CHAPTER I

INTRODUCTORY. RELATIONS OF THE GEOMETRY OF TWO, THREE, FOUR AND FIVE DIMENSIONS

THE present chapter consists of various examples of the interest and importance of the comparison of the geometry of spaces of different dimensions. The first section (pp. 1—32) is concerned with relations between theorems in two and in three dimensions. The second section (pp. 32—40) deals with the representation in four dimensions of some results belonging to ordinary space of three dimensions. The last section (pp. 40—64) deals with the employment of space of five dimensions for the consideration of properties arising both in three and in two dimensions. Some few references occur to space of any number of dimensions.

SECTION I. THEOREMS OF TWO AND THREE DIMENSIONS

The conics touching the fives from six arbitrary lines of a plane. Let three lines, p, q, r, be given in a plane, as well as a fourth line containing two points, I, J; let any conic be drawn touching the four lines; let σ be the conic, through the points I, J, which contains the three intersections of the lines p, q, r; then this conic σ passes through the point, S, in which intersect the tangents from I, J to the former conic. Or, in other words (Vol. II, p. 81), the circle through the intersections of three tangents of a parabola contains the focus of this parabola. Thus if four lines be given, beside the line which contains the points I, J, the conic touching the five lines being then definite, the four conics, all through I, J, each containing the intersections of three of the four given lines, meet in a point, namely, the point, S, in which the tangents from I, J to the former conic intersect (Vol. II, p. 82); namely, these are four circles meeting in the focus of the parabola. If now, finally, five lines be given, beside the line containing the points I, J, there will be five parabolas, each a conic touching the last line and four of the others, and five foci, S_1, S_2, ..., S_5. It is the case that the circle containing any three of these foci passes through the other two, that is, that the seven points S_1, ..., S_5, I, J lie on a conic. Of this theorem a proof was given by Clifford ("A synthetic proof of Miquel's theorem," *Math. Papers*, 1882, p. 38), with the help of certain particular cubic curves.

A circle through the focus, S, of a parabola is a conic containing the intersections of three tangents of the parabola, namely IJ, IS, JS, and may be said to be triangularly circumscribed to the parabola. Conversely, any conic triangularly circumscribed to a conic and meeting a tangent of this in points I, J, contains the intersection, S, of the tangents to this from I and J; it may, therefore, be regarded as a circle through the focus of a parabola, when I and J are taken as the Absolute points. Thus the theorem above referred to may be stated by saying that, if a, b, c, d, l' and l be six arbitrary lines given in a plane, and I, J be two arbitrary given points of the line l, and five conics be defined each as touching l and four of the lines a, b, c, d, l', then there exists a conic passing through I and J which is triangularly circumscribed to these five conics. The symmetry may suggest that this latter is also triangularly circumscribed to the conic which is defined by touching the five lines a, b, c, d, l'; this is in fact the case. We thus have six conics, each touching five of the six given lines; and each of the six lines touches five of the conics. There cannot be two conics through the points, I, J, of l, both triangularly circumscribed to the six conics; such a conic, if existent, is defined by the points I, J and three of the conics touching the line IJ, namely as containing the intersections, S_1, S_2, S_3, of the tangents from I, J to these three conics, respectively.

To prove this symmetrical result we may proceed as follows: Denote the plane of the six given lines, a, b, c, d, l, l' by ϖ. Draw through each of the lines a, b, c, d an arbitrary plane, denoting the intersections of these in threes by A, B, C, D, of which D is the intersection of the planes through a, b, c, and so on. It is assumed that A, B, C, D are not in a plane. Through the six points consisting of A, B, C, D and the two arbitrary points I, J, of the line l, there can be put a definite cubic curve, which we denote by γ. This will meet the plane ϖ in another point beside I and J; say, in E. Then, through the five points A, B, C, D, E there can be drawn another cubic curve, γ', to have the line l' for chord, meeting this, suppose, in the points I', J' (Vol. III, p. 139). We prove that the conic, ω, containing the five points I, J, E, I', J', is triangularly circumscribed to the six conics touching the fives of the six given lines a, b, c, d, l, l'. It is thus independent of the planes drawn through a, b, c, d.

