ATTRACTIONS.

Introductory remarks.

1. **Law of attraction.** If two particles of matter are placed at any sensible distance apart, they attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance.

Let m, m' be the masses of two particles, r their distance apart; if F be the mutual attraction which each exerts upon the other, then F is given by the equation $F = \kappa \dfrac{mm'}{r^2}$

If f be the acceleration produced by the attraction of m at the distance r, then $f = \kappa \dfrac{m}{r^2}$.

The quantity κ is called the *constant of attraction.* Its magnitude depends on the particular units in which the masses m, m', the distance r and the force F are measured. To avoid the continual recurrence of this constant running through every equation, it is usual to so choose the units that $\kappa = 1$. When this is done the units are called *theoretical* or *astronomical units.*

Putting $\kappa = 1$ in the equations, we see that when m and r are both unity the acceleration f is also unity. We infer that the astronomical unit of mass is that mass which, when collected into a particle, produces by its attraction at a unit of distance the unit of acceleration. The expression for F shows that the unit of force is the attraction which a particle whose mass is the astronomical unit of mass exerts on an equal particle at a unit of distance.

To avoid the continual repetition of the same set of words, we shall use the phrase *attraction at a point* to mean the attraction on a unit of mass collected into a particle and placed at that point.

It is convenient to use different systems of units for different purposes. The astronomical units should be used in analytical investigations. In any numerical applications we may choose such units of space and time as we may find convenient, and then introduce into our formulæ the factor κ with its appropriate value.

It may be noticed that in using different units for different purposes we are following the analogy of other mathematical sciences. In practical trigonometry we measure angles in degrees, in theoretical trigonometry we adopt that unit by which our analytical formulæ are most simplified. Also in algebra we have one base in logarithms for use in calculations and another for theoretical investigations; and so on through all the sciences.

2. Numerical estimate. To obtain a numerical estimate of the magnitude of the force of attraction, we must determine by experiment the mutual attraction of some two bodies. We may exhibit the result in either of two forms: (1) we may determine the value of κ when the units of space, mass, &c. have been chosen; (2) we may determine the magnitude of the astronomical unit of mass by expressing it as a multiple of some other known mass.

The two bodies on which the experiment should be tried are obviously the earth and some body at its surface. Regarding the earth as a sphere, whose strata of equal density are concentric spheres, it will be shown further on that its attraction on all external bodies is the same as if its whole mass were collected into a particle and placed at its centre. If then m be the mass of the earth and a its radius, the acceleration of a body at its surface is $\kappa m/a^2$. Let g be the acceleration actually produced by the attraction of the earth on any body placed at its surface. We thus form the equation $\kappa m/a^2 = g$.

Several experiments have been made to determine the mean density of the earth. One of these is the Cavendish experiment, but there have been others conducted on different plans. The result is that the mean density has been variously estimated to be from $5\frac{1}{2}$ to 6 times that of water. According to Baily's repetition of the Cavendish experiment the ratio is 5·67. Representing this ratio by β, we learn that the attraction of a sphere of water, of the same size as that of the earth, will produce in a body, placed at its surface, an acceleration equal to g/β.

3. *To find the value of κ when the units of space, mass, and time are the centimetre, the gramme and the second.* Since the mass of a cubic centimetre of water is one gramme nearly, the mass m of a sphere of water of the same size as the earth is $\frac{4}{3}\pi a^3$ grammes, where the radius a is measured in centimetres. By the experiment just described $\frac{\kappa m}{a^2}=\frac{g}{\beta}$; taking $\beta=5{\cdot}67$, $g=981$ (see Vol. I. Art. 11), we find $\kappa=\frac{3g}{4\pi a\beta}=\frac{1}{1543\times 10^4}$.

If therefore the attracting masses are measured in grammes and the distances in centimetres, the expression for F with this value of κ gives the attraction in dynes.

Let m be the mass, measured in grammes, of a particle which produces by its attraction at the distance of one centimetre a unit of acceleration. Then m is the astronomical unit of mass. The formula $f=\kappa m/r^2$ gives $1=\kappa m$, $\therefore m=1543\times 10^4$ grammes.

