CHAPTER I.

ON FINDING MOMENTS OF INERTIA BY INTEGRATION.

1. In the subsequent pages of this work it will be found that certain integrals continually recur. It is therefore convenient to collect these into a preliminary chapter for reference. Though their bearing on dynamics may not be obvious beforehand, yet the student may be assured that it is as useful to be able to write down moments of inertia with facility as it is to be able to quote the centres of gravity of the elementary bodies.

In addition however to these necessary propositions there are many others which are useful as giving a more complete view of the arrangement of the axes of inertia in a body. These also have been included in this chapter though they are not of the same importance as the former.

2. All the integrals used in dynamics as well as those used in statics and some other branches of mixed mathematics are included in the one form

$$\iiint x^\alpha y^\beta z^\gamma \, dx \, dy \, dz,$$

where (α, β, γ) have particular values. In statics two of these three exponents are usually zero, and the third is either unity or zero, according as we wish to find the numerator or denominator of a co-ordinate of the centre of gravity. In dynamics of the three exponents one is zero, and the sum of the other two is usually equal to 2. The integral in all its generality has not yet been fully discussed, probably because only certain cases have any real utility. In the case in which the body considered is a homogeneous ellipsoid the value of the general integral has been found in gamma functions by Lejeune Dirichlet in Vol. IV. of Liouville's journal. His results were afterwards extended by Liouville in the same volume to the case of a heterogeneous ellipsoid in which the strata of uniform density are similar ellipsoids.

In this treatise, it is intended chiefly to restrict ourselves to the consideration of moments and products of inertia, as being the only cases of the integral which are useful in dynamics.

3. **Definitions.** If the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the *moment of inertia* of the system about that line.

If M be the mass of a system and k be such a quantity that Mk^2 is its moment of inertia about a given straight line, then k is called the *radius of gyration* of the system about that line.

The term "moment of inertia" was introduced by Euler, and has now come into general use wherever Rigid Dynamics is studied. It will be convenient for us to use the following additional terms.

If the mass of every particle of a material system be multiplied by the square of its distance from a given plane or from a given point, the sum of the products so formed is called the moment of inertia of the system with reference to that plane or that point.

If two straight lines Ox, Oy be taken as axes, and if the mass of every particle of the system be multiplied by its *two* co-ordinates x, y, the sum of the products so formed is called the *product of inertia* of the system about those two axes.

This might, perhaps more conveniently, be called the product of inertia of the system with reference to the two co-ordinate planes xz, yz.

The term *moment of inertia with regard to a plane* seems to have been first used by M. Binet in the *Journal Polytechnique*, 1813.

4. Let a body be referred to any rectangular axes Ox, Oy, Oz meeting in a point O, and let x, y, z be the co-ordinates of any particle m, then according to these definitions the moments of inertia about the axes of x, y, z respectively will be

$$A = \Sigma m(y^2 + z^2), \quad B = \Sigma m(z^2 + x^2), \quad C = \Sigma m(x^2 + y^2).$$

The moments of inertia with regard to the planes yz, zx, xy, respectively, will be

$$A' = \Sigma mx^2, \quad B' = \Sigma my^2, \quad C' = \Sigma mz^2.$$

The products of inertia with regard to the axes yz, zx, xy, will be

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad F = \Sigma mxy.$$

Lastly, the moment of inertia with regard to the origin will be

$$H = \Sigma m(x^2 + y^2 + z^2) = \Sigma mr^2,$$

where r is the distance of the particle m from the origin.

5. **Elementary Propositions.** The following propositions may be established without difficulty, and will serve as illustrations of the preceding definitions.

(1) The three moments of inertia A, B, C about three rectangular axes are such that the sum of any two of them is greater than the third.

(2) The sum of the moments of inertia about any three rectangular axes meeting at a given point is always the same; and is equal to twice the moment of inertia with respect to that point.

For $A+B+C=2\Sigma m(x^2+y^2+z^2)=2\Sigma mr^2$, and is therefore independent of the directions of the axes.

(3) The sum of the moments of inertia of a system with reference to any plane through a given point and its normal at that point is constant and equal to the moment of inertia of the system with reference to that point.

Take the given point as origin and the plane as the plane of xy, then $C'+C=\Sigma mr^2$, which is independent of the directions of the axes.

Hence we infer that

$$A'=\tfrac{1}{2}(B+C-A),\ B'=\tfrac{1}{2}(C+A-B),\ \text{and}\ C'=\tfrac{1}{2}(A+B-C).$$

(4) Any product of inertia as D cannot numerically be so great as $\tfrac{1}{2}A$.