The conic ω is the intersection with the plane ϖ of the quadric, Ω, defined by the nine points $A, B, C, D, E, I, J, I', J'$; the cubic curves, γ, γ', each meeting Ω in seven points, lie on this quadric. Cubic curves lying on a quadric are of two families, since such a curve meets all generators of the surface, of one system of generators, in one point, and all generators of the other system in two points;

two cubics of the same family have four common points, but two cubics of different families have five common points (Vol. III, p. 139). Thus the curves γ, γ' are of different families on Ω.

We prove now, first, that the conic ω is triangularly circumscribed to the conic touching the lines a, b, c, d, l; namely by obtaining triads of points of ω whose joins touch this conic (a, b, c, d, l). Let P be any point of ω; through this, and the other four points, A, B, C, D, of Ω, there can be drawn, lying on Ω, a cubic curve of the same family as γ (Vol. III, p. 129). This curve will meet ω in two further points, say Q and R. Either of these, with A, B, C, D, determines the cubic curve, and so determines the other two of the three points, P, Q, R, of ω. Thus, as P varies on ω, the triads P, Q, R form an involution of sets of three points thereon, and the lines QR, RP, PQ all touch a conic (Vol. II, p. 135); this conic, which we may denote by λ, is evidently triangularly inscribed in ω. We prove that λ is the conic touching a, b, c, d, l, by considering different positions of P. When P is at E, the line QR is the line l; thus λ touches l. In general the cubic curve through A, B, C, D, P, Q, R is projected from P by a quadric cone, which may be defined as that containing PA, PB, PC, PD and a particular one of the two generators of Ω at that point P. But when P is at one of the two intersections of the line d with ω, this cone degenerates, becoming the aggregate of the plane ABC, which contains d, and the plane joining PD to the particular generator at P spoken of; one of the two lines PQ, PR, in which the cone meets the plane ϖ, is thus the line d. Therefore the conic λ touches d. By a similar argument it touches a, b, c.

To prove that the conic ω is triangularly circumscribed to the conic touching a, b, c, d and l', we describe a cubic curve through a varying point, P, of ω and through A, B, C, D, lying on Ω, but of the same family as the curve γ'.

That the conic touching a, b, c, and both of l and l', is triangularly inscribed to ω, we likewise prove by obtaining an involution of sets of three points lying on ω. For this, let the points in which the lines DA, DB, DC meet the plane ϖ be denoted, respectively, by A_1, B_1 and C_1, and consider the conics drawn through the four points A_1, B_1, C_1 and E. The sets of three points other than E in which these conics meet ω are then sets of such an involution (Vol. II, p. 138), and the three joins of the points of such a set are tangents of a conic, which we denote by δ. One conic through A_1, B_1, C_1 and E consists of the two lines B_1C_1 and A_1E; thus δ touches B_1C_1, which is the line a. Similarly δ touches b and c. Again, the conics through A_1, B_1, C_1 and E may be defined by quadric cones, of vertex D, containing DA, DB, DC and DE; one such cone, however, is the cone which contains the cubic curve γ, and this meets the conic ω in the points I, J. Thus the conic δ touches the line l. Another such cone is that containing the curve γ', which meets ω in I' and J'. Thus δ also touches l'. It is thus shewn that the conic touching a, b, c and l, l' is triangularly inscribed in ω. An analogous argument proves ω to be triangularly circumscribed to the *three* conics touching l, l' and, respectively, b, c, d; c, a, d; a, b, d.

The theorem stated is thus completely established. It has been remarked that the conic obtained, triangularly circumscribed to the six primary conics, is uniquely determined by its intersections, I, J, with one of the six given lines, l. It is clear, however, from the reasoning given that the conic is also determinate when E is given; its intersections with l being on the tangents from E to the conic touching a, b, c, d, l, and its intersections with l' being on the tangents from E to the conic touching a, b, c, d, l', while the conic passes through E. Thus, whatever be the planes, drawn through a, b, c, d, by which A, B, C, D are determined, the cubic curve drawn through E, A, B, C, D, to have l for chord, meets l in the same two points, I, J; and similarly for l'.