Let F be the force measured in dynes which one astronomical unit of mass exerts on another at the distance of one centimetre. The formula $F=\kappa mm'/r^2$ gives $F=1/\kappa$ since $m=m'$ and $m\kappa=1$. The force F is 1543×10^4 dynes.

4. *To find the value of κ when the units of space and time are the foot and the second, and those of mass and force are the pound and the poundal.* Since the weight of a cubic foot of water is the same as that of $\gamma=61$ pounds nearly, the mass m of a sphere of water of the same size as the earth is $\frac{4}{3}\pi a^3\gamma$, where the radius a is measured in feet. By the experiment just described $\frac{\kappa m}{a^2}=\frac{g}{\beta}$. If we take $a=20926000$ feet this gives $\kappa=\frac{1}{93\times 10^7}$.

If therefore the attracting masses are measured in pounds and the distances in feet, the expression for F with this value of κ gives the attraction in poundals.

The astronomical unit of mass, when the foot and the second are the units of space and time, is 93×10^7 pounds and the astronomical unit of force is 93×10^7 poundals. A poundal is roughly equal to the weight of half an ounce. See Vol. I. Art. 11.

5. *Dimensions of κ and m.* When the unit of mass is arbitrarily chosen the attraction F of a particle of mass m on a particle of equal mass is $F=\kappa m^2/r^2$. It follows that the dimensions of κ are the same as $FL^2\mu^{-2}$ or $L^3\mu^{-1}t^{-2}$ where F, L, μ, t stand for force, length, mass, and time. When the factor κ is omitted the dimensions of astronomical mass include those of κ and become the same as those of $\mu\kappa^{\frac{1}{2}}$ or, which is the same thing, $F^{\frac{1}{2}}L$ or $L^{\frac{3}{2}}\mu^{\frac{1}{2}}t^{-1}$. This also follows at once from the formula $F=m^2/r^2$. These dimensions are the same as those of the electrostatic measure of electricity. See Maxwell's *Electricity*, Arts. 41, 42.

6. Ex. 1. Prove that the mass of the particle which at the distance of one centimetre from a particle of equal mass attracts it with the force of one dyne is 3928 grammes. Everett's *Units and Physical Constants.*

Ex. 2. Show that a cubic foot of water, collected into a particle, attracts an equal particle placed at the distance of one foot with a force equal to the weight of $1/(8\times 10^6)$ pounds.

7. Law of the direct distance. There are other laws besides that of the inverse square which may govern the attraction of bodies in special cases. Some of these will be mentioned as we proceed. But the most useful is that in which the attraction varies as the distance. In this case the attraction of two particles, each on the other, is represented by $F = mm'r$, where m, m' are their masses, and r, the distance between them.

8. When the attraction obeys the law of the direct distance, the resultant attraction of any body at any point is found at once by using Art. 51 of Vol. I. Let O be any point, A_1, A_2, &c. the positions of the attracting particles; let m_1, m_2, &c. be their masses. The component attractions at O are then given by $X = \Sigma mx = \bar{x}\Sigma m$, $Y = \bar{y}\Sigma m$, $Z = \bar{z}\Sigma m$, where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of gravity of the body or system of attracting points.

It immediately follows that the resultant attraction at O is the same as if the whole mass Σm of the attracting system were collected into a single particle placed at the centre of gravity. *The resultant force on a particle at O tends therefore towards the centre of gravity of the attracting system, and is proportional to the distance of the attracted point from it.*

9. In what follows, when no special law of force is mentioned, it is to be understood that the law meant is that of the inverse square. This is often called the Newtonian law.

When the law of attraction is said to be $f(r)$, it is meant that the mutual attraction of two particles whose masses are m, m' placed at a distance apart equal to r is $mm'f(r)$.

Attraction of rods, discs, &c.

10. Attraction of a rod. *To find the attraction of a uniform thin straight rod AB at any external point P.*

Let m be the mass of a unit of length, then m is called the *line density* of the rod. Let p be the length of the perpendicular PN from P on the rod. Let QQ' be any element of the rod, $NQ = x$; let also the angle $NPQ = \theta$, then $x = p\tan\theta$.

