CHAPTER I

INTRODUCTORY ACCOUNT OF RATIONAL AND ELLIPTIC CURVES

THE present volumes V, VI are an introduction to the more important of the algebraical and functional relations which are necessary for a clear and precise understanding of the principles of algebraic geometry. These relations are as the bones of the structure, to be clothed finally with a body of purely geometrical doctrine.

For the expression of these relations we make free use of coordinates, of which the justification has been examined in Vols. I and II. With their use we can define an algebraic construct (curve, surface, manifold, etc.) as the aggregate of points whose coordinates satisfy a set of algebraic equations, taken with points (limiting points, and other) which it may be necessary conventionally to add thereto. And, it is to be understood that all coordinates, and parameters, that enter in the equations employed, are capable of complex values; in particular, an aggregate will be said to be of dimension, or freedom, r, or simply to be ∞^r, when it depends on the values of r parameters not restricted to real values. In the elementary geometry two figures, or two manifolds, are regarded as essentially identical when they are projectively related to one another, that is (as we have seen) transformable into one another by equations which are linear in the (homogeneous) coordinates; in general algebraic geometry, two manifolds are regarded as essentially identical when the coordinates of the points of either are expressible as rational algebraic functions of the coordinates of the points of the other, whether linear functions or not.

The simplest algebraic construct from this point of view is then a line, which may be regarded as the locus of a point identified by one parameter fixing the position of the point upon the line. But the points of a conic, of which one point is known, are equally expressible in terms of a parameter, by taking the intersection of the conic with a variable line drawn through the known point. Or again, a plane cubic curve with a double point of known coordinates, is likewise the locus of a point whose coordinates are rational in one parameter, the intersection of the curve with a variable line through the double point. More generally, there is an unlimited family of curves with the property that the coordinates of any point of the curve are expressible as rational functions of a parameter; and this parameter can be chosen so that, conversely, it is a rational function of the coordinates of the point of the curve with which it is associated.

It will be seen, moreover, though this is a subsidiary property at the present stage, that, in the rational functions of the parameter which express the coordinates of a point of the curve, the coefficients which enter may be taken to be rational in the coefficients of the equations by which the curve is given and in the coordinates of one point of the curve; this is illustrated by the case of a conic. Regarding the variable parameter as representing a point of a line, such a curve as we have spoken of is therefore in (1, 1) birational correspondence with the line. Such a curve is called a *rational* curve. In the present chapter we give some fundamental properties of such curves, partly to illustrate general ideas, and partly because these properties should be known. And we treat also, for the same reasons, of curves which, in order of formal difficulty, next follow, those called *Elliptic Curves*.

Linear series on a line. One fundamental notion, which can be stated for a line, and has application not only to rational curves but to algebraic curves in general, is that of a *series of sets of points*, and, in particular, of a *linear* series. If a general point of a line be given by a parameter θ, a *set* of n points will be given by an equation $\theta^n + a_1\theta^{n-1} + \ldots + a_n = 0$, with given coefficients $a_1, \ldots, a_n$. These coefficients in turn may depend on other parameters $\xi, \eta, \ldots$, so that when these vary the coefficients vary, and consequently the original set of n points of the line also varies, and gives rise to a *series* of sets, of each n points. The coefficients $a_1, \ldots, a_n$ may be *rational* functions of $\xi, \eta, \ldots$, all of these being capable of independent variation; then we have a *rational* series of sets of n points on the line. As a particular case of this, the coefficients $a_1, \ldots, a_n$ may be linear (fractional) functions of $\xi, \eta, \ldots$, all with the same denominator, of the forms $a_i = u_i/u$, where $u = p\xi + q\eta + \ldots$, $u_i = p_i\xi + q_i\eta + \ldots$, $(i = 1, \ldots, n)$, where $p, q, \ldots, p_i, q_i, \ldots$ are constants. Then the sets on the line are given by $u\theta^n + u_1\theta^{n-1} + \ldots + u_n = 0$, and hence by $\xi U + \eta V + \ldots = 0$, where $U = p\theta^n + p_1\theta^{n-1} + \ldots$, $V = q\theta^n + q_1\theta^{n-1} + \ldots$; these $U, V, \ldots$ are then definite polynomials of order n in θ. Such a series is called a *linear* series of sets of points on the line. But in regard to the polynomials $U, V, \ldots$ two facts must be clear. It may happen that they all vanish for a certain number, say k, of definite values of θ; then, as $\xi, \eta, \ldots$ vary, the sets of the series consist of $n-k$ variable points, and of k points common to all the sets, these being fixed. Further, it may happen that $U, V, \ldots$ are not linearly independent, but connected by one or more linear equations, with constant coefficients, satisfied identically for all values of θ; and this will always happen if the number, say $r+1$, of the polynomials $U, V, \ldots$ is greater than $n+1$. When the $r+1$ polynomials $U, V, \ldots$ are linearly independent, we can obviously determine a set of the linear series, of which r points have

