CHAPTER I

ALGEBRAIC CORRESPONDENCE

Part I. Elementary methods. There is said to be an (s, r) correspondence between points P, Q of a single given curve when, if P is assigned, anywhere on the curve, there is determined a set, $Q_1, Q_2, \ldots, Q_r$, of r points, these being the r positions of Q corresponding to P, and, when Q is assigned, there is similarly determined a set $P_1, \ldots, P_s$, of s points, these being the s positions of P corresponding to Q. In particular, the set of points P so determined where Q is at any one of $Q_1, \ldots, Q_r$, say Q_i, is a set $P_1, \ldots, P_s$ of which P is one; and, when Q is at another of $Q_1, \ldots, Q_r$, say Q_j, the set so determined may be a different set, $P_1', \ldots, P_s'$, but, among these, P is still one. The operation by which we pass from P to $Q_1, \ldots, Q_r$ is called the *forward*, or *direct*, operation; it may be denoted by a symbol, say T, so that we may write $Q = TP$, but it must be borne in mind that TP has r significations. The operation by which we pass from Q to $P_1, \ldots, P_s$ is called the *reverse* operation, and may be denoted by T_1 (and, later, by T^{-1}), and we may write $P = T_1 Q$. It is supposed, in what follows, unless the contrary be stated, that the set $Q_1, \ldots, Q_r$ is determined from P by algebraical processes, so that any rational symmetrical function of the coordinates of $Q_1, \ldots, Q_r$, say $(z_1, t_1), \ldots, (z_r, t_r)$, is a rational function (in virtue of the equation of the curve) of the coordinates (x, y) of P. In particular there are two equations, satisfied respectively by $z_1, \ldots, z_r$ and by $t_1, \ldots, t_r$, say

$$u_0 z^r + u_1 z^{r-1} + \ldots + u_r = 0, \quad v_0 t^r + v_1 t^{r-1} + \ldots + v_r = 0,$$

in which $u_0, \ldots, u_r, v_0, \ldots, v_r$ are rational polynomials in x and y. These equations must be consistent with one another in virtue of the equation of the curve, $f(z, t) = 0$; and, if the coordinates be chosen with generality, it should be possible, by means of $f(z, t) = 0$, to obtain from the second equation a rational expression for t in terms of x, y, z. The relations are then expressed by two equations, say $u(x, y, z) = 0$, $t = R(x, y, z)$, where $u = 0$ denotes the first equation, and R is rational in x, y, z, the equation $f(z, t) = 0$, or

$$f[z, R(x, y, z)] = 0,$$

being satisfied in virtue of $f(x, y) = 0$ and $u = 0$. Thus the correspondence is given by $u = 0$, $t = R$, $f(x, y) = 0$, which involve $f(z, t) = 0$. And from these similar equations are determinable,

$$w(z, t, x) = 0, \quad y = S(z, t, x), \quad f(z, t) = 0.$$

A particularly simple case is that for a rational curve. Then any two points (x), (z), of the curve, can be characterised by parameters, say θ and ϕ. The expression of the correspondence is then by a single equation, which we suppose irreducible, say $R(\theta, \phi)=0$, where R is a rational polynomial both in θ and ϕ, of respective orders s and r in these. A special example is given, for the correspondence of the points $(\theta^2, \theta, 1)$, $(\phi^2, \phi, 1)$ of the conic $x-y^2=0$, by the equation

$$\theta^2(a\phi^2+h\phi+g)+\theta(h\phi^2+b\phi+f)+g\phi^2+f\phi+c=0;$$

this expresses that the two points (z_1, t_1), (z_2, t_2) of the conic, which correspond to the point (x, y) thereon, are the intersections of the conic with the polar of (x, y) taken in regard to another fixed conic.

