DYNAMICS.

CHAPTER I.

MOVING AXES AND RELATIVE MOTION.

Moving Axes.

1. In many problems in dynamics it is found that the axes of reference suitable to the initial state of the motion are not well adapted to follow the body under consideration during its whole course of motion. It is therefore sometimes convenient to use axes which themselves move in space so that they always keep those positions which are most appropriate to the instantaneous position of the body. Thus, to take a simple case, in dynamics of a particle we sometimes resolve our forces along the tangent and normal to the path. This is practically the same as using a set of Cartesian axes which move so as to be always parallel to the tangent and normal. This theory has been generalised in Vol. I. Chap. IV. where the motion is referred to any two lines whatever which move in one plane. We now propose to extend the theory still further. We shall discuss the general equations of motion of a particle and then those of a rigid body referred to any rectangular axes which move as we may find convenient.

2. If we make the axes to which we refer the body move, it is clear that we must have some means of determining the position and motion of these axes in space. This might be effected by having another set of axes which are themselves fixed in space and to which in turn we might refer the moving axes. This is the course adopted by Euler; thus in the equations usually called after his name (Vol. I. Chap. V.) he uses two sets of axes. The advantage of giving motion to the axes is however greatly

diminished if we must also use a set of fixed axes throughout the motion. For this reason *we shall now determine the motion of the moving axes by angular velocities* θ_1, θ_2, θ_3 *about themselves.* In other words, we regard the axes as if they were a material system of three straight lines at right angles whose motion at any instant was given by three coexistent angular velocities about axes which instantaneously coincided with them. In this way we do not use any fixed axes except at the beginning or end of the solution, and only in such a manner as we may find convenient.

3. In order to understand how the motion of a body is referred to moving axes let us first suppose that the body is turning about a fixed point. Taking this point as origin we determine the motion of the body by three angular velocities ω_1, ω_2, ω_3 about the axes in the same manner as if the axes were fixed in space. The position of the body at the time $t + dt$ may be constructed from that at the time t by turning the body through the angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$ successively round the instantaneous positions of the axes. But it must be remembered that $\omega_3 dt$ does not now give the angle the body has been turned through relatively to the plane xz, but relatively to some plane fixed in space passing through the instantaneous position of the axis of z. The angle turned through relatively to the plane of xz is $(\omega_3 - \theta_3)\, dt$.

If there be no fixed point we use the construction explained in Vol. I. Chap. V. We represent the motion of the body by the six components u, v, w; $\omega_1, \omega_2, \omega_3$ referred to any origin, the axes being treated as if they were fixed for the moment. Here u, v, w are the resolved parts in the directions of the axes of the velocity of the origin or base point, and ω_1, ω_2, ω_3 are the resolved parts about the same axes of the angular velocity of the body. In the same way the motion of the axes is given by the components of motion p, q, r; $\theta_1, \theta_2, \theta_3$, the moving axes being themselves the instantaneous axes of reference.

In most cases however the axes will be made to turn round some point which either is fixed or may be treated as fixed. Their directions in space are made to vary in a manner suitable to the purpose we have in hand. We then have p, q, r all zero. Since any point may be reduced to rest by the method explained in Vol. I. Chap. IV. this supposition, which will be generally made, does not really limit our choice of axes.

4. **Fundamental Theorem.** *A system of rectangular axes moves in any manner about a fixed point* O, *it is required to establish the kinematical relations between these axes and a system of axes fixed in space and coincident with them at any time* t.

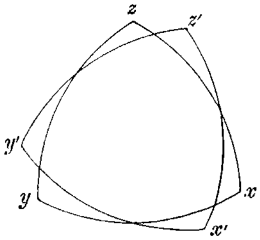
Let Ox, Oy, Oz be the positions of the moving axes at the time t; after an interval dt these assume new positions, which we represent by Ox', Oy', Oz'. The change of position may be represented by a rotation θdt about some instantaneous axis, which we may represent by OI. Let θ_1, θ_2, θ_3 be the components of the angular velocity θ, so that the axes are moved from their positions Ox, Oy, Oz at the time t into their positions Ox', Oy', Oz' at the time $t+dt$ by the three rotations $\theta_1 dt$, $\theta_2 dt$, $\theta_3 dt$ about Ox, Oy, Oz performed in any order.