arbitrarily assigned positions, the other points of the set being thereby determined; in this case we speak of the series as being of *freedom* (or dimension) r, or as being ∞^r. And we may speak of $n-k$ as the *grade* of the series, this being the number of points which vary when $\xi, \eta, \ldots$ vary. It is obviously necessary that the freedom be not greater than the grade. In particular cases it is convenient to consider linear series of which all the sets have a certain number of fixed common points, and it may be convenient to modify the definition of grade accordingly.

A series of sets of points on the line which is not linear, nor rational, may be *algebraical* in a more general sense. This will be so if the coefficients $a_1, \ldots, a_n$, in the equation which determines the points of a set, be algebraic functions, of one or more parameters $\xi, \eta, \ldots$, of such character as not to be capable of being expressed as rational functions of other parameters. The simplest case of this is when they are algebraic functions of one parameter, ξ. This means, in the first instance, that each coefficient, a_i, satisfies an (irreducible) algebraic equation with coefficients which are rational in ξ; in this case, however, it can be shewn that a single algebraic function of ξ can be chosen, say σ, such that all the coefficients $a_1, \ldots, a_n$ are expressible rationally in terms of ξ and σ, say $a_i = \psi_i(\xi, \sigma)$; this can be done so that the aggregate of values of all these coefficients, each determined by its own equation in terms of ξ, is obtained by taking $a_i = \psi_i(\xi, \sigma)$, and allowing ξ, σ to take all possible simultaneous values consistent with the algebraic equation by which σ is determined from ξ. We may thus *define* an algebraic series of sets of points, upon the line, as that given by an equation $\theta^n + \Sigma\psi_i(\xi, \sigma)\theta^{n-i} = 0$, wherein σ, ξ have all values which satisfy a definite (irreducible) polynomial equation $f(\sigma, \xi) = 0$. And we may similarly have an algebraic series of sets depending on r parameters $\xi_1, \ldots, \xi_r$, wherein each coefficient a_i, of the determining equation, is rationally expressible in terms of $\xi_1, \ldots, \xi_r$ and a further variable σ, satisfying a rational polynomial equation, say $f(\sigma, \xi_1, \ldots, \xi_r) = 0$, all values of σ and $\xi_1, \ldots, \xi_r$ which satisfy this equation being taken; it can be shewn that this covers all cases of an algebraic series.

A general linear series of sets of n points, in which the freedom is also n, is that expressed by the original equation $\theta^n + \Sigma a_i \theta^{n-i} = 0$, in which all the coefficients vary independently of one another. Every algebraic series of sets of n points evidently consists of sets selected from this general series; it may belong to a linear series, of freedom less than n, arising by imposing linear restriction of the values allowed to $a_1, \ldots, a_n$ in the general linear series spoken of.

In an algebraic series of freedom 1, there will generally be more than one set which contains a particular point of the line; for an equation $\theta_0^n + \Sigma\psi_i(\xi, \sigma)\theta_0^{n-i} = 0$ will generally be satisfied, with the