The attraction at P of the element QQ' is

$$\frac{mdx}{PQ^2} = \frac{md(p\tan\theta)}{(p\sec\theta)^2} = \frac{md\theta}{p}.$$

Let X, Y be the resolved attractions at P parallel and perpendicular to the length AB. Let the angles NPA, NPB be α, β,

then $$X = \int m\frac{d\theta}{p}\sin\theta = \frac{m}{p}(\cos\alpha - \cos\beta) \;\ldots\ldots\ldots\; (1),$$

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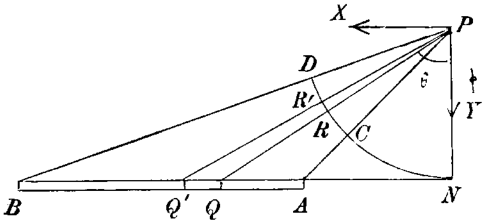
$$Y = \int m \frac{d\theta}{p} \cos\theta = \frac{m}{p}(\sin\beta - \sin\alpha) \quad\text{.........}\ (2).$$



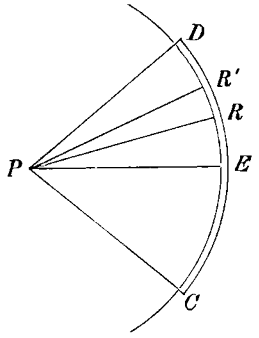
11. Substitute for $\cos\alpha$, $\cos\beta$ their values obtained from the triangles PNA, PNB; the resolved attraction parallel to the rod takes the useful form $$X = \frac{m}{PA} - \frac{m}{PB} \quad\text{..............}\ (3).$$

It should be noticed that this is the attraction at P of the rod AB resolved in the direction from A towards B.

12. Describe a circle with centre P and radius PN and let the portion CD included between the distances PA, PB represent a thin circular rod of the same material and section as the given rod AB.

The attraction at P of the element RR' of the circular rod is therefore $\dfrac{m\,.\,RR'}{PR^2} = m\dfrac{pd\theta}{p^2} = m\dfrac{d\theta}{p}$. But this has just been proved to be the same as the attraction of the element QQ'. Thus each element of the rod AB attracts P with the same force as the corresponding element of the rod CD. *The resultant attraction of the straight rod AB is therefore the same in direction and magnitude as that of the circular rod CD.*

13. The resultant attraction at P of the circular rod CD must clearly bisect the angle CPD. It immediately follows that *the direction of the resultant attraction at P of a straight rod AB bisects the angle APB.*



To find the magnitude of the resultant attraction at P of the circular arc CD, we draw PE bisecting the angle CPD. Let the angle any radius PR makes with PE be ψ. Let 2γ be the angle CPD. Since $RR' = pd\psi$ the attraction of the whole circular arc when resolved along PE is

$\int \frac{mpd\psi}{p^2} \cos\psi = \frac{m}{p} \, 2\sin\gamma$, the limits of the integral being $\psi = -\gamma$ and $\psi = \gamma$. *The magnitude F of the resultant attraction at P of a straight rod AB is given by* $F = \frac{2m}{p} \sin\frac{APB}{2}$.

14. When the rod AB is infinite in both directions the angle APB is equal to two right angles. *The resultant attraction of an infinite rod at any point P is equal to $2m/p$, and it acts along the direction of the perpendicular p drawn from P to the rod.*

This proposition leads to *a useful rule* which helps us *to find the attraction of any cylindrical surface or solid* which is infinitely extended in both directions. We pass a plane through the attracted point P perpendicular to the generating lines and cutting the cylinder in a cross section. If the attracting body be composed of elementary rods of line density m, each of these attracts P as if a mass $2m$ were collected into its cross section and *the law of attraction were changed to the inverse distance. The attraction of the whole cylinder is then equal to that of this cross section.* If the cylinder be solid and of volume density ρ, the cross section is an area of surface density 2ρ; if the cylinder is a surface of surface density σ, the cross section is a curve of line density 2σ. The same rule will apply to a heterogeneous cylinder provided the density along each generator is uniform.