(5) If A, B, F be the moments and product of inertia of a lamina about two rectangular axes in its plane, then AB is greater than F^2.

If t be any quantity we have $At^2+2Ft+B=\Sigma m(yt+x)^2=$ a positive quantity. Hence the roots of the quadratic $At^2+2Ft+B=0$ are imaginary, and therefore $AB>F^2$.

(6) Prove that for any body

$$(A+B-C)(B+C-A)>4E^2,$$

$$(A+B-C)(B+C-A)(C+A-B)>8DEF.$$

(7) The moment of inertia of the surface of a sphere of radius a and mass M about any diameter is $M\tfrac{2}{3}a^2$.

Since every element is equally distant from the centre its moment of inertia about the centre is Ma^2. Hence by (2) the result follows.

(8) The moment of inertia of the surface of a hemisphere of radius a and mass M about a diameter is $M\tfrac{2}{3}a^2$.

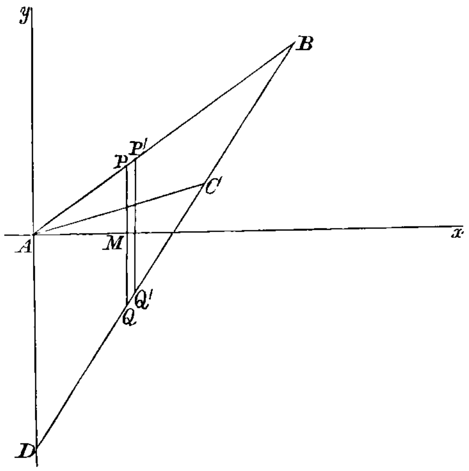
This follows immediately from (7) by completing the sphere.

6. It is clear that the process of finding moments and products of inertia is merely that of integration. We may illustrate this by the following example.

To find the moment of inertia of a uniform triangular plate about an axis in its plane passing through one angular point.

Let ABC be the triangle, Ay the axis about which the moment is required. Draw Ax perpendicular to Ay and produce

BC to meet Ay in D. The given triangle ABC may be regarded as the difference of the triangles ABD, ACD. Let us then first



find the moment of inertia of ABD. Let $PQP'Q'$ be an elementary area whose sides PQ, $P'Q'$ are parallel to the base AD, and let PQ cut Ax in M. Let β be the distance of the angular point B from the axis Ay, $AM = x$ and $AD = l$.

Then the elementary area $PQP'Q'$ is clearly $l\dfrac{\beta - x}{\beta}\,dx$, and its moment of inertia about Ay is $\mu l\dfrac{\beta - x}{\beta}\,dx\,.\,x^2$, where μ is the mass per unit of area. Hence the moment of inertia of the triangle ABD

$$= \mu\int_0^\beta l\left(1 - \frac{x}{\beta}\right)x^2dx = \tfrac{1}{12}\mu l\beta^3.$$

Similarly if γ be the distance of the angular point C from the axis Ay, the moment of inertia of the triangle ACD is $\frac{1}{12}\mu l\gamma^3$. Hence the moment of inertia of the given triangle ABC is $\frac{1}{12}\mu l(\beta^3 - \gamma^3)$. Now $\frac{1}{2}l\beta$ and $\frac{1}{2}l\gamma$ are the areas of the triangles ABD, ACD. Hence if M be the mass of the triangle ABC, the moment of inertia of the triangle about the axis Ay is

$$\tfrac{1}{6}M(\beta^2 + \beta\gamma + \gamma^2).$$

Ex. If each element of the mass of the triangle be multiplied by the nth power of its distance from the straight line through the angle A, then it may be proved in the same way that the sum of the products is

$$\frac{2M}{(n+1)(n+2)}\,\frac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma}.$$

7. *When the body is a lamina the moment of inertia about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two rectangular axes in its plane drawn from the point where the former axis meets the plane.*

For let the axis of z be taken normal to the plane, then, if A, B, C be the moments of inertia about the axes, we have,

$$A = \Sigma m y^2, \quad B = \Sigma m x^2, \quad C = \Sigma m (x^2 + y^2),$$

and therefore $$C = A + B.$$

We may apply this theorem to the case of the triangle. Let β', γ', be the distances of the points B, C from the axis Ax. Then the moment of inertia of the triangle about a normal to the plane of the triangle through the point A is

$$= \tfrac{1}{6} M (\beta^2 + \beta\gamma + \gamma^2 + \beta'^2 + \beta'\gamma' + \gamma'^2).$$

Ex. Prove that the moment of inertia of the perimeter of a circle of radius a and mass M about any diameter is $\frac{1}{2} Ma^2$.

Since every element is equally distant from the axis of the circle, the moment of inertia about that axis is Ma^2. The result follows at once.