same θ_0, by several pairs (ξ_1, σ_1), (ξ_2, σ_2), ... satisfying the fundamental equation $f(\sigma, \xi) = 0$. The number of such sets is called the *index* of the series. This is evidently equal to the number of zeros of the rational function, for the curve $f(\sigma, \xi) = 0$, which is expressed by $\theta_0^n + \Sigma\psi_i(\xi, \sigma)\theta_0^{n-i}$. From this it can be proved that the index cannot be 1 unless the curve $f(\sigma, \xi) = 0$ is rational. And then, for the index to be 1, each of the rational functions $\psi_i(\xi, \sigma)$ must be expressible as a fractional linear function of the parameter by which the curve $f(\sigma, \xi) = 0$ is expressed, with a denominator the same for all. In other words, an algebraic series on the line, of freedom 1 and of index 1, must be given by an equation of the form $\lambda\phi + \mu\psi = 0$, where ϕ, ψ are definite polynomials in the parameter θ of the line, and λ, μ are independently variable. More generally, in an algebraic series of freedom r, on a line, there is generally more than one set of the series of which r points are assigned; when the series is linear there is only one such set.

A familiar application of these ideas arises in the definition of a rational curve. Suppose that the coordinates of any point of a given curve are expressible rationally in terms of a parameter, θ, whose variation gives all the points of the curve. For example, for the curve whose equation is $x^2 + y^2 = z^2$, we may take $x = 1 - \theta^4$, $y = 2\theta^2$, $z = 1 + \theta^4$; then $\theta^2 = y/(x+z)$, and to each point of the curve correspond two points of the line on which θ is represented. We thus have, on the line, an algebraic series of sets of two points, of index 1; but these sets belong to a linear series, expressed by $\lambda + \mu\theta^2 = 0$, where λ, μ are variable. The coordinates of a point of the curve are then expressible rationally in terms of the single parameter λ/μ; and this is a *representative* parameter, as having only one value for a point of the curve, $-y/(x+z)$. Such a parameter can always be chosen to express a rational curve, as we have indicated. (Cf. Vol. II, p. 136.)

Rational curves. Consider now a plane curve, of which the ratios of the coordinates x, y, z of a point, are expressible rationally in terms of a parameter, θ, so chosen that only one value of this belongs to any general point of the curve. Thus, if ρ denote a factor of proportionality, we may say that each of ρx, ρy, ρz is equal to a polynomial in θ; let one at least of these polynomials be of order n (in general, all of this order); then n is the number of values of θ for which an arbitrary linear form $ax + by + cz$ vanishes, namely the number of points of the curve on an arbitrary line, or the *order* of the curve. For illustration, suppose $n = 3$; and imagine the plane (x, y, z) to be in space, wherein the coordinates are X, Y, Z, T. To compare now with the equations $\rho x = f_1$, $\rho y = f_2$, $\rho z = f_3$ for the plane cubic curve, take in the space (X, Y, Z, T) the curve expressed by $\sigma X = f_1$, $\sigma Y = f_2$, $\sigma Z = f_3$, $\sigma T = f_4$, where f_4 is an arbitrary cubic

polynomial, such that no identity $Af_1+Bf_2+Cf_3+Df_4=0$ holds, for values of A, B, C, D independent of θ; this is possible since evidently no identity $af_1+bf_2+cf_3=0$ holds, or the points of the plane cubic would be in a line. The curve in space is likewise a cubic curve, having three points in a plane. Since now any point θ, or (x, y, z), of the plane cubic curve is associated with the point θ, or (X, Y, Z, T), of the cubic curve in space by the equations $X/x=Y/y=Z/z$, we have the conclusion that any rational cubic curve in a plane may be regarded as the projection, from a point, of a rational cubic curve in space. Similarly, suppose we have a rational quartic curve in a plane, given by equations $\rho x=\phi_1$, $\rho y=\phi_2$, $\rho z=\phi_3$, where ϕ_1, ϕ_2, ϕ_3 are linearly independent quartic polynomials in a parameter θ; and suppose the plane to lie in a fourfold space of (homogeneous) coordinates X, Y, Z, T, U. Take two other, arbitrary, quartic polynomials in θ, namely ϕ_4, ϕ_5, and consider the rational curve, in this fourfold space, which is given by $\sigma X=\phi_1$, $\sigma Y=\phi_2$, $\sigma Z=\phi_3$, $\sigma T=\phi_4$, $\sigma U=\phi_5$, it being understood that no identity