More generally, if (x, y), (z, t) be points of a curve $f(x, y)=0$, or $f(z, t)=0$, connected by a relation $R(x, y, z, t)=0$, where R is a rational (non-homogeneous) polynomial in x, y, z, t, then, to a point (x, y) of the curve, there correspond the intersections with $f(z, t)=0$ of the curve $R(x, y, z, t)=0$, in which (z, t) are current coordinates, and to a point (z, t) the intersections with $f(x, y)=0$ of the curve $R(x, y, z, t)=0$, in which (x, y) are current coordinates. Such a relation $R(x, y, z, t)=0$ is necessarily of the form

$$u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0,$$

where u_i, v_j are rational polynomials; this equation expresses that the points (z, t) which correspond to a given (x, y), are points of a set belonging to the linear series, on the curve $f(z, t)=0$, obtained by its intersections with the curves $\lambda_0 v_0(z, t)+\ldots+\lambda_k v_k(z, t)=0$, namely, that set for which $\lambda_0, \ldots, \lambda_k$ are in the ratios of

$$u_0(x, y), \ldots, u_k(x, y).$$

Similarly, the points (x, y) which correspond to a given (z, t) are points of a set of a certain linear series. We have given a special example, of a $(2, 2)$ correspondence on a conic. Another very obvious example is that between the point of contact (x, y), of a tangent at any point of a curve $f(x, y)=0$, and the points (z, t), other than the point of contact, in which the tangent meets the curve. The correspondence is then expressed by an equation of the form $f_1(x, y).z+f_2(x, y).t+f_3(x, y)=0$, in which f_1, f_2, f_3 are the partial derivatives of f. In this case, if $f=0$ be of order n, the $(n-2)$ points (z, t) which correspond to the point (x, y), when taken with the point (x, y) itself, counted twice, are all the points of a particular set of the linear series determined by $\lambda z+\mu t+\nu=0$, where λ, μ, ν are parameters. And so in general, for the curve in (z, t) expressed (with given x, y) by the equation,

$$u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0,$$

it may happen that, beside fixed zeros common to all of

$$v_0(z, t)=0, \ldots, v_k(z, t)=0,$$

(which we ignore), there is a certain number of intersections, with the fundamental curve $f(z, t)=0$, which coincide with (x, y). In such case, if this number be denoted by γ, and the set of points (z, t), or Q, which correspond to (x, y), or P, be denoted by (Q), the composite set $\gamma P+(Q)$ forms a set of a certain linear series; and then, if (Q') be the points Q which correspond to another point (P'), the two sets $\gamma P+(Q)$ and $\gamma P'+(Q')$ belong to the same linear series, or, in the phraseology of Vol. v, Chap. iv, are *equivalent*, which we write as $\gamma P+(Q)\equiv\gamma P'+(Q')$. It is not to be assumed, however, though the set (Q) is determined from γ coincident points P, when the correspondence is established, that the complete linear series to which the composite set $\gamma P+(Q)$ belongs, is of freedom γ. We have indeed had above a simple example on a conic, in which the sets belong to a series of freedom 2, though γ is zero. And it will appear below that there are correspondences for which γ is negative, the equivalence above written being then understood to mean $-\gamma P'+(Q)\equiv -\gamma P+(Q')$. The number γ, in correspondences of the kind now considered, is called the *valency*.

A correspondence which is entirely defined by a single equation of the form $u_0(x, y)v_0(z, t)+\ldots+u_k(x, y)v_k(z, t)=0$, has, clearly, the same valency as its reverse. For, if the left side, regarded as a function of (z, t), vanishes to order γ when (z, t) approaches to (x, y), regarded as a fixed point, then it is equally true that, regarded as a function of (x, y), it vanishes to order γ when the point (x, y) approaches to the point (z, t), regarded as fixed.

Conversely, we may define a correspondence, of indices (s, r), with (positive or zero) valency γ, between the places (x), (z) of the curve, by the properties: (1), that, when (x) is given, there is determined, algebraically, a set of r places $(z_1), \ldots, (z_r)$, any rational symmetrical function of these latter being expressible rationally by the coordinates x, y, of the point (x); and (2), that, when (z), one of $(z_1), \ldots, (z_r)$, is given, there is a similar determination of the set, $(x_1), \ldots, (x_s)$, of places, which correspond to (z) in the reverse correspondence; with (3), the further property, that, in the direct correspondence, the composite set consisting of $(z_1), \ldots, (z_r)$, and the place (x) taken γ times, are the points of a set of a linear series, which is the same for all positions of (x). For, with these hypotheses, we can construct a function, rational in the coordinates of both the places (x) and (z), which, as a function of (z), vanishes at $(z_1), \ldots, (z_r)$ as well as γ times at (x), and, as a function of (x), vanishes at $(x_1), \ldots, (x_s)$ as well as γ times at (z). This function, by its vanishing, expresses then both the direct and the reverse correspondence.

