CHAPTER I.

THE PARALLELOGRAM OF FORCES.

1. THE science of Mechanics treats of the action of forces on bodies. Under the influence of these forces the bodies may either be in motion or remain at rest. That part of mechanics which treats of the motion of bodies is called Dynamics. That part of mechanics in which the bodies are at rest is called Statics.

If the determination of the motion of bodies under given forces could be completely and easily solved, there would be no obvious advantage in this division of the subject into two parts. It is clear that statics is only that particular case of dynamics in which the motions of the bodies are equated to zero. But the particular case in which the motion is zero presents itself as a much easier problem than the general one. At the same time this particular case is one of great importance. It is important not merely for the intrinsic value of its own results but because these are found to assist in the solution of the general case by the help of a theorem due to D'Alembert. It has therefore been generally found convenient to lead up to the general problem of dynamics by considering first the particular case of statics.

2. Since statics is a particular case of dynamics we may begin by discussing the first principles of the more general science. We should consider how the mass of a body is measured, how the velocity and acceleration of any particle are affected by the action of forces. The general principles having been obtained we may then descend to the particular case by putting these velocities equal to zero. In this way the relationship of the two great branches of mechanics is clearly seen and their results are founded on a common basis.

3. There is another way of studying statics which has its own advantages. We might begin by assuming some simple axioms relating to the action of forces on bodies without introducing any properties of motion. In this method we introduce no terms or principles but those which are continually used in statics, leaving to dynamics the study of those terms which are peculiar to it.

Whether this is an advantageous method of studying statics or not depends on the choice of the fundamental axioms. In the first place they must be simple in character. In the second place they must be easily verified by experiment. For example we might take as an axiom the proposition usually called the parallelogram of forces or we might, after Lagrange, start from the principle of work. But neither of these principles satisfies the conditions just mentioned, for they do not seem sufficiently obvious on first acquaintance to command assent.

If we found the two parts of mechanics on a common basis, that basis must be broader than that which is necessary to support merely the principles of statics. We have to assume at once all the experimental results required in mechanics instead of only those required in statics. Now there is an advantage in introducing the fundamental experiments in the order in which they are wanted. We thus more easily distinguish the special necessity for each, we see more clearly what results are deduced from each experiment. The order of proceeding would be to begin with such elementary axioms about forces as will enable us to study their composition and resolution. Presently other experimental results are introduced as they are required and finally when the general problem of dynamics is reached, the whole of the fundamental axioms are summed up and consolidated.

In a treatise on statics it is necessary to consider both these methods. We shall examine first how the elementary principles of statics are connected with the axioms required for the more general problem of dynamics, and secondly how they may be made to stand on a base of their own.

4. In mechanics we have to treat of the action of forces on bodies. The term force is defined by Newton in the following terms.

An impressed force is an action exerted on a body in order to change its state either of rest or of uniform motion in a straight line.

5. Characteristics of a Force. When a force acts on a body the action exerted has (1) a point of application, (2) a direction in space, (3) magnitude.

Two forces are said to be equal in magnitude when, if applied to the same particle in opposite directions, they balance each other. The magnitudes of forces are measured by taking some one force as a unit, then a force which will balance two unit forces is represented by two units and so on.

6. The simplest appeal to our experience will convince us that many at least of the ordinary forces of nature possess these three characteristics. If force be exerted on a body by pulling a string attached to it, the point of attachment of the string is the point of application, and the direction of the string is the direction of the force. The existence of the third element of a force is shown by the fact that we may exert different pulls on the string.

All the causes which produce or tend to produce motion in a body are not known. But as they are studied, it is found that they can be analysed into simpler causes, and these simpler causes are seen to have the three characteristics of a force. If there be any causes of motion which cannot be thus analysed, such causes are not considered as forces whose effects are to be discussed in the science of statics.

7. There are other things besides forces which possess these three characteristics. These other things may be used to help us in our arguments about forces so far as their other properties are common also to forces.

The most important of these analogies is that of a finite straight line. Let this finite straight line be AB. One extremity A will represent the point of application. The direction in space of the straight line will represent the direction of the force and the length of the line will represent the magnitude of the force.

