DE

TRANSFORMATIONE FUNCTIONUM ELLIPTICARUM.

EXPOSITIO PROBLEMATIS GENERALIS DE TRANSFORMATIONE.

1.

Integralia maxime memorabilia, quae Formula exhibentur $\int \frac{d\phi}{\sqrt{1-k^2 \operatorname{Sin} \phi^2}}$, et quae Functionum Ellipticarum, quae dicuntur, primam speciem constituunt, ab Argumento duplice pendent, et ab Amplitudine ϕ et a Modulo k. Eiusmodi functionis inter se comparatis valoribus, quos illa pro diversis Amplitudinibus obtinet, eodem manente Modulo, egregia multa detexerant Analystae, quae Additionem eorum et Multiplicationem spectant. Quam nuper vidimus quaestionem a Cl. *Abel* in Commentatione, nostra laude majore, mirum in modum provectam esse (V. *Crelle* Journal für reine und angewandte Mathematik V. II.).

Alia est quaestio nec minoris momenti — immo sensu latissimo capta illam involvens — de comparatione Functionum Ellipticarum pro Modulis instituenda diversis. Quam quaestionem post praeclara inventa Cl[i] *Legendre* — Theoriae Functionum Ellipticarum Conditoris — ad principia certa nos primi revocavimus, eiusque solutionem dedimus generalem (V. *Astronomische Nachrichten* A. 1827. No. 123. 127). Hanc nostram de Transformatione Theoriam et quae alia inde in Analysin Functionum Ellipticarum redundant, iam fusius exponemus.

A

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2.

Problema, quod nobis proponimus, generale hoc est:

„*Quaeritur Functio rationalis y elementi x ejusmodi, ut sit:*

$$\frac{dy}{\sqrt{A'+B'y+C'y^2+D'y^3+E'y^4}}=\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}}$$

Quod Problema et Multiplicationem videmus amplecti et Transformationem.

Innumera iam diu constabant exempla eiusmodi functionum rationalium y, quae problemati proposito satisfaciunt. Primum notum erat, quicunque datus sit numerus integer impar n, eiusmodi functionem rationalem y exhiberi posse, ut sit:

$$\frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}}=\frac{n\,dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}};$$

quod est de Multiplicatione theorema. Quem in finem adhiberi debet forma:

$$y=\frac{a+a'x+a''x^2+a'''x^3+\ldots+a^{(nn)}x^{nn}}{b+b'x+b''x^2+b'''x^3+\ldots+b^{(nn)}x^{nn}},$$

Coëfficientibus a, a′, a″, ...; b, b′, b″, ... rite determinatis. Satis diu etiam exploratum est, formam hanc:

$$y=\frac{a+a'x+a''x^2}{b+b'x+b''x^2},$$

seu hanc generaliorem:

$$y=\frac{a+a'x+a''x^2+a'''x^3+\ldots+a^{(2^m)}x^{2^m}}{b+b'x+b''x^2+b'''x^3+\ldots+b^{(2^m)}x^{2^m}},$$

quae ex illius substitutionis repetitione ortum ducit, ita determinari posse, ut solvat problema. Nuper admodum etiam probatum est a Cl° Legendre, eum in finem adhiberi posse formam hanc rite determinatam:

$$y=\frac{a+a'x+a''x^2+a'''x^3}{b+b'x+b''x^2+b'''x^3};$$

seu rursus, eadem substitutione repetita, hanc generaliorem:

$$y=\frac{a+a'x+a''x^2+a'''x^3+\ldots+a^{(3^m)}x^{3^m}}{b+b'x+b''x^2+b'''x^3+\ldots+b^{(3^m)}x^{3^m}}.$$

His inter se iunctis formis patet, problemati satisfieri posse, idonea facta Coëfficientium electione, posito:

$$y=\frac{a+a'x+a''x^2+a'''x^3+\ldots+a^{(p)}x^p}{b+b'x+b''x^2+b'''x^3+\ldots+b^{(p)}x^p},$$

siquidem p sit uumerus formae $2^{\alpha}\,3^{\beta}\,(2m+1)^2$. Iam sequentibus probabitur, idem valere, *quicunque sit* p *numerus.*

PRINCIPIA TRANSFORMATIONIS.

3.