To prove this, we argue as follows: the meaning of the hypothesis (3)—if $(c_1), \ldots, (c_r)$ be the positions of $(z_1), \ldots, (z_r)$ corresponding to a position (a) of (x)—is, that there exists a rational function of (z), on the curve, having, for its zeros, the place (x), taken γ times, and the places $(z_1), \ldots, (z_r)$, which correspond to (x), taken each once, and having, for its poles, the place (a), taken γ times, and the places $(c_1), \ldots, (c_r)$, which correspond to (a), each taken once. Such a function is definite, save for a constant multiplier; dividing the function by its value at an arbitrary place (c) of the curve, we may then agree that it reduces to unity at (c). So determined, let the function be denoted by

$$R_c\left(z;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right).$$

The actual expression of such a function is to be found, by, first, forming the most general rational function of (z) which has $(a)^\gamma, (c_1), \ldots, (c_r)$ as poles, and, then, limiting the arbitrary constants, which enter linearly in such a function, by the condition that $(x)^\gamma, (z_1), \ldots, (z_r)$ are its zeros. This limitation is by linear equations; some, relating to the zeros $(x)^\gamma$, being explicitly rational in (x), the others, relating to the zeros $(z_1), \ldots, (z_r)$, being symmetrical in these, and, therefore, by the initial definition, also ultimately rational in (x). The function of (z) thus formed, depending on (x), is thus rational in (x).

Let us now consider this as a function of (x), depending on (z). As such, it vanishes to order γ when (x) approaches to (z). To the place (z) there correspond, by hypothesis, in the reverse correspondence, places $(x_1), \ldots, (x_s)$. When (z) is at any one of the places $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct correspondence, then (x) will be at one of the places $(x_1), \ldots, (x_s)$, which correspond to this position of (z) in the reverse correspondence. Thus, conversely, when (x) is at one of $(x_1), \ldots, (x_s)$, then (z) will be at one of $(z_1), \ldots, (z_r)$; and the function of (x), now under consideration, will vanish to the first order. The function will, we assume, have no other zeros than those mentioned. A like statement is true for the poles: the function of (z), considered above, has a pole of order γ when (z) is at (a), and has also a pole, of the first order, when (z) is at any one of the places $(c_1), \ldots, (c_r)$, which correspond to (a) in the direct correspondence; and this function is of the form

$$R\left(z;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right) \div R\left(c;\ \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right),$$

and is infinite to the first order when (c) is at any one of the places $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct transformation; this arises when, and only when, (x) is at any one of the places

$(a_1), \ldots, (a_s)$, which correspond to (c) in the reverse transformation; the quotient is also infinite, to order γ, when (x) is at (c); and is not otherwise infinite; and it reduces to unity when (x) is at (a), and, therefore, $(z_1), \ldots, (z_r)$ are, respectively, at $(c_1), \ldots, (c_r)$; also, the *infinities* of the numerator and denominator, of the quotient just written do not depend on the position of (x). Thus the function, regarded as depending on (x), may equally be represented by

$$R_a\left(x;\, \begin{matrix} z^\gamma, x_1, \ldots, x_s \\ c^\gamma, a_1, \ldots, a_s \end{matrix}\right).$$

Thus we have the result: *For an existing (s, r) correspondence, between the places (x), (z) of the given curve, assumed to be of valency γ (zero or positive) in the direct correspondence from (x) to $(z_1), \ldots, (z_r)$, and such that any rational symmetric function of $(z_1), \ldots, (z_r)$ is rational in (x), while to (z) there correspond $(x_1), \ldots, (x_s)$ in the reverse correspondence, if (a), (c) be two arbitrary places, we can construct a function $\phi(x, z; a, c)$, rational in regard to both (x) and (z), and in regard to both (a) and (c); this function, regarded as a function of (z), is zero to order γ at the place (x), and, to the first order, at each of the places $(z_1), \ldots, (z_r)$ which correspond to (x) in the direct correspondence; it is also infinite to order γ at the place (a), and, to the first order, at each of the places $(c_1), \ldots, (c_r)$, which correspond to (a) in the direct correspondence. This function of (z) expresses, by its zeros, the direct correspondence. But, regarded as a function of (x), it expresses the reverse correspondence also, being zero to order γ at the place (z), and, to the first order, at the places $(x_1), \ldots, (x_s)$, which correspond to (z) in the reverse correspondence, beside having a pole of order γ at the place (c), and poles, of the first order, at the places $(a_1), \ldots, (a_s)$, which correspond to (c) in the reverse correspondence. Thus the reverse correspondence is equally of valency γ.*