Three laws of attraction are therefore especially useful. These are (1) the law of the inverse square, (2) that of the inverse distance, and (3) that of the direct distance.

15. When the point P moves about and comes to the other side of the attracting rod AB, crossing AB produced but not passing through any portion of the attracting rod, the components X, Y remain continuous functions of the coordinates of P, and will continue to represent the component attractions. When P lies in AB produced Y takes the singular form $0/0$, but it is evident that it changes sign through zero. The resultant attraction is then given by (3) which is free from singularity.

When P passes through the material of the rod the case is somewhat different. When P approaches the thin rod, the angles β and α become ultimately $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, the Y component becomes infinite while X remains finite. The attraction is therefore ultimately perpendicular to the rod and finally changes sign through infinity. When P is inside the indefinitely thin rod the Y component is zero by symmetry and the X component represents the attraction.

In the preceding analysis we have regarded the linear dimensions of the transverse section of each element QQ' as infinitesimal when compared with the distance from P. This however is not true for any material rod when P approaches very closely to any point of it. The rod (or at least the portion which is near to P) must then be regarded as a cylindrical solid.

16. Ex. 1. If two forces be applied at P acting along AP, PB taken in order, and each equal to m/p, prove that their resultant is equal in magnitude to the attraction of the rod AB and acts in a direction perpendicular to that attraction.

Ex. 2. The sides of a triangle are formed of three thin uniform rods of equal density. Prove that a particle attracted by the sides is in equilibrium if placed at the centre of the inscribed circle.

If one side of the triangle repel while the other two attract the particle, prove that the centre of an escribed circle is a position of equilibrium. [Math. T.]

This follows at once from Art. 12. Draw straight lines from the centre I of the inscribed circle to the corners A, B, C of the triangle, cutting the circle in A', B', C'. The attractions of the sides AB, BC, CA are the same as those of the arcs $A'B'$, $B'C'$, $C'A'$, that is their resultant attraction is the same as that of the whole circle on the centre. This attraction is clearly zero.

Ex. 3. Four uniform straight rods of equal density form a quadrilateral, and their lengths are such that the sum of two opposite sides is equal to the sum of the other two opposite sides. Find the position of equilibrium of a particle under the attraction of the four sides.

Ex. 4. Every particle of three similar uniform rods of infinite length, lying in the same plane, attracts with a force varying inversely as the square of the distance; prove that a particle will be in equilibrium if it be placed at the centre of gravity of the triangle ABC enclosed by the rods. [Math. Tripos, 1859.]

The attractions at P are perpendicular to the sides of the triangle and therefore, when P is in equilibrium, their magnitudes are proportional to those sides. Hence by Art. 14 the areas APB, BPC, CPA are equal and therefore P is the centre of gravity.

Ex. 5. A particle is placed at any point P on the bisector of the angle C of a triangle. Show that the direction of the resultant attraction of the three sides at P bisects the angle APB and is equal in magnitude to $2m\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)\sin\frac{APB}{2}$, where α and γ are the perpendiculars from P on the sides BC, AB respectively.

Describe a circle centre P to touch the sides AC, BC. The resultant attraction of these two sides is equal and opposite to that of the arc of the circle which lies between the straight lines AP, BP on the side remote from C (Art. 12).

Ex. 6. Two uniform parallel straight rods AB, CD attract each other: show that the components of their mutual attraction, respectively perpendicular and parallel to the rods, are

$$Y=\frac{mm'}{p}(BC-BD-AC+AD), \qquad X=mm'\log\frac{BC'+BC}{AC'+AC}\cdot\frac{AD'+AD}{BD'+BD},$$

where C', D' are the projections of C, D on the rod AB, p the distance between the rods, and m, m' the masses per unit of length.

Ex. 7. P is a particle in the diagonal AC of a square $ABCD$, and within the square; show that the attraction of the perimeter of the square upon P is equal to $M\cdot\frac{OP}{PA\,.\,PB\,.\,PC}$; where M is the mass of the perimeter, O the centre of the square. [Trin. Coll., 1882.]