8. **Reference Table.** The following moments of inertia occur so frequently that they have been collected together for reference. The reader is advised to commit to memory the following table:

The moment of inertia of

(1) A rectangle whose sides are $2a$ and $2b$

about an axis through its centre in its plane perpendicular to the side $2a$ $\Big\} = \text{mass } \dfrac{a^2}{3}$,

about an axis through its centre perpendicular to its plane $\Big\} = \text{mass } \dfrac{a^2 + b^2}{3}$.

(2) An ellipse semi-axes a and b

about the major axis $a = \text{mass } \dfrac{b^2}{4}$,

about the minor axis $b = \text{mass } \dfrac{a^2}{4}$,

about an axis perpendicular to its plane through the centre $\Big\} = \text{mass } \dfrac{a^2 + b^2}{4}$.

In the particular case of a circle of radius a, the moment of inertia about a diameter $= \text{mass } \dfrac{a^2}{4}$, and that about a perpendicular to its plane through the centre $= \text{mass } \dfrac{a^2}{2}$.

(3) An ellipsoid semi-axes a, b, c

$$\text{about the axis } a = \text{mass } \frac{b^2 + c^2}{5}.$$

In the particular case of a sphere of radius a the moment of inertia about a diameter = mass $\frac{2}{5} a^2$.

(4) A right solid whose sides are $2a$, $2b$, $2c$

$$\left.\begin{array}{r}\text{about an axis through its centre perpendicular}\\ \text{to the plane containing the sides } b \text{ and } c\end{array}\right\} = \text{mass } \frac{b^2 + c^2}{3}.$$

These results may be all included in one rule, which the author has long used as an assistance to the memory.

$$\left.\begin{array}{r}\text{Moment of inertia}\\ \text{about an axis}\\ \text{of symmetry}\end{array}\right\} = \text{mass } \frac{\text{(sum of squares of perpendicular semi-axes)}}{3, 4 \text{ or } 5}.$$

The denominator is to be 3, 4 or 5, according as the body is rectangular, elliptical or ellipsoidal.

Thus, if we require the moment of inertia of a circle of radius a about a diameter, we notice that the perpendicular semi-axes in its plane is the radius a, and that the semi-axis perpendicular to its plane is zero, the moment of inertia required is therefore $M\frac{a^2}{4}$, if M be the mass. If we require the moment about a perpendicular to its plane through the centre, we notice that the perpendicular semi-axes are each equal to a and the moment required is therefore $M\frac{a^2 + a^2}{4} = M\frac{a^2}{2}$.

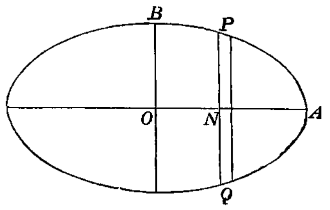
9. As the process for determining these moments of inertia is very nearly the same for all these cases, it will be sufficient to consider only two instances.

To determine the moment of inertia of an ellipse about the minor axis.

Let the equation to the ellipse be $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Take any elementary area PQ parallel to the axis of y, then clearly the moment of inertia is

$$4\mu \int_0^a x^2 y\, dx = 4\mu \frac{b}{a} \int_0^a x^2 \sqrt{a^2 - x^2}\, dx,$$

where μ is the mass of a unit of area.



To integrate this, put $x = a \sin \phi$, then the integral becomes

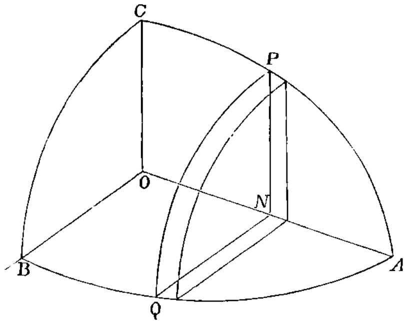
$$a^4 \int_0^{\frac{\pi}{2}} \cos^2 \phi \sin^2 \phi \, d\phi = a^4 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\phi}{8} d\phi = \frac{\pi a^4}{16};$$

$$\therefore \text{ the moment of inertia} = \mu \pi ab \frac{a^2}{4} = \text{mass } \frac{a^2}{4}.$$

In the same way we may show that the product of inertia of an elliptic quadrant about its axes = mass $\frac{ab}{2\pi}$.

To determine the moment of inertia of an ellipsoid about a principal diameter.