$$A\phi_1+B\phi_2+C\phi_3+D\phi_4+E\phi_5=0$$

holds, for constant values of A, B, C, D, E. This new curve is related to the given quartic by the equations $X/x=Y/y=Z/z$; and, in space of four dimensions, the three equations $X=0$, $Y=0$, $Z=0$ represent a line. Hence we say, a rational plane quartic curve may be looked upon as arising by projection *from a line*, from a rational curve in space of four dimensions; this curve is also of order 4, since, with a, b, c, d, e arbitrary, the equation

$$aX+bY+cZ+dT+eU=0$$

is satisfied by 4 values of θ. A rational quartic curve in space of three dimensions is similarly derivable from a rational quartic curve in four dimensions, by taking only one additional coordinate, the derivation in this case being by projection from a point.

In general, a rational curve in space of r dimensions, where the homogeneous coordinates are $x_0, x_1, \ldots, x_r$, is given by equations $\rho x_i=f_i$, in which the functions f_i are polynomials in a parameter, θ. One at least of these polynomials must be of order as great as r, since otherwise constants $a_0, \ldots, a_r$ can be found to render the equation $\Sigma a_i f_i=0$ identically true; in which case there exist one or more linear equations $\Sigma a_i x_i=0$ connecting the coordinates of a point of the curve, which then lies in space of less than r dimensions. On the other hand, if one (or all) of the polynomials f_i is of order greater than r, the curve may be regarded as derived by projection from a rational curve lying in space of more than r dimensions, in the manner we have illustrated. We may then suppose that these polynomials are of order r, and are linearly independent. The order of the curve represented by the equations, being, by definition, the

number of points lying on a general prime of the space, whose equation is of the form $\Sigma c_i x_i = 0$, is then r. Conversely, any algebraic curve of order r, lying in space of r dimensions, is rational; for a variable prime, drawn through $(r-1)$ fixed points of the curve, meets the curve in one further point; and this point is identified by the single parameter which fixes the particular prime; moreover, by supposing the $(r-1)$ points to coincide at one point of the curve, we see that the only irrationality entering into the rational expression of the points of the curve, beyond those which determine the curve, is that of the coordinates of some particular point of the curve.

We see then that all rational curves are reducible to curves of order r, in space of r dimensions. Such a curve is called a rational normal curve; it is given, so far, by equations of the form $\rho x_i = f_i$, where the $(r+1)$ polynomials f_i are linearly independent; by taking suitable linear functions of $x_0, \ldots, x_r$, say $\xi_0, \ldots, \xi_r$, it is thus capable of being expressed by $\xi_0/\theta^r = \xi_1/\theta^{r-1} = \ldots = \xi_r/1$. Moreover, we see that we can pass from any rational curve of order r, in space of r dimensions, to any other such curve, by linear transformation of the coordinates. Upon such a curve there can be no point such that the value of the parameter θ appropriate to this point occurs as a multiple root in the equation, of order r, which gives the intersections of the curve with a general prime of the space; for such a prime can be put through $r-1$ arbitrary other points of the curve, beside this one. In other words, the curve has no multiple point.

But a rational plane curve, of order greater than 2, must needs have multiple points. We prove in fact that a rational plane curve of order n has $\frac{1}{2}(n-1)(n-2)$ double points, or multiple points equivalent to as many double points; and, conversely, that an algebraic plane curve of order n with $\frac{1}{2}(n-1)(n-2)$ double points, is necessarily rational. To prove the former, assume the curve given by an equation $f(x, y, z) = 0$, and to have δ nodes, and κ cusps. We prove $\delta + \kappa = \frac{1}{2}(n-1)(n-2)$. The equation of the tangent of the curve, at an ordinary point (x, y, z), being known, the t tangent lines of the curve, from an arbitrary point (ξ, η, ζ), touch the curve at the ordinary intersections of the curve with the curve of order $(n-1)$ given by $\xi \partial f/\partial x + \eta \partial f/\partial y + \zeta \partial f/\partial z = 0$; but it is easy to prove that this curve has 2 intersections with the original curve at a node, and has 3 intersections at a cusp. Thus we have $t + 2\delta + 3\kappa = n(n-1)$. On the other hand, assuming the expression of the coordinates of a point of the curve in terms of a parameter, θ, if we join (ξ, η, ζ) to an arbitrary point (x_1, y_1, z_1) of the curve, for which the value of the parameter is θ_1, the parameters of the remaining $(n-1)$ intersections of the joining line, with the curve, are given by the equation $\xi(yz_1 - y_1 z) + \eta(zx_1 - z_1 x) + \zeta(xy_1 - x_1 y) = 0$; if herein, x, y, z and x_1, y_1, z_1 are replaced by their values in terms of θ and θ_1, respec-