A simple illustration is that remarked above, where the direct correspondence leads from a place (x) to the $(n-2)$ places $(z_1), \ldots, (z_{n-2})$, in which the curve is met by the tangent of the curve at the place (x). The reverse correspondence then leads from a place (z), of the curve, to the $(n'-2)$ places $(x_1), \ldots, (x_{n'-2})$, which are the points of contact of the tangents to the curve drawn from (z). The equation of the tangent at (x), in which (z) is used as current coordinate, is equally the equation, with (x) as current coordinate, of the first polar of a point (z). The fact that the valency of the reverse correspondence, like that of the direct correspondence, is 2, is the fact that the first polar, of a point (z) of the curve, touches the curve at this point.

There are two general definitions in regard to correspondence which it is convenient to employ: (a), If we have two correspondences, in which there correspond to P, in the direct correspondences, respectively the sets (Q) and (R), then we shall regard the composite set, consisting of the sets (Q), (R) together, say the set $(Q)+(R)$, as being in correspondence with P, directly, in a correspondence which we call the *sum* of the two given correspondences. It can then be seen that the reverse of this sum correspondence is likewise the sum of the reverses of the two original correspondences. For the correspondence reverse to T, which we denote by T_1, is defined by the fact that T_1, applied to any one of the points TP, gives rise to a set of which P is one point. Now consider $(T+U)P$, which is defined as the aggregate of the sets $(Q)=TP$ and $(R)=UP$. The operation $(T+U)_1$, reverse to $(T+U)$, is that which, applied to any one of the set TP, or to any one of the set UP, gives a set including P. This is satisfied only if $(T+U)_1=T_1+U_1$. In symbols,

$$(T_1+U_1)(T+U)P=(T_1+U_1)(TP+UP)$$
$$=T_1TP+T_1UP+U_1TP+U_1UP,$$

and P is among the latter aggregate. Beside this definition of the sum of two correspondences, is, (*b*), that of the *product* of two correspondences: If to P, in a direct correspondence T, corresponds the set (Q), of which Q_i denotes every element in turn; and to Q_i, in another direct correspondence U (independent of i), corresponds a set $(R)_i$, the set (Q) consisting of q points, and each set $(R)_i$ consisting of r points, then we regard the aggregate of rq points forming all the sets $(R)_i$, as corresponding to P in a direct correspondence, which we call the product of T by U, and denote by UT, the operation T being that first applied. It is easy to see that the reverse correspondence $(UT)_1$, is T_1U_1.

For correspondences with valency, as considered above, the valency of $T+U$ is the sum of the valencies of T and U. For, if, in a notation employed already,

$$(Q)+\gamma P\equiv(Q')+\gamma P', \quad (R)+\delta P\equiv(R')+\delta P',$$

then also

$$(Q)+(R)+(\gamma+\delta)P\equiv(Q')+(R')+(\gamma+\delta)P'.$$

For the product of two correspondences, however, the valency of UT is the negative product of the valencies of T and U. For, from $\Sigma Q_i+\gamma P\equiv\Sigma Q_i'+\gamma P'$, $(R)_i+\delta Q_i\equiv(R')_i+\delta Q_i'$, of which the latter gives $\Sigma(R)_i+\delta\Sigma Q_i\equiv\Sigma(R')_i+\delta\Sigma Q_i'$, we have

$$\Sigma(R)_i-\delta\gamma P\equiv\Sigma(R')_i-\delta\gamma P'.$$

Thus, also, the (generally different) correspondences UT and TU have the same valency, which is also the valency of U_1T, TU_1, UT_1, T_1U, U_1T_1, T_1U_1.