Ex. 8. Let the finite rod AB be produced both ways to infinity and let the portion beyond A attract and the portion beyond B repel P, the portion between A

and B exerting no force at P. Prove that the resultant force at P bisects the angle external to APB and is equal to $\frac{2m}{p}\cos\frac{APB}{2}$.

Describe a circle, centre P, to touch AB and intersect PA in C and BP produced in H. The resultant force at P is therefore equal to the attraction of the arc CH. Art. 12.

Ex. 9. The law of attraction of a uniform thin straight rod is the inverse κth power. Prove that the components of attraction at a point P parallel and perpendicular to the rod are respectively

$$X=\frac{m}{\kappa-1}\left(\frac{1}{PA^{\kappa-1}}-\frac{1}{PB^{\kappa-1}}\right),\qquad Y=\frac{m}{p^{\kappa-1}}\int_a^\beta (\cos\theta)^{\kappa-1}\,d\theta,$$

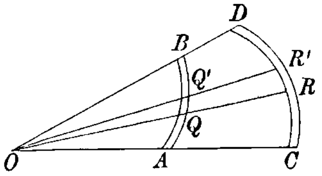
the latter integral can be found by a formula of reduction in the usual way.

Ex. 10. The law of attraction of a cylinder infinitely extended in both directions is the inverse κth power. Prove that the attraction at a point P is equal to that of the cross section provided (1) the law of attraction of the section is the inverse $(\kappa-1)$th power and (2) ratio of its density to the cylindrical density is $2\int(\cos\theta)^{\kappa-1}\,d\theta$, the limits being 0 to $\frac{1}{2}\pi$, (see Art. 14).

17. Curvilinear rods. The method by which the attraction of the straight rod AB is replaced by that of the circular arc CD in Art. 12 may be extended to other curves.

Two curvilinear rods AB, CD are so related that if any two radii vectores OAC, OBD are drawn, the attractions of the intercepted arcs AB, CD at the origin O are the same in direction and magnitude. It is required to find the relation between the densities of the rods.

Since the attractions are equal for all arcs, they are equal for infinitesimal arcs. Let OQR, $OQ'R'$ be two consecutive radii vectores; ds, ds' the arcs QQ', RR'; m, m' the masses at Q, R per unit of length. Then if the law of attraction is the inverse κth power of the distance we have



$$\frac{mds}{r^\kappa}=\frac{m'ds'}{r'^\kappa},$$

where $r=OQ$, $r'=OR$. If ϕ, ϕ' be the angles the radius vector OQR makes with the tangents at Q and R, this gives

$$\frac{m}{r^{\kappa-1}\sin\phi}=\frac{m'}{r'^{\kappa-1}\sin\phi'}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

The densities of the curvilinear rods at corresponding points must therefore be proportional to $r^{\kappa-1}\sin\phi$.

If the law of attraction be the inverse square, two curvilinear rods in one plane equally attract the origin, if the densities at corresponding points in the two rods are proportional to the perpendiculars from the origin on the tangents.

18. If the two curves are so related that *each is the inverse of the other*, we have $OQ \cdot OR = OQ' \cdot OR'$. A circle can therefore be described about the quadrilateral $QRR'Q'$. In the limit when QQ', RR' become tangents this gives $\sin\phi = \sin\phi'$. If also $\kappa = 1$, we see that $m = m'$. It follows therefore that when the law of attraction is the inverse distance, any curvilinear rod and its inverse, if of equal uniform line density, equally attract the origin.

19. Ex. 1. Let the law of attraction be the inverse distance and let P be any point attracted by a uniform straight rod AB. Draw PN perpendicular to the rod and describe a circle on PN as diameter. Prove that the attraction of AB at P is the same as that of the corresponding arc CD of the circle intercepted between the straight lines PA, PB, if the line densities are equal. Compare Art. 12.

Ex. 2. Two rigid and equal semicircular arcs of matter with uniform section and density are hinged together at both extremities. The matter attracts according to the law of gravitation. If equal and opposite forces applied along the line joining the middle points of the semicircles keep them apart with their planes at right angles, the magnitude of each force will be $4m^2 \log(1+\sqrt{2})$, where m is the mass of unit length of arc. [Math. Tripos, 1874.]