Designentur per U, V functiones rationales integrae elementi x; sit porro $y = \frac{U}{V}$; fit:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{VdU - UdV}{\sqrt{Y}},$$

brevitatis causa posito:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4.$$

Fractionem $\frac{VdU - UdV}{\sqrt{Y}}$ in formam simpliciorem redigere licet, quoties Y factores duplices habet; quin adeo, ubi praeter quatuor factores lineares inter se diversos e reliquorum numero bini inter se aequales existunt, fractio illa sponte in Differentiale Functionis Ellipticae redit $\frac{dx}{U\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$, designante U functionem elementi x rationalem. Quem accuratius examinemus casum ac videamus, quot et quales sibi poscat Conditiones.

Sint functiones U, V altera p^{ti}, altera m^{ti} ordinis, ita ut $m \leqq p$: erit Y $(4p)^{ti}$ ordinis. Iam ut, quatuor factoribus linearibus exceptis, e reliquis functionis Y factoribus, quorum est numerus $4p - 4$, bini inter se aequales evadant, $(2p - 2)$ Conditionibus satisfaciendum erit. Quot enim functio proposita duplices habere debet factores lineares, tot inter Coëfficientes eius intercedere debent Aequationes Conditionales.

At functionibus U, V Quantitates Constantes Indeterminatae insunt $m + p + 2$, seu potius $m + p + 1$, quippe e quarum numero unam aliquam $= 1$ ponere licet. Quarum igitur numero vel aequatur numerus Conditionum $2p - 2$ vel ab eo superatur, modo supponatur, m esse aliquem e numeris $p - 3$, $p - 2$, $p - 1$, p, quibus casibus numerus Indeterminatarum fit resp. $2p - 2$, $2p - 1$, $2p$, $2p + 1$. Duos priores casus reiiciendos esse cum infra demonstrabitur, tum hunc in modum patet. Namque inventis functionibus U, V, quae functioni Y formam illam praescriptam conciliant, ubi loco x substituitur $\alpha + \beta x$, neque ordo mutatur functionum U, V, Y, neque numerus factorum duplicium functionis Y: unde in solutionem inventam statim duas Quantitates Arbitrarias inferre licet. Itaque numerus Indeterminatarum numerum Conditionum duabus saltem unitatibus superare debet, unde casus $m = p - 3$, $m = p - 2$ reiiciendi sunt. Porro videmus, loco x posito $\frac{\alpha + \beta x}{1 + \gamma x}$, tertium casum ad quartum reduci et quartum minime mutari, quo igitur casu Indeterminatarum tres et arbitrariae manent et manere debent.

Iam igitur evictum est, quantum quidem e numero Indeterminatarum et numero Conditionum inter se comparatis concludere licet, *quicunque sit p numerus, formam:*

$$y = \frac{a + a'x + a''x^2 + \ldots + a^{(p)}x^p}{1 + b'x + b''x^2 + \ldots + b^{(p)}x^p}$$

ita determinari posse, ut sit:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{M\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

designante M functionem rationalem ipsius x; imo solutionem tres Quantitates Arbitrarias involvere posse.

4.

Ut determinetur functio illa M, sit $Y = (A + Bx + Cx^2 + Dx^3 + Ex^4)\,TT$, designante T functionem elementi x integram rationalem: erit

$$M = \frac{T}{V\frac{dU}{dx} - U.\frac{dV}{dx}}.$$

Ipsa T erit ordinis $(2p - 2)^{ti}$; nec maioris esse potest $V\frac{dU}{dx} - U\frac{dV}{dx}$. Iam casibus quibusdam constat, scilicet ubi numerus p formam illam habet $2^\alpha 3^\beta (2n + 1)^2$, M adeo fieri Constantem. Idem generaliter probabitur sequentibus, quicunque sit p numerus.

Functiones U, V supponere possumus factorem communem non habere; adiecto enim factore communi, fractio $\frac{U}{V} = y$ non mutatur. Resolvamus expressionem

$$A' + B'y + C'y^2 + D'y^3 + E'y^4$$

in factores lineares, ita ut sit:

$$A' + B'y + C'y^2 + D'y^3 + E'y^4 = A'.(1 - \alpha'y).(1 - \beta'y)(1 - \gamma'y)(1 - \delta'y),$$

unde etiam:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4 = A'(V - \alpha'U)(V - \beta'U)(V - \gamma'U)(V - \delta'U).$$

Iam existere non potest factor, qui quantitatibus $V-\alpha' U$, $V-\beta' U$, $V-\gamma' U$, $V-\delta' U$ vel omnibus vel imo duabus tantum ex earum numero communis sit; idem enim et V et U simul metiretur, quas factorem communem non habere supposuimus. Itaque ubi factor aliquis linearis functionem Y bis metitur, idem unam aliquam e quantitatibus $V-\alpha' U$, $V-\beta' U$, $V-\gamma' U$, $V-\delta' U$ et ipsam bis metiatur necesse est.