tively, and the result divided by $\theta-\theta_1$, there remains an equation $(\theta, \theta_1)=0$, of order $n-1$ in regard both to θ and θ_1, and symmetrical in regard to these. There are therefore $2(n-1)$ values of θ_1 for which the points (θ), (θ_1) coincide, given by the equation $(\theta_1, \theta_1)=0$, which, for a general position of (ξ, η, ζ), will be of aggregate order $2(n-1)$ in θ_1. Such coincidences arise, however, only in two ways, if we assume that the curve has no multiple points beside nodes and cusps: (i) when (θ_1) is a point of the curve at which the tangent line passes through (ξ, η, ζ); (ii) when (θ_1) is a cusp. Thus we infer that $t+\kappa=2(n-1)$. From the former equation obtained we therefore have $\delta+\kappa=\frac{1}{2}(n-1)(n-2)$. The corresponding equation when the curve has higher singularities requires an appropriate definition of δ and κ, into which we do not enter now.

To prove the converse result, that a plane curve of order n, whose only multiple points are nodes and cusps, whose aggregate number is $\frac{1}{2}(n-1)(n-2)$, can have the coordinates of its points expressed rationally by a reversible parameter, we shew that such a curve can be changed, by transformations which are rational in the coordinates, and also rational in the coefficients in the given equation of the curve, either to a straight line, or to a conic. For this, consider, in conjunction with the given curve f, of order n, the most general plane curve ψ, of order $n-2$, which passes through each of the $\frac{1}{2}(n-1)(n-2)$ double points of f. The number of terms in the equation of the general plane curve of order $n-2$ is $\frac{1}{2}n(n-1)$; for this to contain a double point of f one linear condition must be imposed upon the coefficients in ψ. The form of ψ under consideration will thus contain $\frac{1}{2}n(n-1)-\frac{1}{2}(n-1)(n-2)$, or $n-1$, homogeneously entering arbitrary coefficients, or *more* if the $\frac{1}{2}(n-1)(n-2)$ conditions for the double points are not independent; thus the curve ψ will have an equation of the form

$$\lambda_0\psi_0+\ldots+\lambda_{n-2}\psi_{n-2}+\mu_1V_1+\mu_2V_2+\ldots=0,$$

where $\psi_0, \ldots, \psi_{n-2}, V_1, V_2, \ldots$ are definite polynomials of order $n-2$ in the coordinates, linearly independent of one another upon the curve f, and $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}, \mu_1, \mu_2, \ldots$ are arbitrary. As the double points of f are the common solutions of three rational equations $\partial f/\partial x=0$, $\partial f/\partial y=0$, $\partial f/\partial z=0$, symmetrical functions of the coordinates of all these points are expressible rationally by the coefficients in the equation of f, and therefore the coefficients in the polynomials $\psi_0, \ldots, \psi_{n-2}, V_1, \ldots$ are rational in the coefficients in f. The curve ψ will have intersections with f not at the double points, of number $n(n-2)-(n-1)(n-2)$, or $n-2$; the number of coefficients left arbitrary in ψ cannot therefore be enough to enable us to prescribe a particular ψ having, other than at the double points, more than $n-2$ intersections with the given curve f; thus the terms

$\mu_1 V_1 + \mu_2 V_2 + \ldots$ are unnecessary; and the double points of f do furnish independent conditions for ψ. Moreover, the curves $\psi_0 = 0, \ldots, \psi_{n-2} = 0$ cannot have common zeros on the curve f, other than at the double points of f, because the number of intersections remaining would then be less than the number which can be prescribed arbitrarily by proper choice of $\lambda_0, \ldots, \lambda_{n-2}$. In particular $\psi_0, \ldots, \psi_{n-2}$ have no common factor.