20. Some inverse problems. Ex. 1. A uniform rod is bent into the form of a curve such that *the direction of the attraction of any arc PQ at the origin O bisects the angle POQ*. Show that the curve is either a straight line or a circle whose centre is O.

The data lead to the differential equation $\int \frac{ds}{r^2} \sin\theta = \tan\frac{\theta}{2} \int \frac{ds}{r^2} \cos\theta$. The limits of the integrals being 0 and θ. The equation may be solved by differentiation.

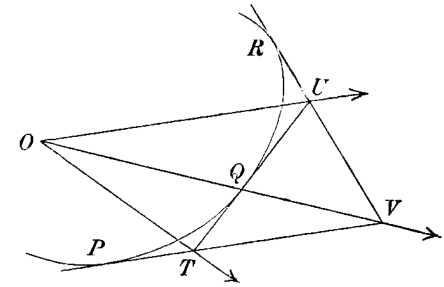
Ex. 2. Find the law of density of a curvilinear rod of given form that the *direction of the attraction at a given point O of any arc PQ may bisect the angle POQ*. If the law of attraction be the inverse κth power of the distance, the result is that the line density m at P must be proportional to $pr^{\kappa-2}$ where $r = OP$ and p is the perpendicular on the tangent at P.

Draw any circle, centre O, intersecting OP, OQ in C, D. The attraction of CD (regarded as a uniform rod) at O is by hypothesis the same in direction as that of PQ and may (by giving CD the proper density) be made the same in magnitude also. Include the additional elements QQ', DD'. It is clear that unless their attractions at O are equal the attraction of PQ' cannot coincide in direction with that of CD'. The attractions at O of corresponding elements of the two rods are therefore equal. Hence as in Art. 17 the density m at every point of PQ varies as $pr^{\kappa-2}$ The proposition may also be proved analytically as indicated in the last example.

Ex. 3. A uniform rod is bent into a curve such that the direction of the attraction at the origin of any arc PQ passes through the centre of gravity of the arc. Prove that, either the law of attraction is the direct distance, or the curve is a straight line which passes through the origin.

Ex. 4. If any uniform arc of an equiangular spiral attract a particle placed at the pole, prove that the resultant attraction acts along the line joining the pole to the intersection of the tangents at the extremities of the arc.

Prove also that if any other given curve possess this same property, the law of attraction must be $F = \frac{\mu}{p^2}\frac{dp}{dr}$, where p is the perpendicular drawn from the attracted particle on the tangent at the point of which the radius vector is r.



Reversing the attracting forces, we may regard the rod as acted on by a centre of repulsive force. Since the resultant force on any arc PQ acts along OT, where T is the intersection of the tangents at P and Q, we may resolve that force into two components which act along TP and TQ. It follows that the resultant force on any arc PQ may be balanced by two forces or tensions acting along the tangents at P and Q.

To complete the analogy of the force at P to a tension, we must show that that force is always the same whatever the length of the arc PQ may be. To prove this let PQ, QR be two contiguous arcs, and let the tangents at P, Q meet in T, those at Q, R in U, those at P, R in V. Resolving the forces at T, U, V as before, the components along PT, QT and RU, QU must together be equivalent to the components along PV, RV. We have to deduce from this that the components along PT and PV are equal. This follows at once by taking moments about U.

The conditions of equilibrium of the rod are therefore the same as those of a string acted on by a central force. Referring to Art. 474, Vol. I., the tension is obviously $T = A/p$ and the force $f(r)$ has the value given above. See the *Solutions of the Senate House problems for the year* 1860, page 61. The analytical solution leads to an interesting differential equation which can be solved without great difficulty.

21. Attraction of a circular disc. *To find the attraction of a uniform thin circular disc at any point in its axis.*

Let O be the centre, ABA' the disc seen in perspective; OZ the axis, i.e. a straight line drawn through O perpendicular to the plane of the disc. Let a be the radius of the disc, m the mass per unit of area, usually called *the surface density*. Let P