Iam notentur aequationes sequentes:

$$(V-\alpha' U)\frac{dU}{dx}-\frac{d(V-\alpha' U)}{dx}.U = V\frac{dU}{dx}-U\frac{dV}{dx}$$
$$(V-\beta' U)\frac{dU}{dx}-\frac{d(V-\beta' U)}{dx}.U = V.\frac{dU}{dx}-U\frac{dV}{dx}$$
$$(V-\gamma' U)\frac{dU}{dx}-\frac{d(V-\gamma' U)}{dx}.U = V\frac{dU}{dx}-U\frac{dV}{dx}$$
$$(V-\delta' U)\frac{dU}{dx}-\frac{d(V-\delta' U)}{dx}.U = V.\frac{dU}{dx}-U\frac{dV}{dx},$$

e quibus sequitur, factorem qui unam aliquam e quantitatibus $V-\alpha' U$, $V-\beta' U$, $V-\gamma' U$, $V-\delta' U$ bis ideoque etiam eius differentiale metiatur, eundem metiri expressionem $V\frac{dU}{dx}-U\frac{dV}{dx}$. Productum vero ex omnibus istis factoribus, ipsam etiam Y bis metientibus, conflatum posuimus $=T$, unde T ipsam $V\frac{dU}{dx}-U\frac{dV}{dx}$ metietur. At T inferioris ordinis non est quam ipsa $V\frac{dU}{dx}-U\frac{dV}{dx}$, unde videmus

$$M=\frac{T}{V\frac{dU}{dx}-U\frac{dV}{dx}}$$

abire in Constantem.

Ceterum adnotemus, ubi functionum U, V altera inferioris ordinis fuisset quam $(p-1)^{ti}$, ipsam etiam $V\frac{dU}{dx}-U\frac{dV}{dx}$ inferioris ordinis fuisse quam T, quae tamen illam metiri debet; quod cum absurdum sit, reiici debebant casus $m=p-2$, $m=p-3$.

Iam igitur demonstratum est, *formam:*

$$y=\frac{a+a'x+a''x^2+\ldots+a^{(p)}x^p}{b+b'x+b''x^2+\ldots+b^{(p)}x^p},$$

quicunque sit numerus p, ita determinari posse, ut prodeat:

$$\frac{dy}{\sqrt{A'+B'y+C'y^2+D'y^3+E'y^4}}=\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}}$$

Quod est Principium in Theoria Transformationum Functionum Ellipticarum Fundamentale.

PROPONITUR EXPRESSIO $\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$ IN FORMAM SIMPLICIOREM REDIGENDA $\frac{dx}{M\sqrt{(1-x)(1-k^2x^2)}}$.

5.

Trium Constantium Arbitrariarum ope, quas solutionem Problematis nostri admittere vidimus, expressio $A+Bx+Cx^2+Dx^3+Ex^4$ in simpliciorem redigi potest hanc: $A(1-x^2)(1-k^2x^2)$. Ut hoc et reliqua, quae modo demonstrata sunt, exemplis etiam monstrentur, propositum sit, datam expressionem:

$$\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

facta substitutione:

$$y=\frac{a+a'x+a''x^2}{b+b'x+b''x^2}$$

in simpliciorem transformare hanc:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Quaeritur de substitutione adhibenda, de Modulo k et de factore Constante M e datis quantitatibus α, β, γ, δ determinandis.