Let the equation to the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Take any elementary area PNQ parallel to the plane of yz. Its area is evidently $\pi PN . QN$. Now PN is the



value of z when $y = 0$, and QN the value of y when $z = 0$, as obtained from the equation to the ellipsoid; $\therefore PN = \frac{c}{a}\sqrt{a^2 - x^2}$, $QN = \frac{b}{a}\sqrt{a^2 - x^2}$;

$$\therefore \text{ the area of the element} = \frac{\pi bc}{a^2}(a^2 - x^2).$$

Let μ be the mass of the unit of volume, then the whole moment of inertia

$$= \mu \int_{-a}^{a} \frac{\pi bc}{a^2}(a^2 - x^2) \frac{PN^2 + QN^2}{4} dx$$

$$= \mu \frac{\pi}{4} \frac{bc}{a^2} \int_{-a}^{a} (a^2 - x^2) \frac{b^2 + c^2}{a^2} (a^2 - x^2) \, dx$$

$$= \mu \frac{4}{3} \pi abc \frac{b^2 + c^2}{5} = \text{mass } \frac{b^2 + c^2}{5}.$$

In the same way we may show that the product of inertia of the octant of an ellipsoid about the axes of $(x, y) = \text{mass } \frac{2ab}{5\pi}$.

Ex. 1. The moment of inertia of an arc of a circle whose radius is a and which subtends an angle 2α at the centre

(a) about an axis through its centre perpendicular to its plane $= Ma^2$,

(b) about an axis through its middle point perpendicular to its plane

$$= 2M\left(1 - \frac{\sin \alpha}{\alpha}\right) a^2,$$

(c) about the diameter which bisects the arc $=M\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\frac{a^2}{2}$.

Ex. 2. The moment of inertia of the part of the area of a parabola cut off by any ordinate at a distance x from the vertex is $\frac{3}{7}Mx^2$ about the tangent at the vertex, and $\frac{1}{5}My^2$ about the principal diameter, where y is the ordinate corresponding to x.

Ex. 3. The moment of inertia of the area of the lemniscate $r^2=a^2\cos 2\theta$ about a line through the origin in its plane and perpendicular to its axis is $M\frac{3\pi+8}{48}a^2$.

Ex. 4. A lamina is bounded by four rectangular hyperbolas, two of them have the axes of co-ordinates for asymptotes, and the other two have the axes for principal diameters. Prove that the sum of the moments of inertia of the lamina about the co-ordinate axes is $\frac{1}{4}(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)$, where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the hyperbolas.

Take the equations $xy=u$, $x^2-y^2=v$, then the two moments of inertia are $A=\iint x^2 J\,du\,dv$ and $B=\iint y^2 J\,du\,dv$, where $1/J$ is the Jacobian of (u, v) with regard to (x, y). This gives at once $A+B=\frac{1}{2}\iint du\,dv$, where the limits are clearly $u=\frac{1}{2}\alpha^2$ to $\frac{1}{2}\alpha'^2$, $v=\beta^2$ to $v=\beta'^2$.

Ex. 5. A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two similar hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the co-ordinate axes. Prove that the product of inertia about the co-ordinate axes is $\frac{(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)}{4(m^2+n^2)}$, where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the curves.

Ex. 6. If $d\sigma$ be an element of the surface of a sphere referred to any rectangular axes meeting at the centre, prove that $\int x^{2n}d\sigma=\frac{4\pi}{2n+1}r^{2n+2}$, where r is the radius of the sphere and n is integral.

Ex. 7. Taking the same axes as in the last example, prove that

$$\int x^{2f}y^{2g}z^{2h}d\sigma=\frac{4\pi}{2n+1}r^{2n+2}\frac{L(f)\,L(g)\,L(h)}{L(n)},$$

where $n=f+g+h$ and $L(f)$ stands for the quotient of the product of all the natural numbers up to $2f$ by the product of the same numbers up to f, both included.

To prove this, we notice that by the last example we have

$$\int(\lambda x+\mu y+\nu z)^{2n}d\sigma=(\lambda^2+\mu^2+\nu^2)^n\frac{4\pi r^{2n+2}}{2n+1}.$$

Expand both sides and equate the coefficients of $\lambda^{2f}\mu^{2g}\nu^{2h}$.

If we multiply the result by Ddr we have the value of the integral for any homogeneous shell of density D and thickness dr. Regarding D as a function of r, and integrating with regard to r, we can find the value of the integral for any heterogeneous sphere in which the strata of equal density are concentric spheres.

Ex. 8. If $d\sigma$ be an element of the surface of an ellipsoid referred to its principal diameters, and if p be the perpendicular from the centre on the tangent plane, prove

$$\int x^{2f}y^{2g}z^{2h}p\,d\sigma=\frac{4\pi}{2n+1}\frac{L(f)\,L(g)\,L(h)}{L(n)}a^{2f+1}b^{2g+1}c^{2h+1},$$

where a, b, c are the semi-axes and the rest of the notation is the same as before.