A simple correspondence is obtained by considering, on the fundamental curve, any linear series of sets of points which has freedom unity. Any point P of the curve then determines the other points (Q), of the set to which P belongs; and there is thus a correspondence of valency unity, in which the points corresponding to P are these other points (Q), of the set of the linear series determined by P. In this case, since a set of the series is equally determined by any one of its points, the reverse correspondence is the same as the original. Further, by what has been said, the product of any two such simple correspondences, determined by any two linear series of freedom unity, is a correspondence whose valency is negative unity. By the sum and product of such simple correspondences we can thus set up a correspondence of any arbitrary valency, positive, zero or negative.

The chief utility of the theory of correspondence, in geometry, arises by the consideration of the *united* points, or *coincidences*; these are the positions of P in which one of the points Q, corresponding thereto, coincides with P.

Consider first the easy case of a rational curve: let a correspondence, between two points represented by parameters θ, ϕ, be given by an equation $(\theta, \phi)=0$, of order s in θ and order r in ϕ. The coincidences arise then corresponding to the roots of the equation $(\theta, \theta)=0$, and are $s+r$ in number, provided the original relation contains the term $\theta^s\phi^r$, that is, provided the parameters are so chosen that $\theta=\infty$ is not a coincidence. Further, the set of points forming the coincidences is equivalent with the aggregate of the two sets (Q), $(Q)_1$, which correspond to a point P in the direct and reverse correspondences; indeed, on a rational curve, any two sets of the same number of points are equivalent with one another. The same equivalence, of the set of coincidences with the aggregate of the sets corresponding to any point in the direct and reverse correspondences, holds on any curve, for the coincidences in a correspondence which is of valency zero.

The proof of this remark, as to the coincidences for a correspondence, on any curve, which is of valency zero, arises from the expression of the correspondence by the vanishing of a polynomial $\Sigma u_i(x, y)v_i(z, t)$, the numerator of the function $\phi(x, z; a, c)$ investigated above for any correspondence. This, as a polynomial in (z, t), has, when the valency is zero, no zeros coinciding with (x, y). The remark made is the same as the statement that the zeros of $\Sigma u_i(x, y)v_i(x, y)$ are a set equivalent to the aggregate of the set (Q), given by $\Sigma u_i(x_0, y_0)v_i(z, t)=0$ with $f(z, t)=0$, and the set $(Q)_1$, given by $\Sigma u_i(x, y)v_i(x_0, y_0)=0$ with $f(x, y)=0$. The theorem is contained in what follows, but is necessary as a lemma.

In general, however, the set of coincidences (U), of a correspondence of valency γ, in which the direct and reverse sets corresponding to a point P are (Q) and $(Q)_1$, upon a curve on which a canonical set, of $2p-2$ points, is denoted by (K), satisfies the equivalence $(U) \equiv (Q)+(Q)_1+\gamma[2P+(K)]$. To prove this important result, remark, first, that, if true for each of two correspondences, it is true for the correspondence which is their sum; for the coincidences in the sum are evidently the aggregate of those in the separate correspondences. Then, next, the formula is clearly true for the simple correspondence, of valency unity, in which P and (Q) together, as also P and $(Q)_1$ together, are a set of a linear series of freedom unity, on the curve. For, in this case, it becomes the fundamental result $(J) \equiv (L)+(M)+(K)$, in which (L) and (M) are any two sets of the linear series, and (J) is the Jacobian set of the series (Vol. v, p. 84). From these facts it follows that the formula is true for any correspondence, of positive valency γ, which can be obtained as the sum of γ such simple correspondences. Being assumed to be true, in virtue of the lemma preceding, for any correspondence of zero valency, it is therefore true, for any *general* correspondence, of negative valency $-\gamma$, not necessarily assumed to be obtained by composition of the simple correspondences referred to. Hence, also by the lemma, it is true for any general correspondence of positive valency γ.