Let us represent by the symbol R any directed quantity or vector, such as a force, a velocity, the moment of a couple about its axis, or an angular momentum. Let us suppose that the vector may be resolved and compounded according to the "parallelogram law." Let us represent its components parallel to the three axes Ox, Oy, Oz by the symbols U, V, W. In the time dt the vector R has changed its magnitude and direction; in the same time the axes have also changed. The components of the vector at the time $t+dt$ in the then direction of the axes of reference, i.e. in the directions Ox', Oy', Oz' are $U+dU$, $V+dV$, $W+dW$.

We wish to find the increase in the time dt of the component in the direction of the axis Ox *supposed fixed in space*. Describe a sphere of unit radius whose centre is at O and let the axes cut the sphere in the points x, y, z, x', y', z'. Thus we have two spherical triangles xyz, $x'y'z'$, all whose sides are right angles. The resolved part of the vector at the time $t+dt$ along the axis Ox is

$$(U+dU)\cos xx' + (V+dV)\cos xy' + (W+dW)\cos xz'.$$

The rotations about Ox and Oy cannot alter the arc xy, but the rotation about Oz will move y' away from x by the arc $\theta_3 dt$. In the same way the rotations about Ox and Oz cannot alter the arc xz but the rotation about Oy will move z' towards x by the arc $\theta_2 dt$. Therefore



$$xy' = xy + \theta_3 dt,$$
$$xz' = xz - \theta_2 dt.$$

Also the cosine of the arc xx' differs from unity by the square of a small quantity. Substituting λ, we find that at the time $t+dt$ the component of the vector along Ox is

$$U + dU - V\theta_3 dt + W\theta_2 dt.$$

The rate of increase of the component of the vector in the direction Ox is

$$U_1 = \frac{dU}{dt} - V\theta_3 + W\theta_2.$$

In the same way the rates of increase of the components in the directions Oy, Oz are

$$V_1 = \frac{dV}{dt} - W\theta_1 + U\theta_3,$$

$$W_1 = \frac{dW}{dt} - U\theta_2 + V\theta_1.$$

We have here practically used two sets of axes. One set Ox, Oy, Oz moves about the fixed origin according to the law determined by the angular velocities θ_1, θ_2, θ_3, these are the axes of reference. Another set coincides with Ox, Oy, Oz at the time t, but is fixed in space and is therefore left behind by the axes of reference as they move in the time dt. The symbols U, V, W represent the resolved parts of the vector along either set of axes at the time t. The symbols $U + dU$, $V + dV$, $W + dW$ represent the components along the *moving axes* at the time $t + dt$; and $U + U_1\,dt$, $V + V_1\,dt$, $W + W_1\,dt$, represent the components along the *fixed axes* at the same time $t + dt$.

5. **Important Applications.** We may now apply this general theorem to a variety of vectors*.

(1) Let the vector R be the radius vector of a moving point P. Then U, V, W represent the co-ordinates x, y, z; while U_1, V_1, W_1 represent the component velocities in space. These we now represent by u, v, w. Therefore

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

(2) Let the vector R be the velocity of a moving point P. Then U, V, W represent the component velocities u, v, w; while U_1, V_1, W_1 represent the accelerations. These we represent by X, Y, Z. Therefore

* The sets of equations (1) (2) (3) were given in this form by the late Prof. Slesser (*Cambridge Quarterly Journal*, Vol. II., 1858) to whom the two special cases given further on in Art. 12 had been previously shown by the author, together with their application to the motion of spheres. Other proofs were given of them in the following number of the *Quarterly Journal* by Rev. P. Frost. All four sets of equations were given by R. B. Hayward in Vol. X. of the *Cambridge Transactions*, 1856. Similar results were also given in *Liouville's Journal*, 1858.