This result follows at once from the corresponding one for a spherical shell by the *method of projections.*

Ex. 9. Show that the volume V, the surface S, and the moment of inertia I with regard to the plane perpendicular to the co-ordinate x_1, of the sphere in space of n dimensions, whose equation is $x_1^2+x_2^2+\ldots+x_n^2=r^2$, are given by

$$V=r^n(\Gamma\tfrac{1}{2})^n/\Gamma(\tfrac{1}{2}n+1),\quad S=\frac{n}{r}V,\quad I=V\frac{r^2}{n+2}.$$

These results follow easily from Dirichlet's theorem. See also Art. 5 (2).

10. **Method of Differentiation.** Many moments of inertia may be deduced from those given in Art. 8 by the method of differentiation. Thus the moment of inertia of a solid ellipsoid of uniform density ρ about the axis of a is known to be $\frac{4}{3}\pi abc\rho\frac{b^2+c^2}{5}$. Let the ellipsoid increase indefinitely little in size, then the moment of inertia of the enclosed shell is

$$d\left\{\frac{4}{3}\pi abc\rho\frac{b^2+c^2}{5}\right\}.$$

This differentiation can be effected as soon as the law according to which the ellipsoid alters is given. Suppose the bounding ellipsoids to be similar, and let the ratio of the axes in each be given by $b=pa$, $c=qa$. Then

$$\text{moment of inertia of solid ellipsoid}=\frac{4}{3}\pi\rho pq\frac{p^2+q^2}{5}a^5;$$

$$\therefore\ \text{moment of inertia of shell}=\frac{4}{3}\pi\rho pq(p^2+q^2)a^4da.$$

In the same way the mass of solid ellipsoid $=\frac{4}{3}\pi\rho pqa^3$;

$$\therefore\ \text{mass of shell}=4\pi\rho pqa^2da.$$

Hence the moment of inertia of an indefinitely thin ellipsoidal shell of mass M bounded by similar ellipsoids is $M\frac{b^2+c^2}{3}$.

By reference to Art. 8, it will be seen that this is the same as the moment of inertia of the circumscribing right solid of equal mass. These two bodies therefore have equal moments of inertia about their axes of symmetry at the centre of gravity.

11. The moments of inertia of a heterogeneous body whose boundary is a surface of uniform density may sometimes be found by the method of differentiation. Suppose the moment of inertia of a homogeneous body of density D, bounded by any surface of uniform density, to be known. Let this when expressed in terms of some parameter a be $\phi(a)D$. Then the moment of inertia of a stratum of density D will be $\phi'(a)Dda$. Replacing D by the variable density ρ, the moment of inertia required will be $\int\rho\phi'(a)da$.

Ex. 1. Show that the moment of inertia of a heterogeneous ellipsoid about the major axis, the strata of uniform density being similar concentric ellipsoids, and the density along the major axis varying as the distance from the centre, is $\frac{2}{9}M(b^2+c^2)$.

Ex. 2. The moment of inertia of a heterogeneous ellipse about the minor axis, the strata of uniform density being confocal ellipses and the density along the minor axis varying as the distance from the centre, is $\frac{3M}{20}\,\frac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}$.

Other methods of finding moments of inertia.

12. The moments of inertia given in the table in Art. 8 are only a few of those in continual use. The moments of inertia of an ellipse, for example, about its principal axes are there given, but we shall also frequently want its moments of inertia about other axes. It is of course possible to find these in each separate case by integration. But this is a tedious process, and it may be often avoided by the use of the two following propositions.

The moments of inertia of a body about certain axes through its centre of gravity, which we may take as axes of reference, are regarded as given in the table. In order to find the moment of inertia of *that body* about any other axis we shall investigate,

(1) A method of comparing the required moment of inertia with that about a parallel axis through the centre of gravity. This is the theorem of parallel axes.

(2) A method of determining the moment of inertia about this parallel axis in terms of the given moments of inertia about the axes of reference. This is the theorem of the six constants of a body.

13. **Theorem of Parallel Axes.** *Given the moments and products of inertia about all axes through the centre of gravity of a body, to deduce the moments and products about all other parallel axes.*

The moment of inertia of a body or system of bodies about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus the moment of inertia of the whole mass collected at the centre of gravity about the original axis.

The product of inertia about any two axes is equal to the product of inertia about two parallel axes through the centre of gravity plus the product of inertia of the whole mass collected at the centre of gravity about the original axes.

Firstly, take the axis about which the moment of inertia is required as the axis of z. Let m be the mass of any particle of