We remark in passing that we can now at once see that the coordinates of a point of the curve f are expressible rationally by a parameter, if the coordinates of some arbitrarily taken point of this curve be assumed known. For, if, first, $n-3$ arbitrary points be taken on f, the curves ψ through these points and through the double points will, by what we have seen, have an equation of the form $\theta u - v = 0$, where u, v are definite polynomials in x, y, z, involving the coordinates of the $n-3$ points taken, and θ is variable. The combination of this equation with the equation of f will lead to the coordinates of the only remaining intersection of this curve with f, expressed rationally in terms of θ. If we now suppose the $n-3$ arbitrarily taken fixed points of f to be made to coincide at one point of f, we thus have an expression rational in θ, and in the coordinates of one point of f; and θ is conversely rational in the coordinates of the point which it represents, being equal to v/u.

But we may proceed by a succession of steps, from the general equation of ψ involving $n-1$ homogeneous parameters $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}$. Take $(n-4)$ arbitrary points of the plane, whose coordinates may then be reckoned rational. The curves ψ passing through these will then have an equation of the form $c_0\Psi_0 + c_1\Psi_1 + c_2\Psi_2 = 0$, where Ψ_0, Ψ_1, Ψ_2 are definite linear functions of $\psi_0, \ldots, \psi_{n-2}$, likewise, therefore, rational in the coefficients in f; while c_0, c_1, c_2 are arbitrary. Take now ξ, η, ζ, so that $\xi/\Psi_0 = \eta/\Psi_1 = \zeta/\Psi_2$; then, as (x, y, z) describes the curve f, the point of which ξ, η, ζ are the coordinates will describe another curve, ϕ. The order of this curve ϕ, equal to the number of zeros of a general form $u\xi + v\eta + w\zeta = 0$, on ϕ, is equal to the number of variable zeros of $u\Psi_0 + v\Psi_1 + w\Psi_2$ on f, other than the common zeros of Ψ_0, Ψ_1, Ψ_2 at the double points of f, namely is $n-2$ in general, that is, when the double points are distinct. To any ordinary point of f will correspond a single point of ϕ; to a node of f will in general correspond two points of ϕ, each obtained by one of the modes of approach on f to this double point; but these will be fixed points on ϕ. To a general point of ϕ, say (ξ, η, ζ), there will, by the construction, correspond a point (x, y, z) of the curve f; but there will not correspond *two* points. For equations of the form

$$\Psi_0(x', y', z')/\Psi_0(x, y, z) = \Psi_1(x', y', z')/\Psi_1(x, y, z) = \Psi_2(x', y', z')/\Psi_2(x, y, z)$$

would involve that every curve $c_0\Psi_0+c_1\Psi_1+c_2\Psi_2=0$ which passes through the point (x, y, z) of f passes likewise through (x', y', z'); it is not true that every curve ψ of order $n-2$, through the double points of f, which is drawn through a general point of f likewise passes through another point of f determined thereby, since, as we have seen, such a curve ψ involves a number $n-2$ of arbitrary parameters equal to the number of its unassigned intersections with f; we may therefore assume, if the $n-4$ fixed points of the plane are taken with sufficient generality, that the same is true of the system $c_0\Psi_0+c_1\Psi_1+c_2\Psi_2=0$. Wherefore, the two equations

$$\xi/\Psi_0=\eta/\Psi_1=\zeta/\Psi_2,$$

taken with $f=0$, lead, for a general point (ξ, η, ζ) of the curve ϕ, to a single point (x, y, z) of f. This point may be determined by rational processes of elimination; so that the ratios of x, y, z are not only rational in (ξ, η, ζ), but equally rational in the coefficients in Ψ_0, Ψ_1, Ψ_2, and hence also in the coefficients in f.