Ponatur $a+a'x+a''x^2=U$, $b+b'x+b''x^2=V$, $y=\frac{U}{V}$: e principiis modo expositis fieri debet:

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V)=K(1-x^2)(1-k^2x^2)(1+mx)^2(1+nx)^2$$

designante K Constantem aliquam arbitrariam. Hinc videmus duos e numero factorum $U-\alpha V$, $U-\beta V$, $U-\gamma V$, $U-\delta V$, qui erunt secundi ordinis, adeo fieri quadrata. Ponamus igitur:

$$U-\gamma V=C(1+mx)^2$$
$$U-\delta V=D(1+nx)^2.$$

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Iam quod reliquos attinet factores $U-\alpha V$, $U-\beta V$, poni poterit, aut:

$$U-\alpha V = A(1-x^2),\quad U-\beta V = B(1-k^2x^2),\ \text{aut:}$$
$$U-\alpha V = A.(1-x)(1-kx),\quad U-\beta V = B(1+x)(1+kx),$$

designantibus A, B, C, D quantitates Constantes. Prius reiiciendum erit. Prodiret enim $\frac{U-\alpha V}{U-\beta V} = \frac{y-\alpha}{y-\beta} = \frac{A}{B}\cdot\frac{1-x^2}{1-k^2x^2}$, unde sequeretur, elemento x in $-x$ mutato y immutatum manere, quod absurdum esse patet ex aequationibus:

$$\frac{U-\alpha V}{U-\gamma V} = \frac{y-\alpha}{y-\gamma} = \frac{A}{B}\cdot\frac{1-k^2}{(1+mx)^2}$$
$$\frac{U-\alpha V}{U-\delta V} = \frac{y-\alpha}{y-\delta} = \frac{A}{D}\cdot\frac{1-k^2}{(1+nx)^2}.$$

Poni igitur debet:

1) $U-\alpha V = A(1-x).(1-kx)$
2) $U-\beta V = B(1+x).(1+kx)$
3) $U-\gamma V = C(1+mx)^2$
4) $U-\delta V = D(1+nx)^2$.

Adnotare convenit, e Constantibus A, B, C, D unam aliquam ex arbitrio determinari posse.

6.

Videmus ex aequatione 1), et posito $x=1$ et posito $x=\frac{1}{k}$ fieri $U=\alpha V$. Hinc ex aequatione:

$$\frac{U-\gamma V}{U-\beta V} = \frac{C}{B}\cdot\frac{(1+mx)^2}{(1+x)(1+kx)},$$

posito $x=1$, prodit:

$$\frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\cdot\frac{(1+m)^2}{2(1+k)};$$

posito $x=\frac{1}{k}$:

$$\frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\cdot\frac{\left(1+\frac{m}{k}\right)^2}{2\left(1+\frac{1}{k}\right)},$$

unde:

$$(1+m)^2 = k\left(1+\frac{m}{k}\right)^2.$$

Prorsus simili modo invenitur:

$$(1+n)^2=k\left(1+\frac{n}{k}\right)^2,$$

unde $m=\sqrt{k}$, $n=-\sqrt{k}$. Neque enim aequales ponere licet m et n; tum enim expressio $\frac{U-\gamma V}{U-\delta V}=\frac{y-\gamma}{y-\delta}$, ideoque ipsa y abiret in Constantem.

Iam in aequatione:

$$\frac{U-\gamma V}{U-\delta V}=\frac{y-\gamma}{y-\delta}=\frac{C}{D}\cdot\left\{\frac{1+\sqrt{k.x}}{1-\sqrt{k.x}}\right\}^2$$

ponatur primum $x=+1$, quo casu $U=\alpha V$; deinde $x=-1$, quo casu $U=\beta V$: prodeunt duae aequationes sequentes:

$$\frac{\alpha-\gamma}{\alpha-\delta}=\frac{C}{D}\left\{\frac{1+\sqrt{k}}{1-\sqrt{k}}\right\}^2$$

$$\frac{\beta-\gamma}{\beta-\delta}=\frac{C}{D}\left\{\frac{1-\sqrt{k}}{1+\sqrt{k}}\right\}^2.$$

Quibus in se ductis aequationibus, fit:

$$\frac{C}{D}=\sqrt{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}},$$

unde ponere licet:

$$C=\sqrt{(\alpha-\gamma)(\beta-\gamma)}$$

$$D=\sqrt{(\alpha-\delta)(\beta-\delta)};$$

nam e quantitatibus A, B, C, D una ex arbitrio determinari poterat.