Other things besides forces may also be represented graphically by a finite straight line. Thus in dynamics it will be seen that both the velocity and the momentum of a particle have direction and magnitude and may in the same way be represented by a finite straight line. One extremity A is placed at the particle, the direction of the straight line represents the direction of the velocity and the length represents the magnitude. Generally this analogy is useful whenever the things considered obey what we shall presently call the *parallelogram law*.

8. In order to represent completely the direction of a force by the direction of the straight line AB, it is necessary to have some convention to determine whether the force pulls A in the direction AB or pushes A in the direction BA. This convention is supplied by the use of the terms positive and negative. The positive and negative directions of straight lines being defined by some convention or rule, the forces which act in the positive directions of their lines of action are called positive and those in the opposite directions are called negative. These conventions are often indicated by the conditions of the problem under consideration, but they usually agree with the rules adopted in the differential calculus. Thus the direction of the radius vector drawn from the origin is usually taken as the positive direction, and so on for all lines.

Sometimes instead of using the term positive, the direction or sense of a force is indicated by the order of the letters, thus a force AB is a force acting in the direction A to B, a force BA is a force acting from B towards A.

9. The third element of a force is its magnitude. This is represented by the length of the representative straight line. A unit of force is represented by a unit of length on any scale we please; a force of n such units of force is then represented by a straight line of n units of length.

10. **Measure of a force.** A force must be measured by its effects. Since a force may produce many effects there are several methods open to us. If we wish the measure of two equal forces acting together to be twice that of a single force equal to either, the effect which is to measure the force must be properly chosen.

We may measure a force by the weight of the mass which it will support. Placing two equal masses side by side, they will be supported by equal forces. Joining these together we see that a double force will support a double mass. Thus the effect is proportional to the magnitude of the cause.

We may also measure a force by the motion it will produce in a given body in a given time. If by motion is here meant velocity then it may be shown by the experiments usually quoted to prove the second law of motion that a double force will produce a double velocity. So here also the effect chosen as the measure is proportional to the magnitude of the cause. This measure requires some experimental results, necessary for dynamics, but not used afterwards in statics.

If we agree to measure a force by the weight it will support the unit will depend on the force of gravity at the place where the experiment is made. Such a unit will therefore present several inconveniences. If also we measure a force by the velocity generated in a unit of mass in a unit of time, it is necessary to discuss how these other units are to be chosen.

It is not necessary for us, at this stage of our argument, to decide on the best method of measuring a force. It will be presently seen that our equations are concerned for the most part with the ratios of forces rather than with the forces themselves. The choice of the actual unit is therefore unimportant at present, and we can leave this choice until the proper occasion arrives. The comparative effects of forces will then have been discussed, and the reader will the better understand the reasons why any particular choice is made.

When therefore we speak of several forces equal to the weight of one, two or three pounds &c., acting on a body and determine the conditions of equilibrium, we shall find that the same conditions are true for forces equal to the weight of one, two or three oz. &c., and generally of all forces in the same ratio.

11. One system of units is that based on the foot, pound, and second as the three fundamental units of length, mass, and time. The unit force is that force which acting on a pound of matter for one second generates a velocity of one foot per second. This unit of force is called the poundal.

The foot and the pound are defined by certain standards kept in a place of security for reference. Thus the imperial yard is the distance between two marks on a certain bar, preserved in the Tower of London, when the whole bar has a temperature of 62° Fah. The unit of time is a certain known fraction of a mean solar day.

The units committee of the British Association recommended the general adoption of the centimetre the gramme and the second as the three fundamental units of space, mass and time. These they proposed should be distinguished from absolute units, otherwise derived, by the letters C. G. S. prefixed, these being the initial letters of the names of the three fundamental units. The C. G. S. unit of force is called a dyne. This is the force which acting on a gramme for a second generates the velocity of a centimetre per second.

It is found by experiment that a body, say a unit of mass, falling in vacuo for one second acquires very nearly a velocity of 32·19 feet per second. This velocity is the same as 981·17 centimetres per second. It follows therefore that a poundal is about $\frac{1}{32}$th part of the weight of one pound, and a dyne is the weight of $\frac{1}{981}$th part of a gramme. These numerical relations strictly apply only to the place of observation, for the force of gravity is not the same at all places on the earth. The difference between the greatest and least values of gravity is about $\frac{1}{196}$th of its mean value.

