Cambridge University Press
978-1-108-05068-5 - A Synopsis of Elementary Results in Pure and Applied Mathematics: Containing Propositions, Formulae, and Methods of Analysis, with Abridged Demonstrations: Volume 2
George Shoobridge Carr
Excerpt
More information

DIFFERENTIAL CALCULUS.

INTRODUCTION.

1400 *Functions.*—A quantity which depends for its value upon another quantity x is called a *function* of x. Thus, $\sin x$, $\log x$, a^x, a^2+ax+x^2 are all functions of x. The notation $y=f(x)$ expresses generally that y is a function of x. $y=\sin x$ is a particular function.

1401 $f(x)$ is called a *continuous* function between assigned limits, when an indefinitely small change in the value of x always produces an indefinitely small change in the value of $f(x)$.

A *transcendental* function is one which is not purely algebraical, such as the exponential, logarithmic, and circular functions a^x, $\log x$, $\sin x$, $\cos x$, &c.

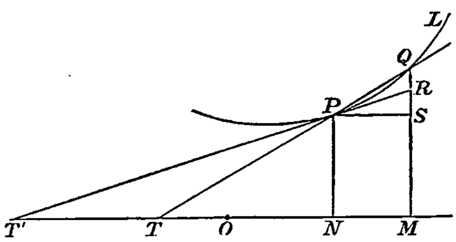
If $f(x)=f(-x)$, the function is called an *even* function. If $f(x)=-f(-x)$, it is called an *odd* function.

Thus, x^2 and $\cos x$ are even functions, while x^3 and $\sin x$ are odd functions of x; the latter, but not the former, being altered in value by changing the sign of x.

1402 *Differential Coefficient or Derivative.*—Let y be any function of x denoted by $f(x)$, such that any change in the value of x causes a definite change in the value of y; then x is called the *independent variable*, and y the *dependent variable*. Let an indefinitely small change in x, denoted by dx, produce a corresponding small change dy in y; then the ratio $\frac{dy}{dx}$, in the limit when both dy and dx are vanishing, is called the *differential coefficient*, or *derivative*, of y with respect to x.

1403 THEOREM.—The ratio $dy : dx$ is *definite* for each value of x, and generally *different* for different values.

PROOF.—Let an abscissa ON (Defs. 1160) be measured from O equal to x, and a perpendicular ordinate NP equal to y. Then, *whatever may be the form of the function* $y = f(x)$, as x varies, the locus of P will be *some line* PQL. Let $OM = x'$, $MQ = y'$ be values of x and y near to the former values. Let the straight line QP meet the axis in T; and when Q coincides with P, let the final direction of QP cut the axis in T'.



Then $\frac{QS}{SP}$ or $\frac{y'-y}{x'-x} = \frac{PN}{NT}$. And, ultimately, when QS and SP vanish, *they vanish in the ratio of* $PN : NT'$. Therefore $\frac{dy}{dx} = \frac{PN}{NT'} = \tan PT'N$, a *definite* ratio at each point of the curve, but *different* at different points.

1404 Let NM, the increment of x, be denoted by h; then, when h vanishes,
$$\frac{dy}{dx} = \frac{f(x+h)-f(x)}{h} = f'(x),$$
a new function of x, called also the *first derived function*. The process of finding its value is called *differentiation*.

1405 *Successive differentiation.*—If $\frac{dy}{dx}$ or $f'(x)$ be differentiated with respect to x, the result is the *second differential coefficient* of $f(x)$, or the *second derived function*; and so on to any number of differentiations. These successive functions may be represented in any of the three following systems of notation :—

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3},\quad \frac{d^4y}{dx^4},\quad \ldots\ldots\quad \frac{d^ny}{dx^n};$$
$$f'(x),\quad f''(x),\quad f'''(x),\quad f^{\text{iv}}(x),\quad \ldots\ldots\quad f^n(x);$$
$$y_x,\quad y_{2x},\quad y_{3x},\quad y_{4x},\quad \ldots\ldots\quad y_{nx}.^*$$

The operations of differentiating a function of x once, twice, or n times, are also indicated by prefixing the symbols
$$\frac{d}{dx},\ \frac{d^2}{dx^2},\ \ldots\ \frac{d^n}{dx^n};\ \text{or}\ \frac{d}{dx},\ \left(\frac{d}{dx}\right)^2,\ \ldots\ \left(\frac{d}{dx}\right)^n;$$
or, more concisely, $d_x,\ d_{2x},\ \ldots\ d_{nx}$.