We have thus found a new curve ϕ, of points (ξ, η, ζ), of order m say, where m is in general $n-2$, which is wholly in birational correspondence with f. We have seen above, in passing, that the coordinates of the points of f are expressible rationally in terms of a reversible parameter (the expressions involving the coordinates of a point of f); the points of ϕ are therefore expressible rationally by a parameter. From this, by what was proved above, it follows that ϕ has the equivalent of $\frac{1}{2}(m-1)(m-2)$, in general $\frac{1}{2}(n-3)(n-4)$ double points. If we assume that these are distinct, the same process can be applied to ϕ as was applied to f; it can be placed in $(1, 1)$ birational correspondence with a curve of order $n-4$; and the argument can be repeated, until finally we reach either a line (when n is odd), or a conic. The coordinates of the points of a conic can be expressed rationally in terms of a parameter and the coordinates of one particular point arbitrarily taken on the conic; the coordinates of the points of a line are wholly rational in terms of a parameter. The conclusion which has been stated thus follows in general. But the reasoning assumes that at each stage the curve obtained has distinct nodes or cusps. This condition is evidently unnecessary when, for instance, there arises, instead, a k-ple point with separated tangents; for by prescribing the reducing curve to have there a $(k-1)$-ple point, equivalent to $\frac{1}{2}k(k-1)$ conditions, we thereby prescribe $k(k-1)$ intersections, or twice the number of conditions; the multiple point thus has just the effect of $\frac{1}{2}k(k-1)$ separated double points. The examination of the corresponding necessary modification of the reasoning in more complicated cases must be omitted at this stage. Another possibility may be illustrated by considering the simple case where the process is applied to reduce a

curve of order k which has a general $(k-1)$-ple point O. The reducing curves would then be taken to be curves of order $k-2$ with a $(k-2)$-ple point at the given multiple point O; these reducing curves would then consist of $k-2$ lines through this point, and the system $u\Psi_0+v\Psi_1+w\Psi_2=0$ would consist of two variable lines through O, together with $(k-4)$ fixed lines through O. The process thus reduces the given curve to the conic $\xi\zeta-\eta^2=0$, which is rational without the assignment of any point thereon. The original curve is in fact obviously rational, being met in one variable point by a variable line through O, even when k is even. The proof that essentially no other case needs remark must be omitted here.

Greatest possible number of double points of a plane curve. It is natural to suppose, since the condition for a curve of order n to be rational has been shewn to be the possession of $\frac{1}{2}(n-1)(n-2)$ double points, in general, that this is the maximum number possible. We prove now definitely that this is so. More generally, for an irreducible curve of order n with multiple points, of which the general one is of multiplicity denoted by k (varying from point to point), we prove that $\frac{1}{2}\Sigma k(k-1)\leqslant\frac{1}{2}(n-1)(n-2)$. This inequality we prove in the equivalent form

$$n(n-1)\geqslant\Sigma k(k-1)+\tfrac{1}{2}(n-1)(n+2)-\tfrac{1}{2}\Sigma k(k-1).$$

For the curve $f=0$, of order n, it can easily be proved that the so-called first polar curve, whose equation is

$$\xi\partial f/\partial x+\eta\partial f/\partial y+\zeta\partial f/\partial z=0,$$

has $k(k-1)$ intersections with $f=0$ at an ordinary k-ple point of f; it may have more when the tangents at the multiple point are not distinct (for instance, at a cusp there are 3 intersections). As the total number of intersections is $n(n-1)$, we infer that

$$\Sigma k(k-1)\leqslant n(n-1),$$

so that we have $\frac{1}{2}(n-1)(n+2)-\frac{1}{2}\Sigma k(k-1)\geqslant n-1$, and, for $n>1$, the number, say ξ, occurring on the left is positive. Now consider the most general curve of order $n-1$, prescribed to have a $(k-1)$-ple point at every k-ple point of the curve f. If these conditions at the multiple points are independent, the curve will have $\xi+1$ homogeneously entering arbitrary coefficients; for a curve of order m has $\frac{1}{2}m(m+3)+1$ homogeneously entering coefficients, and, for a curve to have an h-ple point at a given point requires $\frac{1}{2}h(h+1)$ conditions.

Thus such curve of order $n-1$ has always at least $\xi+1$ homogeneous coefficients, and, by choice of these, can be made to pass through at least ξ other points of the curve f. By being made to have a $(k-1)$-ple point at a k-ple point of f, a number of intersections at least $k(k-1)$ is secured; but it may happen, for special kinds of the multiple point, that there are more than this. Thus the