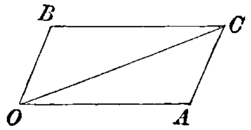
The relations which exist between these and other units in common use are given at length in Everett's treatise on *units and Physical Constants* and in Lupton's *numerical tables*. We have nearly

one inch = 2·54 centimetres, one pound = 453·59 grammes

It follows from what precedes that one poundal = 13825 dynes.

12. The parallelogram of velocities. This proposition is preliminary to Newton's laws of motion.

The velocity of a particle when uniform is measured by the space described in a given time. A straight line whose length is equal to this space will represent the velocity in direction and magnitude; Art. 8. Suppose a particle to be carried uniformly in the given time from O to C, then OC represents its velocity. This change of place may be effected by moving the particle in the same time from O to A along the straight line OA, if while this is being done we move the straight line OA (with the particle sliding on it) parallel to itself from the position OA to the position BC. The uniform motion of the particle from O to A is expressed by the statement that its velocity is represented by OA. The displacement produced by the uniform motion of the straight line is expressed by the statement that the particle has a velocity represented in direction and magnitude by either of the sides OB or AC. It is evident by the properties of similar figures that the path of the particle in space is the straight line OC.



It follows that when a particle moves with two simultaneous velocities represented in direction and magnitude by the straight lines OA, OB *its motion is the same as if it were moved with a single velocity represented in direction and magnitude by the diagonal* OC *of the parallelogram described on* OA, OB *as sides. This proposition is usually called the parallelogram of velocities.*

Let a particle move with three simultaneous velocities represented in direction and magnitude by the three straight lines OA_1, OA_2, OA_3. We may replace the two velocities OA_1, OA_2 by the single velocity represented in direction and magnitude by the diagonal OB_1 of the parallelogram described on OA_1, OA_2 as sides. The particle now moves with the two simultaneous velocities represented by OB_1 and OA_3. We may again use the same rule. We replace these two velocities by the single velocity represented in direction and magnitude by the diagonal OB_2 described on OB_1 and on OA_3 as sides. We have thus replaced the three given simultaneous velocities by a single velocity.

In the same way any number of simultaneous velocities may be replaced by a single velocity.

If the simultaneous velocities represented by OA_1, OA_2 &c. were all altered in the same ratio, it is evident from the properties of similar figures that the resulting single velocity will also be altered in the same ratio.

Let the simultaneous velocities OA_1, OA_2 &c. be such that their resulting velocity is zero. It follows that if all the velocities OA_1, OA_2 &c. are altered in any, the same, ratio the resulting velocity is still zero.

13. Newton's laws of Motion. These are given in the introduction to the Principia.

1. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

2. Change of motion is proportional to the force applied and takes place in the direction of the straight line in which the force acts.

3. To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal and oppositely directed.

The full significance of these laws cannot be understood until the student takes up the subject of dynamics. The experiments which suggest these laws, and their further verification, are best studied in connection with that branch of the science, and are to be found in books on elementary dynamics. The student who has not already read some such treatise is advised to assume the truth of these laws for the present. We shall accordingly not enter into a full discussion of them in this treatise, but we shall confine our remarks to those portions which are required in statical problems.

14. *The first law asserts the inertness of matter.* A body at rest will continue at rest unless acted on by some external force. At first sight this may appear to be a repetition of the definition of force, since any cause which tends to move a body at rest is called a force. But it is not so. Here we assert as the result of observation or experiment the inertness of each particle of matter. It has no tendency to move itself, it is moved only by the action of causes *external* to itself.

15. *In the second law of motion the independence of forces which act on a particle is asserted.* If the effect of a force is always proportional to the force impressed it is clearly meant that each force must produce its own effect in direction and magnitude as if it acted singly on the particle placed at rest.

Let us consider the meaning of this statement a little more fully. Let a given force act on a given particle placed at rest at a point O and generate in a given time a velocity which we may represent graphically by the straight line OA. Let a second force act on the same particle again placed at rest at O and generate in the same time a velocity which we may represent by OB. If both forces act simultaneously on the particle both these velocities are generated. The actual velocity of the particle is then represented by the diagonal OC of the parallelogram described on OA, OB as sides, Art. 12. In the same way, if any number of forces act simultaneously on a particle at rest, the law directs that we are to determine the velocity generated by each as if it acted alone for a given time. These separate velocities are then to be combined into a single velocity in the manner described in Art. 12. This single velocity is asserted to be the effect of the simultaneous action of the forces.