* See note to (1487).

1406 If, *after* differentiating a function for x, x be made zero in the result, the value may be indicated in any of the following ways: $\frac{dy}{dx_0}$, $f'(0)$, y_{x0}, $\frac{d}{dx_0}$, d_{x0}.

If any other constant a be substituted for x in y_x, the result may be indicated by $y_{x,a}$.

1407 ***Infinitesimals and Differentials.*** — The evanescent quantities dx, dy are called *infinitesimals;* and, with respect to x and y, they are called *differentials.* dx^2, d^2y are the *second differentials* of x and y; dx^3, d^3y the third, and so on.

1408 The successive differentials of y are expressed in terms of dx by the equations

$$dy = f'(x)\,dx;\quad d^2y = f''(x)\,dx^2;\quad \&c.,\ \text{and}\ d^ny = f^n(x)\,dx^n.$$

Since $f'(x)$ is the coefficient of dx in the value of dy, it has therefore been named *the differential coefficient of y or $f(x)$.** For similar reasons $f''(x)$ is called the *second*, and $f^n(x)$ the n^{th} *differential coefficient* of $f(x)$, &c.

1409 Two infinitesimals are of the *same order* when their ratio is neither zero nor infinity.

If dx, dy are infinitesimals of the same order, dx^2, dy^2, and $dx\,dy$ will be infinitesimals of the *second* order with respect to dx, dy; dx^3, dx^2dy, &c. will be of the *third* order, and so on.

dx, dx^2, &c. are sometimes denoted by $\dot{x}$, $\ddot{x}$, &c.

1410 Lemma.—In estimating the ratio of two quantities, any increment of either which is infinitely small in comparison with the quantities may be neglected.

Hence the ratio of two infinitesimals of the same order is not affected by adding to or subtracting from either of them an infinitesimal of a higher order.

Example.— $\frac{dy-dx^2}{dx} = \frac{dy}{dx} - dx = \frac{dy}{dx}$, for dx is zero in comparison with the ratio $\frac{dy}{dx}$. Thus, in Fig. (1403), putting $PS = dx$, $QS = dy$; we have ultimately, by (1258), $QR = k\,dx^2$, where k is a constant. Therefore $\frac{PN}{NT'} = \frac{RS}{PS} = \frac{dy-k\,dx^2}{dx} = \frac{dy}{dx}$ in the limit, by the principle just enunciated; that is, *QR vanishes in comparison with* PS *or* QS *even when those lines themselves are infinitely small.*

* The name is slightly misleading, as it seems to imply that $f'(x)$ is in some sense a coefficient of $f(x)$.

DIFFERENTIATION.

DIFFERENTIATION OF A SUM, PRODUCT, AND QUOTIENT.

Let u, v be functions of x, then

1411 $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

1412 $$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

1413 $$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \div v^2.$$

PROOF.—(i.) $d(u+v) = (u+du+v+dv)-(u+v) = du+dv.$

(ii.) $d(uv) = (u+du)(v+dv)-uv = v\,du+u\,dv-du\,dv,$

and, by (1410), $du\,dv$ disappears in the ultimate ratio to dx.

(iii.) $$d\left(\frac{u}{v}\right) = \frac{u+du}{v+dv} - \frac{u}{v} = \frac{v\,du-u\,dv}{(v+dv)\,v},$$

therefore &c., by (1410); $v\,dv$ vanishing in comparison with v^2.

Hence, if u be a constant $= c$,

1414 $$\frac{d(cv)}{dx} = c\frac{dv}{dx} \quad \text{and} \quad \frac{d}{dx}\left(\frac{c}{v}\right) = -\frac{c}{v^2}\frac{dv}{dx}.$$

DIFFERENTIATION OF A FUNCTION OF A FUNCTION.