We may also deduce this result for the coincidences directly, by use of the function $\phi(x, z; a, c)$ obtained above. Suppose the valency to be positive. Consider the function of (z) which is expressible, in the notation employed (p. 4, above), in either of the forms

$$(z-x)^{-\gamma} R_c\left(z; \begin{matrix} x^\gamma, z_1, \ldots, z_r \\ a^\gamma, c_1, \ldots, c_r \end{matrix}\right), \quad (z-x)^{-\gamma} R_a\left(x; \begin{matrix} z^\gamma, x_1, \ldots, x_s \\ c^\gamma, a_1, \ldots, a_s \end{matrix}\right),$$

(x) being a general place; and therein suppose (z) to approach to coincidence with (x). For simplicity of statement, suppose that the curve, $f(x, y)=0$, of order n, has n distinct places at $x=\infty$. Putting, in terms of the representative parameter θ for the neighbourhood of (x), $z-x=\theta(dz/dx)+\frac{1}{2}\theta^2(d^2z/dx^2)+\ldots$, where (dz/dx), etc., are the values at (x), a factor θ^γ, in R_c, will cancel a factor $\theta^{-\gamma}$ in $(z-x)^{-\gamma}$. Further, as this function R_c vanishes to the first order when (z) is at any of the places, $(z_1), \ldots, (z_r)$, which correspond to (x) in the direct correspondence, the function of (x) obtained by putting (z) at (x) will vanish when (x) is at any of the places $(z_1), \ldots, (z_r)$; also, as we see with the help of the second form of the function, this function of (x) will have poles of order γ both at (a) and (c), and poles of the first order at each of $(c_1), \ldots, (c_r), (a_1), \ldots, (a_s)$. The function (dz/dx), regarded as a function of (x), vanishes to order $k-1$ when (x) approaches a branch place of index k (where there is a cycle of k values); in all, then, (dz/dx) has $\Sigma(k-1)$, or, say, w zeros, at such places—whose aggregate we may denote by (w); and it is known that $w = 2n + 2p - 2$; the function of $x, (dz/dx)$, is also infinite to the second order at each of the n places at $x = \infty$. We may denote the aggregate of these n places by (∞).

On the whole, then, when (z) coincides with (x) we obtain a function of (x) whose zeros and poles are those represented by the symbols, in the numerator and denominator, respectively, in the fractional scheme

$$\frac{(U)(\infty)^{2\gamma}}{(w)^{\gamma}(a^{\gamma})(c^{\gamma})(c_1)\dots(c_r)(a_1)\dots(a_s)}.$$

In making this deduction, it is assumed that the function of (x) which is the factor of θ^{γ} in R_c, when (z) approaches to (x), gives only those zeros and poles which we have stated. From this scheme, if ν be the number of coincidences of the correspondence, equating the numbers of zeros and poles of this rational function, we hence deduce $\nu + 2\gamma n = (2n + 2p - 2)\gamma + 2\gamma + r + s$, or $\nu = r + s + 2\gamma p$.

Further, if u denote any everywhere finite integral of the curve, the rational function of (x), given by du/dx, has zeros and poles which are indicated by the scheme

$$\frac{(K)(\infty)^2}{(w)},$$

where (K) denotes a canonical set. Dividing then the previous function of (x) by $(du/dx)^{\gamma}$, we obtain a rational function whose zeros and poles are those denoted by the scheme

$$\frac{(U)}{(K)^{\gamma}(a^{\gamma})(c^{\gamma})(c_1)\dots(c_r)(a_1)\dots(a_s)},$$

so that we have the equivalence $(U) \equiv (Q) + (Q)_1 + \gamma[2P + (K)]$ originally stated. A function of the character of $\phi(x, z;\, a, c)$ can be constructed when the valency γ is negative, and then a similar argument leads to the same equivalence for this case.

Remark. A correspondence, such as the simple correspondences we have constructed from a linear series of freedom unity, in which the distinction between the direct and the reverse correspondence is lost, or say $T_1 = T$, is called *symmetrical*. The number of distinct points of coincidence of such a correspondence is $\frac{1}{2}\nu$, namely is $r + p\gamma$.