$$X=\frac{du}{dt}-v\theta_3+w\theta_2,$$

$$Y=\frac{dv}{dt}-w\theta_1+u\theta_3,$$

$$Z=\frac{dw}{dt}-u\theta_2+v\theta_1.$$

(3) Let the vector R be the angular velocity Ω of a body. Then U, V, W are the components of ω about the moving axes, let us call these $\omega_1, \omega_2, \omega_3$. Let $\omega_x, \omega_y, \omega_z$ be the components about the fixed axes. Then we have

$$\frac{d\omega_x}{dt}=\frac{d\omega_1}{dt}-\omega_2\theta_3+\omega_3\theta_2,$$

$$\frac{d\omega_y}{dt}=\frac{d\omega_2}{dt}-\omega_3\theta_1+\omega_1\theta_3,$$

$$\frac{d\omega_z}{dt}=\frac{d\omega_3}{dt}-\omega_1\theta_2+\omega_2\theta_1.$$

(4) Let the vector R be the angular momentum of a body. Let h_1, h_2, h_3 be its components about the moving axes; h_x, h_y, h_z the components about fixed axes. Then

$$\frac{dh_x}{dt}=\frac{dh_1}{dt}-h_2\theta_3+h_3\theta_2,$$

$$\frac{dh_y}{dt}=\frac{dh_2}{dt}-h_3\theta_1+h_1\theta_3,$$

$$\frac{dh_z}{dt}=\frac{dh_3}{dt}-h_1\theta_2+h_2\theta_1.$$

If the origin of co-ordinates is also in motion, these equations require some modifications. Let (p, q, r) be the resolved parts of the velocity of the origin in the directions of the axes. If (u, v, w) represent the resolved velocities of the centre of gravity in space i.e. referred to axes fixed in space we must add p, q, r respectively to the expressions for u, v, w given by (1). Supposing (u, v, w) to continue to represent the velocities referred to axes fixed in space, the expressions (2) will be unaltered. On the same supposition we must add $m(-vr+wq)$, $m(-wp+ur)$, $m(-uq+vp)$ respectively to the expressions for dh_x/dt &c. given by (4), where m is the mass of the body.

To prove this let us determine the parts of dh_x and dh_1 due to the translational and rotational motion of the axes separately. Those of the latter are given by the formulae (4); to find those of the former, let H_x, H_y, H_z be the angular momenta about parallel axes through the centre of gravity. Then, by Vol. I. Chap. I.,

$$h_x=h_1=H_x-mvz+mwy.$$

The differential coefficient dh_x/dt is obtained from this on the supposition that we write $r + dz/dt$, $q + dy/dt$ for dz/dt and dy/dt, because these are the resolved velocities in space of the centre of gravity. The differential coefficient dh_1/dt is obtained without the addition of r and q. We therefore have

$$dh_x/dt = dh_1/dt - mvr + mwq.$$

We may notice that, if the moving set of axes be fixed in the body and move with it, $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, $\theta_3 = \omega_3$. The third set of equations then show that

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}, \quad \frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}, \quad \frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}.$$

These simplified forms are the ones used by Euler in obtaining his equations of motion of a rigid body about a fixed point. See Vol. I. Chap. V.

6. The above results may be obtained in other ways, but there is an obvious advantage in deducing them all by one method.

The equations connecting (u, v, w) with the co-ordinates (x, y, z) may be obtained as follows. The resolved velocities in space of a point P are not given by dx/dt, dy/dt, dz/dt. These are the resolved velocities *relatively* to the moving axes. To find the motion *in space* we must add to these the resolved velocities due to the motion of the axes. If we supposed the particle to be rigidly connected with the axes, its velocities would be expressed by the forms $\theta_2 z - \theta_3 y$, &c. given in Vol. I. Chap. V. By adding the parts together the actual resolved velocities of the particle are found to be those given above.