Let a system of forces be such that when they act simultaneously on a particle placed at rest the resulting velocity of the particle is zero. These forces are then in equilibrium. Let a second system of forces be also such that when they act on the particle placed at rest, the resulting velocity of the particle is again zero. Then this second system of forces is also in equilibrium. Let these two systems act simultaneously, then since the forces do not interfere with each other, the resulting velocity of the particle is still zero. We thus arrive at the following important proposition.

Let us suppose that there are two systems of forces each of which when acting alone on a particle would be in equilibrium. Then when both systems act simultaneously there will still be equilibrium.

This is sometimes called *the principle of the superposition of forces in equilibrium.* When we are trying to find the conditions of equilibrium of some system of forces, the principle enables us to simplify the problem by adding on or removing any particular forces which by themselves are in equilibrium.

Let the forces P_1, P_2 &c. acting on a given particle for a given time generate velocities v_1, v_2 &c. respectively. If the same or equal forces were made to act on a different particle the velocities generated in the same time may be different. But since the effect of each force is proportional to its magnitude the velocities generated by the several forces are to each other in the ratios of v_1 to v_2 to v_3 &c. If then a system of forces is in equilibrium when acting on any one particle, that system will also be in equilibrium when applied to any other particle (Art. 12).

16. We notice also that it is *the change of motion* which is the effect of force. A given force produces the same change of motion in a particle whether that particle is in motion or at rest.

In this way we can determine whether a moving particle is acted on by any external force or not. If the velocity is uniform and the path rectilinear there is no force acting on the particle. If either the velocity is not uniform, or the path not rectilinear, there must be some force acting to produce that change.

Let two equal forces act one on each of two particles and generate in the same time equal changes of velocity; these particles are said to have equal mass. If the force acting on one particle must be n times that on the other in order to generate equal changes of velocity in equal times, the mass of the first particle is n times that of the second. It follows that the mass of a particle is proportional to the force required to generate in it a given change of velocity in a given time. Now all bodies falling from rest in a vacuum under the attraction of the earth are found to have the same velocity at the end of the first second of time, Art. 11. We therefore infer that the masses of bodies are proportional to their weights. The units of mass and force are so chosen that the unit of force acting on the unit of mass will generate a unit of velocity in a unit of time.

The product of the mass of a particle into its velocity is called its *momentum.* It follows from what has just been said that the expression "change of motion" means change of momentum produced in a given time.

These results are peculiarly important in dynamics, but in statics, where the particles acted on are all initially at rest and remain so, they have not the same significance.

17. *In the third law the principle of the transmissibility of force is implied.* The principle is more clearly stated in the remarks which Newton added to his laws of motion. The law asserts the equality of action and reaction. If a force acting at a point A pull a body which has some point B held at rest, the reaction at B is asserted to be equal and opposite to the force acting at A. In general, when two forces act at different points of a body there will be equilibrium if the lines of action coincide, the directions of the forces are opposite, and their magnitudes equal.

From this we deduce that *when a force acts on a body, its effect is the same whatever point of its line of action is taken as the point of application, provided that point is connected with the rest of the body in some invariable manner.*

For let a force P act at A and let B be another point in its line of action. We have just seen that the force P acting at A may be balanced by an equal force Q acting at B in the opposite direction. But the force Q acting at B may also be balanced by an equal force P' acting at B in the same direction as P (Art. 15). Thus the two equal forces P and P' acting respectively at A and B in the same directions can be balanced by the same force Q. Thus the force P acting at A is equivalent to an equal force P' acting at B.

18. Statical Axioms. If we wish to found the science of statics on a basis independent of the ideas of motion we require some elementary axioms concerning matter and force.

In the first place we assume as before the principle of the inertness of matter.

We also require the two principles of the independence and transmissibility of force.

The first of these principles is regarded as a matter of common experience. When our attention is called to the fact, we notice