Ex. 1. To illustrate the formation of the function $\phi(x, z;\, a, c)$, and the limit of $(z-x)^{-\gamma}\phi(x, z;\, a, c)$ when (z) approaches to (x), we may consider the simple example of the correspondence, on a plane cubic curve $f(x, y) = 0$, in which, to (x) corresponds the point (z), in which the tangent at (x) meets the curve again, while, to (z) correspond the four points $(x_1), \dots, (x_4)$, in which the tangents drawn from (z) touch the curve. Then $\gamma = 2$. Denoting the function $(z-x)\partial f/\partial x + (t-y)\partial f/\partial y$, or, say, $(z-x)f_1 + (t-y)f_2$, by $T_{x,z}$, the function $\phi(x, z;\, a, c)$ is

$$T_{x,z}T_{a,c}/T_{a,z}T_{x,c}.$$

If, further, H denote the Hessian form $|\partial^2 F/\partial\xi_i\partial\xi_j|$, where

$$F=\xi_3^3 f(\xi_1\xi_3^{-1}, \xi_2\xi_3^{-1}),$$

we easily compute that $H=-4\xi_3^7(f_1^2f_{22}-2f_1f_2f_{12}+f_2^2f_{11})$, and, when (z) approaches to (x),

$$(z-x)^{-2}T_{x,z}=-\tfrac{1}{2}f_2^{-2}(f_1^2f_{22}-2f_1f_2f_{12}+f_2^2f_{11}).$$

Thus, putting $\xi_3=1$, $\xi_1=x$, $\xi_2=y$, $(z-x)^{-2}\phi(x, z; a, c)$ becomes

$$[\tfrac{1}{8}H/(\partial f/\partial y)^2][T_{a,c}/T_{a,x}T_{x,c}].$$

Here, the Hessian H vanishes in the $1+4+2.1.2$, or 9, inflexions of the curve, which are the coincidences of the correspondence, while $T_{a,x}$ vanishes twice at (a), and once at its tangential (c_1), and $T_{x,c}$ vanishes twice at (c), and once at the four points of contact of tangents from (c); while $\partial f/\partial y$ vanishes once at each of the six points of contact, (w), of tangents parallel to $x=0$, and, being of the second order, is infinite to the second order at the three places (∞). The function which expresses the equivalence of the inflexions with the aggregate $(a^2)(c^2)(c_1)(a_1)\ldots(a_4)$ is then $H/T_{a,x}T_{x,c}$.

Ex. 2. Considering, upon a curve of genus p, a linear series g_r^n, of sets of n points, of freedom r, and the sets of this series in which r points are taken coincident, there is, between each such point of coincidence, P, and the $n-r$ remaining points (Q) of the set determined by P, a correspondence of valency r; in this, there correspond to P the $n-r$ points (Q), and to Q correspond, reversely, a certain number, say ξ, of points P. Let the number of coincidences of a point Q with P be denoted by $x_{n,r}$; this is the number of sets of the series g_r^n for which $(r+1)$ points of the set are coincident. The sets from g_r^n of which a given point Q is one point, form a linear series g_{r-1}^{n-1}; in this are $x_{n-1,r-1}$ sets which contain r coincident points; this is then the number, ξ, required in the correspondence (P, Q), of points P which correspond reversely to Q. Thus, using the formula found above for the number of coincidences in a correspondence, we see that $x_{n,r}=x_{n-1,r-1}+n-r+2pr$; if we put

$$y_{n,r}=x_{n,r}-(r+1)[n+r(p-1)],$$

this equation is the same as $y_{n,r}=y_{n-1,r-1}$. But

$$y_{n,1}=2n+2p-2-2[n+p-1]=0;$$

thus, in general, $y_{n,r}=0$, and the required number $x_{n,r}$ is given by

$$x_{n,r}=(r+1)[n+r(p-1)].$$

This formula gives the total number of cases; in any particular application it is necessary to consider whether (when r is not a prime) the r-fold coincidences can arise by repetitions of r_1-fold coincidences, in which r_1 is a factor of r. Further, it is to be remarked that it is not assumed that the series g_r^n is complete; if, for example, we had $r=n-p$, then $x_{n,r}$ would reduce to $(r+1)^2p$.

Ex. 3. If we have, upon the curve, two correspondences, (s, r, γ) and (s', r', γ'), of respective valencies γ and γ', in which there correspond to P the respective sets (Q) and (Q'), of r and r' points, then the number of times it happens that, for the same P, a point Q coincides with a point Q', is $rs'+r's-2p\gamma\gamma'$.