Since acceleration is the rate of increase of velocity, just as velocity is the rate of increase of space, it is clear that the relations which hold between accelerations and velocities must be the same as those which hold between velocities and spaces. Thus the relations (2) between (X, Y, Z) and (u, v, w) follow at once from those between (u, v, w) and (x, y, z).

7. Ex. 1. Let the motion be referred to *oblique* moving axes so that the sides of the spherical triangle xyz are a, b, c, and the angles A, B, C. Let the equal quantities $\sin a \sin b \sin C$, $\sin b \sin c \sin A$, $\sin c \sin a \sin B$ be called μ. Prove that, if the velocity be represented by the three *components* u, v, w parallel to these axes, then the *resultant* acceleration parallel to the axis of z is

$$Z = \frac{dw}{dt} + \frac{du}{dt}\cos b + \frac{dv}{dt}\cos a - u\theta_2\mu + v\theta_1\mu,$$

with similar expressions for X and Y.

This may be done by the use of the spherical triangles xyz, $x'y'z'$, by first proving $zx' = b + \theta_2 dt \sin c \sin A$, $zy' = a - \theta_1 dt \sin c \sin B$, and then substituting as before.

Ex. 2. Prove in the same way that, if x, y, z be the co-ordinates referred to oblique axes moving about a fixed origin, and u', v', w' the *resultant* velocities parallel to the axes, $w' = \frac{dz}{dt} + \frac{dx}{dt}\cos b + \frac{dy}{dt}\cos a - x\theta_2\mu + y\theta_1\mu$, with similar expressions for u' and v'.

Ex. 3. Prove also that the equations connecting the components, u, v, w with the co-ordinates x, y, z referred to axes with a fixed origin are

$$w = \frac{dz}{dt} + \begin{vmatrix} \mu^{-1}\sin^2 c & -\cot B & -\cot A \\ \theta_3 & \theta_1 & \theta_2 \\ z & x & y \end{vmatrix}$$

with two similar expressions for u and v.

Since w' is the component parallel to z of $(u, v, w,)$, we have $u \cos b + v \cos a + w = w'$, with similar expressions for u' and v'. By solving these we get the required values of u, v, w.

Ex. 4. If the whole acceleration be represented by the three components X, Y, Z parallel to the axes, prove that the expressions for these in terms of u, v, w may be obtained from those given in the last example by changing x, y, z into u, v, w and u, v, w into X, Y, Z.

8. *To explain another general method of obtaining the kinematical relation between fixed and moving axes.*

Let U, V, W be, as before, the components of a vector R. Let OL be any straight line *fixed in space* making with the moving axes the angles α, β, γ. Let R_1 be the resolved part of the vector along OL. Then

$$R_1 = U \cos \alpha + V \cos \beta + W \cos \gamma,$$

$$\therefore \frac{dR_1}{dt} = \frac{dU}{dt} \cos \alpha + \frac{dV}{dt} \cos \beta + \frac{dW}{dt} \cos \gamma$$

$$- U \sin \alpha \frac{d\alpha}{dt} - V \sin \beta \frac{d\beta}{dt} - W \sin \gamma \frac{d\gamma}{dt}.$$

Since OL is any fixed line in space, let it be so chosen that the moving axis of z coincides with it at the time t. Then $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, $\gamma = 0$, also $dR_1/dt = W_1$. Since α is the angle OL makes with the moving axis of x, $d\alpha/dt$ expresses the rate at which the axis of x is separating from a fixed straight line coincident with the axis of z and this is clearly θ_2. Similarly $d\beta/dt = -\theta_1$, hence

$$W_1 = \frac{dW}{dt} - U\theta_2 + V\theta_1$$

where W_1 expresses the rate of increase of the component W along the fixed axis of z. The other two equations follow in the same way. The principle of this method is due to the late Prof. Slesser.