If y be a function of z, and z a function of x,

1415 $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

PROOF.—Since, in all cases, the change dx causes the change dz, and the change dz causes the change dy; therefore the change dx causes the change dy *in the limit*.

Differentiating the above as a product, by (1412), the successive differential coefficients of y can be formed. The first four are here subjoined for the sake of reference. Observe that $(y_z)_x = y_{2z}z_x$.

1416 $y_x = y_z z_x.$

1417 $y_{2x} = y_{2z}z_x^2 + y_z z_{2x}.$

1418 $y_{3x} = y_{3z}z_x^3 + 3y_{2z}z_x z_{2x} + y_z z_{3x}.$

1419 $y_{4x} = y_{4z}z_x^4 + 6y_{3z}z_x^2 z_{2x} + y_{2z}(3z_{2x}^2 + 4z_x z_{3x}) + y_z z_{4x}.$

DIFFERENTIATION OF A COMPOSITE FUNCTION.

If u and v be explicit functions of x, so that $u = \phi(x)$ and $v = \psi(x)$,

1420 $$\frac{dF(u,v)}{dx} = \frac{dF}{du}\frac{du}{dx} + \frac{dF}{dv}\frac{dv}{dx}.$$

Here dF in the first term on the right is the change in $F(u, v)$ produced by du, the change in u; and dF in the second term is the change produced by dv, so that the total change $dF(u, v)$ may be written as in (1408)

$$dF_1 + dF_2 = \frac{dF}{du}du + \frac{dF}{dv}dv.$$

DIFFERENTIATION OF THE SIMPLE FUNCTIONS.

Since $\frac{dy}{dx} = \frac{f(x+h)-f(x)}{h}$ when h vanishes, we have the following rule for finding its value:

1421 Rule.—*Expand* f(x+h) *by some known theorem in ascending powers of* h; *subtract* f(x); *divide by* h; *and in the result put* h *equal to zero.*

The differential coefficients which follow are obtained by the rule and the theorems indicated.

1422 $$y = x^n. \qquad \frac{dy}{dx} = nx^{n-1}.$$

Proof.—Here $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + C(n,2)\,x^{n-2}h^2 +}{h}$ (125) $= nx^{n-1} + C(n,2)\,x^{n-2}h + \ldots = nx^{n-1}$, when h vanishes.

1423 Cor.—

$$\frac{d^ry}{dx^r} = n(n-1)\ldots(n-r+1)\,x^{n-r}. \qquad \frac{d^ny}{dx^n} = \lfloor n.$$

1424 $$y = \log_a x; \qquad \frac{dy}{dx} = \frac{1}{x\log_e a}.$$

Proof.—By (145), $\frac{\log_a(x+h)-\log_a x}{h} = \frac{1}{x\log_e a}\left\{\log_e\left(1+\frac{h}{x}\right)\right\} \div \frac{h}{x}.$

Expand the logarithm by (155).

1425 Cor.— $$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1}\lfloor n-1}{x^n \log a}.$$ Put $n=-1$ in (1423) and $r = n-1$.

1426 $$y = a^x; \qquad \frac{dy}{dx} = a^x \log_e a.$$

PROOF.— $\dfrac{a^{x+h}-a^x}{h} = \dfrac{a^x(a^h-1)}{h}$. Expand a^h by (149).

