algebra: being found in the works of his predecessor BRAHMEGUPTA, and being there a quotation from a more ancient treatise; for it is injudiciously censured, and a less satisfactory method by unrestricted arbitrary assumption given in its place. BHA'SCARA has retained both.

The next instance, which will be here noticed, is the general solution of indeterminate problems of the first degree. It was first given among moderns by BACHET *de* MEZIRIAC in 1624.[1] Having shown how the solution of equations of the form $ax-by=c$ is reduced to $ax-by=\pm 1$, he proceeds to resolve this equation: and prescribes the same operation on a and b as to find the greatest common divisor. He names the residues c, d, e, f, &c. and the last remainder is necessarily unity: a and b being prime to each other. By retracing the steps from $e\mp 1$ or $f\pm 1$ (according as the number of remainders is even or odd) $e\mp 1=\varepsilon$, $\frac{\varepsilon d\pm 1}{e}=\delta$, $\frac{\delta c\mp 1}{d}=\gamma$, $\frac{\gamma b\pm 1}{c}=\beta$, $\frac{\beta a\mp 1}{b}=\alpha$

or $f\pm 1=\zeta$, $\frac{\zeta e\mp 1}{f}=\varepsilon$, $\frac{\varepsilon d\pm 1}{e}$ δ, &c.

The last numbers β and α will be the smallest values of x and y. It is observed, that, if a and b be not prime to each other, the equation cannot subsist in whole numbers unless c be divisible by the greatest common measure of a and b.

Here we have precisely the method of the Hindu algebraists, who have not failed, likewise, to make the last cited observation. See *Brahm.* Algebra,

Therefore $C^2+D^2=(B\varkappa+B\delta=B \text{ into } \varkappa+\delta=)\ B^2$.

The Indian demonstration, with the same symbols, is

$$\left.\begin{array}{l} B : C :: C : \varkappa \\ B : D :: D : \delta \end{array}\right\} \text{Therefore} \left\{\begin{array}{l} \varkappa=\frac{C^2}{B} \\ \delta=\frac{D^2}{B} \end{array}\right.$$

Therefore $B=\varkappa+\delta=\frac{C^2}{B}+\frac{D^2}{B}$ and $B^2=C^2+D^2$.

[1] Problèmes plaisans et délectables qui se font par les nombres. 2d Edit. (1624). LAGRANGE's additions to EULER's Algebra, ij. 382. (Edit. 1807.)

section 1. and *Bhásc. Líl.* ch. 12. *Víj.* ch. 2. It is so prominent in the Indian Algebra as to give name to the oldest treatise on it extant; and to constitute a distinct head in the enumeration of the different branches of mathematical knowledge in a passage cited from a still more ancient author. See *Líl.* § 248.

Confining the comparison of Hindu and modern Algebras to conspicuous instances, the next for notice is that of the solution of indeterminate problems of the 2d degree: for which a general method is given by BRAHMEGUPTA, besides rules for subordinate cases: and two general methods (one of them the same with BRAHMEGUPTA's) besides special cases subservient however to the universal solution of problems of this nature; and, to obtain whole numbers in all circumstances, a combination of the method for problems of the first degree with that for those of the second, employing them alternately, or, as the Hindu algebraist terms it, proceeding in a circle.

BHÁSCARA's second method (*Víj.* § 80—81) for a solution of the problem on which all indeterminate ones of this degree depend, is exactly the same, which Lord BROUNCKER devised to answer a question proposed by way of challenge by FERMAT in 1657. The thing required was a general rule for finding the innumerable square numbers, which multiplied by a proposed (non-quadrate) number, and then assuming an unit, will make a square. Lord BROUNCKER's rule, putting n for any given number, r^2 for any square taken at pleasure, and d for difference between n and r^2 ($r^2 \backsim n$) was $\frac{4r^2}{d^2}\left(=\frac{2r}{d}\times\frac{2r}{d}\right)$ the square required. In the Hindu rule, using the same symbols, $\frac{2r}{d}$ is the square root required.[1] But neither BROUNCKER, nor WALLIS, who himself contrived another method, nor FERMAT, by whom the question was proposed, but whose mode of solution was never made known by him, (probably because he had not found anything better than WALLIS and BROUNCKER discovered,[2]) nor FRENICLE, who treated the subject without, however, adding to what had been done by WALLIS and BROUNCKER,[3] appear to have been aware of the importance of the problem and its universal use: a discovery, which, among the moderns, was reserved for EULER in the middle of the last century. To him, among the moderns, we owe the remark, which the Hindus had made more than a thousand years before,[4] that the problem was requisite to find all the

[1] *Víj.-gań.* § 80—81. [2] Wallis, Alg. c. 98. [3] Ibid.
[4] *Bháscara Víj.* § 173, and § 207. See likewise *Brahm.* Alg. sect. 7.