We may obtain the relations between the second and higher differential coefficients in the same way, though the expressions become more complicated. Since U_1, V_1, W_1 follow the parallelogram law, we have

$$\frac{dR_1}{dt} = \left(\frac{dU}{dt} - V\theta_3 + W\theta_2\right) \cos \alpha + \left(\frac{dV}{dt} - W\theta_1 + U\theta_3\right) \cos \beta + \left(\frac{dW}{dt} - U\theta_2 + V\theta_1\right) \cos \gamma.$$

Repeating the same reasoning, we finally obtain

$$\frac{dW_1}{dt} = \frac{d}{dt}\left(\frac{dW}{dt} - U\theta_2 + V\theta_1\right) - \theta_2\left(\frac{dU}{dt} - V\theta_3 + W\theta_2\right) + \theta_1\left(\frac{dV}{dt} - W\theta_1 + U\theta_3\right).$$

9. We have now obtained a method of transforming the equations of motion with regard to fixed axes into those with regard to axes moving about a fixed origin.

Let any general equation true for *all* fixed axes having a given origin be

$$\psi\,\{\omega_x,\ d\omega_x/dt,\ \&\text{c.}\ldots\} = 0,$$

where ω_x, ω_y, ω_z are the angular velocities about the fixed axes.

Since the fixed axes are *arbitrary in position*, let them be so chosen that the three moving axes are passing through them at the moment under consideration; thus at that instant the two sets are coincident. The equations relative to the moving axes may then be deduced by replacing ω_x, ω_y, ω_z in the general equation $\psi = 0$ by the corresponding quantities ω_1, ω_2, ω_3 for the moving axes; and $d\omega_x/dt$, &c. by the equivalents written above in Art. 5.

The same remarks apply if, instead of ω_x, ω_y, ω_z, the components of any other vector entered into the equation.

10. **General equations of Motion.** *To state the general equations of motion of a system of moving bodies referred to any rectangular axes moving about a fixed origin.*

Let m be the mass of any one body of the system. Let the impressed forces on the body be represented by the three forces mX, mY, mZ acting at its centre of gravity and the three couples L, M, N. We suppose that the unknown reactions of the other bodies of the system are included in these expressions.

Let (u, v, w) be the resolved velocities in space of the centre of gravity of the body. The equations of motion for fixed axes are $u = dx/dt$, $X = du/dt$, &c. When the axes move, these become

$$u = \frac{dx}{dt} - \theta_3 y + \theta_2 z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$X = \frac{du}{dt} - \theta_3 v + \theta_2 w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

with corresponding expressions for the other coordinate axes.

Let (h_1, h_2, h_3) be the angular momenta of the body about parallels to the co-ordinate axes drawn through the centre of gravity.

The equations of moments for fixed axes are $dh_x/dt = L$, &c., Vol. I. Chap. II. When the axes are in motion these become

$$L = \frac{dh_1}{dt} - h_2\theta_3 + h_3\theta_2 \quad \text{.....................(4)},$$

with similar expressions for M and N.

The expressions for (h_1, h_2, h_3) in terms of the angular velocities of the body are given in Vol. I. Chap. v. If ω_1, ω_2, ω_3 be the augular velocities of the body about the parallels to the axes through the centre of gravity, and A, F, &c. the moments and products of inertia, the fundamental relation is

$$h_1 = A\omega_1 - F\omega_2 - E\omega_3$$

with similar expressions for h_2 and h_3. But there are many others which cannot be repeated here.

Besides the dynamical equations there will be the geometrical equations which express the connections of the system. As every such forced connection is accompanied by some reaction, the number of geometrical equations will be the same as the number of unknown reactions. Thus we have sufficient equations to determine the motion.