1427 COR.— $$\frac{d^ny}{dx^n} = a^x(\log_e a)^n.$$

	Function.	*Derivative.*	*Method of Proof by Rule* (1421) *and Limits* (753).
1428	$\sin x$.	$\cos x$.	Expand by (627, 629), and put $1-\cos h = 2\sin^2\frac{h}{2}$.
1429	$\cos x$.	$-\sin x$.	
1430	$\tan x$.	$\sec^2 x$.	Expand by (631), observing (1410).
1431	$\cot x$.	$-\operatorname{cosec}^2 x$.	By $\cot x = \frac{1}{\tan x}$, and (1415).
1432	$\sec x$.	$\tan x \sec x$.	By $\sec x = \frac{1}{\cos x}$, and (1415).
1433	$\operatorname{cosec} x$.	$-\cot x \operatorname{cosec} x$.	Similarly.
1434	$\left.\begin{matrix}\sin^{-1}x\\ \cos^{-1}x\end{matrix}\right\}$.	$\pm\frac{1}{\sqrt{(1-x^2)}}$.	If $\sin^{-1}x = y$, $x = \sin y$, therefore $\frac{dx}{dy} = \cos y = \sqrt{1-x^2}$; therefore $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.
1436	$\left.\begin{matrix}\tan^{-1}x\\ \cot^{-1}x\end{matrix}\right\}$.	$\pm\frac{1}{1+x^2}$.	
1438	$\left.\begin{matrix}\sec^{-1}x\\ \operatorname{cosec}^{-1}x\end{matrix}\right\}$.	$\pm\frac{1}{x\sqrt{(x^2-1)}}$.	Similarly for the rest.

EXAMPLES.

1440 $$(\sqrt{x})_x = (x^{\frac{1}{2}})_x = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}. \qquad (1422)$$

1441 $$\left(\frac{1}{x^n}\right)_x = (x^{-n})_x = -nx^{-n-1} = -\frac{n}{x^{n+1}}. \qquad (1422)$$

1442 $$\{(a+x^2)^3(b+x^3)^2\} = 3(a+x^2)^2 2x(b+x^3)^2 + 2(b+x^3)3x^2(a+x^2)^3$$
$$= 6x(a+x^2)^2(b+x^3)(b+ax+2x^3). \qquad (1412, ’15, ’22)$$

1443 $$\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)_x = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2}.$$
(1413, 1426)

1444 $$d_x(\log\tan x)^2 = 2\log\tan x\frac{1}{\tan x}\sec^2 x = \frac{4\log\tan x}{\sin 2x}. \qquad (1415, ’24, ’30).$$

Some differentiations are rendered easier by taking the logarithm of the function. For example,

1445 $y=\sqrt{\dfrac{1-x^2}{(1+x^2)^3}}$; therefore $\log y=\frac{1}{2}\log(1-x^2)-\frac{3}{2}\log(1+x^2)$;

therefore $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{-2x}{(1-x^2)}-\dfrac{3}{2}\dfrac{2x}{(1+x^2)}$;

therefore $\dfrac{dy}{dx}=y\dfrac{-2x(2-x^2)}{1-x^4}=\dfrac{-2x(2-x^2)}{(1+x^2)^{\frac{5}{2}}(1-x^2)^{\frac{1}{2}}}$.

1446 $y=(\sin x)^x$; therefore $\log y=x\log\sin x$;

therefore $\dfrac{1}{y}y_x=\log\sin x+\dfrac{x}{\sin x}\cos x$; (1415, '24, '28)

therefore $y_x=(\sin x)^x(\log\sin x+x\cot x)$.

Otherwise, by (1420), $y_x=x(\sin x)^{x-1}\cos x+(\sin x)^x\log\sin x$ (1426)
$=(\sin x)^x(x\cot x+\log\sin x)$.

SUCCESSIVE DIFFERENTIATION.

1460 *Leibnitz's Theorem.*—If n be any integer,

$$(yz)_{nx}=y_{nx}z+ny_{(n-1)x}z_x+C(n,2)\,y_{(n-2)x}z_{2x}+\dots$$
$$\dots+C(n,r)\,y_{(n-r)x}z_{rx}+\dots+yz_{nx}.$$

PROOF.—By Induction (233). Differentiate the two consecutive terms

$$C(n,r)\,y_{(n-r)x}z_{rx}+C(n,r+1)\,y_{(n-r-1)x}z_{(r+1)x},$$

and four terms are obtained, the second and third of which are

$$C(n,r)\,y_{(n-r)x}z_{(r+1)x}+C(n,r+1)\,y_{(n-r)x}z_{(r+1)x}$$
$$=\{C(n,r)+C(n,r+1)\}\,y_{(n-r)x}z_{(r+1)x}=C(n+1,r+1)\,y_{(\overline{n+1}-\overline{r+1})x}z_{(r+1)x},$$
by (102).

This is the general term of the series with n increased by unity. Similarly, by differentiating all the terms the whole series is reproduced with n increased by unity.