Ex iisdem aequationibus, altera per alteram divisa, obtinemus:

$$\frac{1+\sqrt{k}}{1-\sqrt{k}}=\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}}{\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}};$$

unde:

$$\sqrt{k}=\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}.$$

Adnotetur adhuc formula:

$$\sqrt{k}+\frac{1}{\sqrt{k}}=2.\frac{\sqrt{(\alpha-\gamma)(\beta-\delta)}+\sqrt{(\alpha-\delta)(\beta-\gamma)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}},$$

unde:

$$(1-\sqrt{k})\left(1-\frac{1}{\sqrt{k}}\right)=\frac{-4\sqrt{(\alpha-\delta)(\beta-\gamma)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}$$

$$(1+\sqrt{k})\left(1+\frac{1}{\sqrt{k}}\right)=\frac{+4\sqrt{(\alpha-\gamma)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}.$$

Ut Constantes A, B, definiantur, observo, ex aequationibus 1), 2), 3), posito $x=\frac{1}{\sqrt{k}}$, quo facto $U=\delta V$, erui:

$$\frac{\delta-\alpha}{\delta-\gamma}=\frac{A(1-\sqrt{k})\left(1-\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\delta)}}$$

$$\frac{\delta-\beta}{\delta-\gamma}=\frac{B(1+\sqrt{k})\left(1+\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}},$$

unde:

$$A=\frac{-\sqrt{(\alpha-\gamma)(\alpha-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right\}$$

$$B=\frac{\sqrt{(\beta-\gamma)(\beta-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right\}$$

7.

E principiis generalibus supra a nobis stabilitis sequitur, in exemplo nostro expressionem $V\frac{dU}{dx}-U\frac{dV}{dx}$ aequalem fore producto $(1+\sqrt{k}x)(1-\sqrt{k}x)$ in quantitatem constantem ducto, quod ita facto calculo comprobatur.

Fit, uti evolutione facta constat:

$$(\gamma-\delta)\left(V\frac{dU}{dx}-U\frac{dV}{dx}\right)=(V-\gamma U)\frac{d(V-\delta U)}{dx}-(V-\delta U)\frac{d(V-\gamma U)}{dx}.$$

Nacti autem sumus:

$$V-\gamma U=C(1+\sqrt{k}.x)^2$$
$$V-\delta U=D(1-\sqrt{k}.x)^2,$$

unde:

$$\frac{d(V-\gamma U)}{dx}=\ 2C(1+\sqrt{k}.x)\sqrt{k}$$
$$\frac{d(V-\delta U)}{dx}=-2D(1-\sqrt{k}.x)\sqrt{k}.$$

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Unde prodit:

$$(\gamma-\delta)\left(V\frac{dU}{dx}-U\frac{dV}{dx}\right)=-4\sqrt{k}.CD(1+\sqrt{k}.x)(1-\sqrt{k}.x).$$

His omnibus rite collectis, obtinemus:

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}=\frac{-4\sqrt{k}}{\gamma-\delta}\cdot\sqrt{\frac{CD}{-AB}}\cdot\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}};$$

unde:

$$M=\frac{\gamma-\delta}{-4\sqrt{k}}\sqrt{\frac{-AB}{CD}}=\frac{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}{4\sqrt{k}}$$

$$=\left\{\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right\}^2,$$

unde:

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}=\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}=$$

$$\frac{dx}{\sqrt{1-xx}\sqrt{\left(\left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4-\left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4xx\right)}}.$$

Posito $(\alpha-\gamma).(\beta-\delta)=G$, $(\alpha-\delta)(\beta-\gamma)=G'$, fit:

$$\frac{dx}{M.\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{dx}{\sqrt{1-x^2}\sqrt{\left(\frac{\sqrt[4]{G}+\sqrt[4]{G'}}{2}\right)^4-\left(\frac{\sqrt[4]{G}-\sqrt[4]{G'}}{2}\right)^4x^2}}.$$

Sit $G=mm$, $G'=nn$, sit porro:

$$m'=\frac{1}{2}(m+n),\quad n'=\sqrt{mn}$$

$$m''=\frac{1}{2}(m'+n'),\quad n''=\sqrt{m'n'},$$

erit, posito $x=\operatorname{Sin}\varphi$:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{d\varphi}{\sqrt{m''m''\operatorname{Cos}\varphi^2+n''n''\operatorname{Sin}\varphi^2}}.$$

Ceterum valor ipsius x facillime computatur ope formulae:

$$\frac{1-\sqrt{k}.x}{1+\sqrt{k}.x}=\sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}}\cdot\sqrt{\frac{y-\delta}{y-\gamma}};$$

ubi:

$$\sqrt{k}=\frac{\sqrt[4]{G}-\sqrt[4]{G'}}{\sqrt[4]{G}+\sqrt[4]{G'}}=\sqrt[4]{\frac{m''m''-n''n''}{m''m''}}.$$