Ex. A heavy rigid body is spitted on a smooth circularly-cylindrical rod, on which it can slide, and which passes through its centre of gravity, and the rod is made to rotate uniformly with angular velocity ω in a right circular cone, semi-vertical angle a, about a vertical axis. If C is the moment of inertia about the rod, A and B about two lines fixed in the body perpendicular to the rod, one of which is inclined at an angle ϕ to the plane through the vertical axis and the rod, and if D, E, F are the products of inertia; prove that

$$Cd^2\phi/dt^2 = \omega^2\sin^2 a\,\{(B-A)\sin\phi\cos\phi + F\cos 2\phi\} - \omega^2 \sin a\cos a\,(E\sin\phi + D\cos\phi).$$

By resolving the angular velocity ω we find $\omega_1 = -\omega\sin a\cos\phi$, $\omega_2 = \omega\sin a\sin\phi$, $\omega_3 = \dot\phi + \omega\cos a$. Substituting these in the expressions for $h_1h_2h_3$ given in Art. 10, and equating to zero the moment of the effective forces about the vertical, the result follows at once. [Math. Tripos, 1885.

11. The motion of the moving axes has been supposed to be determined by the three angular velocities θ_1, θ_2, θ_3. To find their actual position in space we use the Eulerian geometrical equations already given in Vol. I. Chap. v. Let θ, ψ, ϕ be the Eulerian angular coordinates of the moving axes referred to any axes fixed in space. We then have

$$\theta_1 = \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi,$$

$$\theta_2 = \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi,$$

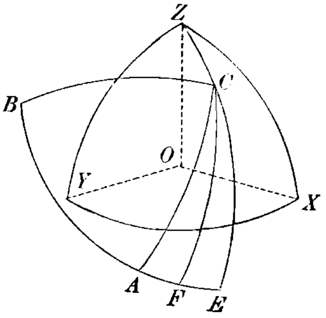
$$\theta_3 = \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta.$$

These geometrical equations determine θ, ϕ, ψ when $\theta_1, \theta_2, \theta_3$ are known.

12. **Two important special cases.** There are two cases in which the equations of motion just found admit of great simplification. As these often occur, it is worth while to discuss them separately.

In the first case we suppose the body to be turning round some point O fixed in space and to be such that *two of the principal moments of inertia at the fixed point are equal.*

Let OC be the axis of unequal moment of inertia and let us take this as the moving axis of z. Let us choose as the other axes of reference two other axes OA, OB which turn round OC in any manner we please. To fix this let χ be the angle the plane COA makes with some plane OCF fixed in the body and passing through OC. Then we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, and $\theta_3 = \omega_3 + d\chi/dt$. Also $h_1 = A\omega_1$, $h_2 = B\omega_2$, $h_3 = C\omega_3$. The equations of moments, Art. 10, are now



$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A - C)\,\omega_2\omega_3 &= L \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A - C)\,\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad &= N \end{aligned}\right\}.$$

In this case the most convenient geometrical equations to express the relations of these moving axes to axes OX, OY, OZ fixed in space are those usually called Euler's geometrical equations. They are given at length in the last article, where ω_1, ω_2 and $\omega_3 + d\chi/dt$ must of course be written on the left-hand sides for θ_1, θ_2, θ_3. In the figure $ZC = \theta$, $XZC = \psi$, $ECA = \phi$.

13. Since $d\chi/dt$ is arbitrary, it may be chosen to simplify either the dynamical equations or the geometrical equations.

I. If we put $d\chi/dt = -\omega_3$, the moving axes of reference move round the axis of OC with an angular velocity relatively to the body equal and opposite to that of the body, so that if the axis OC were fixed in space the axes of reference would be also fixed in space. The dynamical equations then become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + C\omega_2\omega_3 &= L \\ A\frac{d\omega_2}{dt} - C\omega_1\omega_3 &= M \\ C\frac{d\omega_3}{dt} \qquad\quad &= N \end{aligned}\right\}$$