DIFFERENTIAL COEFFICIENTS OF THE n^{th} ORDER.

1461 $(\sin ax)_{nx}=a^n\sin(ax+\frac{1}{2}n\pi)$. By Induction

1462 $(\cos ax)_{nx}=a^n\cos(ax+\frac{1}{2}n\pi)$. and (1428).

1463 $(e^{ax})_{nx}=a^ne^{ax}$. (1426)

1464 $(e^{ax}y)_{nx}=e^{ax}(a+d_x)^n\,y$,

where, in the expansion by the Binomial Theorem, $d_x^r y$ is to be replaced by y_{rx}. (1460, '63)

1465 $$(e^{ax}\cos bx)_{nx} = r^n e^{ax}\cos(bx+n\phi),$$
where $$a = r\cos\phi \quad\text{and}\quad b = r\sin\phi.$$

PROOF.—By Induction. Differentiating once more, we obtain

$$r^n e^{ax}\{a\cos(bx+n\phi) - b\sin(bx+n\phi)\}$$
$$= r^{n+1}e^{ax}\{\cos\phi\cos(bx+n\phi) - \sin\phi\sin(bx+n\phi)\}$$
$$= r^{n+1}e^{ax}\cos(bx+\overline{n+1}\,\phi).$$

Thus n is increased by one.

1466 $$(x^{n-1}\log x)_{nx} = \lfloor n-1 \div x. \qquad (1460),\ (283)$$

1467 $$\left(\frac{1-x}{1+x}\right)_{nx} = \frac{(-1)^n\,2}{(1+x)^{n+1}}\lfloor n. \qquad (1423)$$

1468 $$(\tan^{-1}x)_{nx} = (-1)^{n-1}\lfloor n-1 \sin^n\theta\sin n\theta,$$
where $$\theta = \cot^{-1}x.$$

PROOF.—By Induction. Differentiating again, we obtain (omitting the coefficient)

$$(n\sin^{n-1}\theta\cos\theta\sin n\theta + n\cos n\theta\sin^n\theta)\,\theta_x$$
$$= n\sin^{n-1}\theta\,(\sin n\theta\cos\theta + \cos n\theta\sin\theta)\ (-\sin^2\theta).$$

Since, by (1437), $\theta_x = -(1+x^2)^{-1} = -\sin^2\theta.$

Therefore $(\tan^{-1}x)_{(n+1)x} = (-1)^n \lfloor n \sin^{n+1}\theta\sin(n+1)\theta,$
n being increased by one.

1469 $$\left(\frac{1}{1+x^2}\right)_{nx} = (-1)^n\lfloor n \sin^{n+1}\theta\sin(n+1)\theta. \qquad (1436, 1468)$$

1470 $$\left(\frac{x}{1+x^2}\right)_{nx} = (-1)^n\lfloor n \sin^{n+1}\theta\cos(n+1)\theta.$$

PROOF.—By (1460), $\left(\frac{x}{1+x^2}\right)_{nx} = x\left(\frac{1}{1+x^2}\right)_{nx} + n\left(\frac{1}{1+x^2}\right)_{(n-1)x}$
Then by (1469).

1471 *Jacobi's Formula.*
$$d_{(n-1)x}(1-x^2)^{n-\frac{1}{2}} = (-1)^{n-1}1.3\ldots(2n-1)\sin(n\cos^{-1}x) \div n.$$

PROOF.—Let $y = 1-x^2$; therefore

$$(y^{n+\frac{1}{2}})_{nx} = -(2n+1)(xy^{n-\frac{1}{2}})_{(n-1)x}. \quad \text{Also}\quad (y^{n+\frac{1}{2}})_{nx} = (yy^{n-\frac{1}{2}})_{nx}.$$

Expand each of these values by (1460) and eliminate $(y^{n-\frac{1}{2}})_{(n-2)x}$, the derivative of lowest order. Call the result equation (1). Now assume (1471) true for the value n. Differentiate and substitute the result, and also (1471) on the right side of equation (1) to obtain a proof by Induction.

1472 *Theorem.*—If y, z are functions of x, and n a positive integer,

$$zy_{nx} = (yz)_{nx} - n\,(yz_x)_{(n-1)\,x} + C\,(n, 2)\,(yz_{2x})_{(n-2)\,x} \dots + (-1)^n\,yz_{nx}.$$

PROOF.—By Induction. Differentiate for x, substituting for $z_x y_{nx}$ on the right its value by the formula itself.

PARTIAL DIFFERENTIATION.

1480 If $u = f(x, y)$ be a function of two *independent* variables, any differentiation of u with respect to x *requires that y should be considered constant* in that operation, and *vice versâ*. Thus, $\frac{d^2u}{dx^2}$ or u_{2x} signifies that u is to be differentiated successively twice with respect to x, y being considered constant.

1481 The notation $\frac{d^5u}{dx^2dy^3}$ or $u_{2x\,3y}$* signifies that u is to be differentiated successively twice for x, y being considered constant, and the result three times successively for y, x being considered constant.

1482 The *order* of the differentiations does not affect the final result, or $u_{xy} = u_{yx}$.

PROOF.—Let $u = f(xy)$; then $u_x = \frac{f(x+h,\,y) - f(x,\,y)}{h}$ in limit. (1484)

$$u_{xy} = \frac{du_x}{dy} = \frac{f(x+h,\,y+k) - f(x,\,y+k) - f(x+h,\,y) + f(x,\,y)}{hk} \text{ in limit.}$$

Now, if u_y had been first formed, and then u_{yx}, the same result would have been obtained. The proof is easily extended. Let $u_x = v$; then $u_{2xy} = v_{xy} = v_{yx} = u_{y2x}$; and so on.

THEORY OF OPERATIONS.

1483 Let the symbols Φ, Ψ, prefixed to a quantity, denote operations upon it of the *same class*, such as multiplication or differentiation. Then the law of the operation is said to be *distributive*, when

$$\Phi\,(x+y) = \Phi\,(x) + \Phi\,(y);$$

* See note to (1487).

2 M

that is, the operation may be performed upon an undivided quantity, or it may be distributed by being performed upon parts of the quantity separately with the same result.

1484 The law is said to be *commutative* when

$$\Phi\Psi x = \Psi\Phi x;$$

that is, the order of operation may be changed, Φ operating upon Ψx producing the same result as Ψ operating upon Φx.

1485 $\Phi^m x$ denotes the repetition of the operation Φ m times, and is equivalent to $\Phi\Phi \dots x$ to m operations. This definition involves the index law,

$$\Phi^m\Phi^n x = \Phi^{m+n} x = \Phi^{n+m} x,$$

which merely asserts, that to perform the operation n times in succession upon x, and afterwards m times in succession upon the result, is equivalent to performing it $m+n$ times in succession upon x.

1486 The three laws of *Distribution*, *Commutation*, and the law of *Indices* apply to the operation of *multiplication*, and also to that of *differentiation* (1411, '12). Therefore any algebraic transformation which proceeds at every step by one or more of these laws *only*, has a valid result when for the operation of *multiplication* that of *differentiation* is substituted.

1487 In making use of this principle, the symbol of differentiation employed is $\frac{d}{dx}$, or simply d_x prefixed to the quantity upon which the operation of differentiating with respect to x is to be performed. The repetition of the operation is indicated by $\frac{d^2}{dx^2}$, $\frac{d^3}{dx^3}$, $\frac{d^5}{dx^2dy^3}$, &c., prefixed to the function. An abbreviated notation is d_x, d_{2x}, d_{3x}, d_{2x3y}, &c. Since $d_x \times d_x = d_x^2$ in the symbolic operation of multiplication, it will be requisite, in transferring the operation to differentiation, to change all such *indices* to *suffixes* when the abbreviated notation is being used.

NOTE.—The notation y_x, y_{2x}, u_{2x3y}, d_{2x3y}, &c. is an innovation. It has, however, the recommendations of definiteness, simplicity, and economy of time in writing, and of space in printing. The expression $\frac{d^5u}{dx^2dy^3}$ requires at least fourteen distinct types, while its equivalent u_{2x3y} requires but seven. For