Ex. 1. If E, I, J be three points of a cubic curve in space, of which A, B, C, D are four other points, the two axial pencils of planes joining EI and IJ to A, B, C, D are related to one another; and the former is related to the range on EI determined by the four planes containing A, B, C, D, the latter being similarly related to the range which these four planes determine on IJ. Wherefore, there is a conic, in the plane EIJ, touching the four lines in which this plane is met by the four planes containing A, B, C, D, and also touching EI, IJ. By parity of reasoning this same conic touches also the line EJ.

Ex. 2. If, in the construction above, the lines a, b, c, d, l, lying in the plane $t=0$, be taken to be

$$x=0,\ y=0,\ z=0,\ ax+by+cz=0,\ lx+my+nz=0,$$

the planes DBC, DCA, DAB, ABC being

$$x=0,\ y=0,\ z=0,\ ax+by+cz-t=0,$$

and the point E be $(\xi, \eta, \zeta, 0)$, prove that the points I, J lie on the conic in the plane $t=0$ given by

$$x^{-1}\xi(mb^{-1}-nc^{-1})+y^{-1}\eta(nc^{-1}-la^{-1})+z^{-1}\zeta(la^{-1}-mb^{-1})=0;$$

this lies on the cone, of vertex D, containing the cubic curve through A, B, C, D, E which has the line l for chord.

Ex. 3. Hence infer that if the six given lines of the original plane be of equations

$$x=0,\ y=0,\ z=0,\ a_r x+b_r y+c_r z=0,\ (r=1, 2, 3),$$

then the general conic which is triangularly circumscribed to the six conics touching the fives of these lines has for equation

$$\Delta (a_1x + b_1y + c_1z)(a_2a_3xYZ + b_2b_3yZX + c_2c_3zXY) \\ + (A_1X + B_1Y + C_1Z)(A_2A_3Xyz + B_2B_3Yzx + C_2C_3Zxy) = 0,$$

where Δ is the determinant (a_1, b_2, c_3), and $A_1, B_2, \ldots$ are the minors of $a_1, b_2, \ldots$ therein, while (X, Y, Z) is an arbitrary point, such that $(a_1^{-1}A_1X, b_1^{-1}B_1Y, c_1^{-1}C_1Z)$ is the point E, or (ξ, η, ζ), of the foregoing theory. This equation, it may easily be seen, is unaltered by cyclical change of the suffixes, X, Y, Z being unaltered; it is derived from a form obtained algebraically by Mr F. P. White (*Camb. Phil. Proc.* XXII, 1924, p. 11). If we take

$$X' = A_1X + B_1Y + C_1Z, \quad Y' = A_2X + B_2Y + C_2Z, \quad Z' = A_3X + \ldots,$$

and also

$$x' = a_1x + b_1y + c_1z, \quad y' = a_2x + \ldots, \quad z' = a_3x + \ldots,$$

the equation is also capable of the form

$$x(B_1C_1x'Y'Z' + B_2C_2y'Z'X' + B_3C_3z'X'Y') \\ + \Delta X(b_1c_1X'y'z' + b_2c_2Y'z'x' + b_3c_3Z'x'y') = 0,$$

and of two other forms, obtained from this by cyclical interchange of a, b, c, without change of X', Y', Z'.

Further, if we take the three points P_1, P_2, P_3, where P_r is of coordinates $(a_r^{-1}A_rX, b_r^{-1}B_rY, c_r^{-1}C_rZ)$, and denote the line $a_rx + b_ry + c_rz = 0$ by l_r, and the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ by A_1, B_1, C_1, and then define a conic, S_1, as that through $(A_1, B_1, C_1, P_2, P_3)$, a conic, S_2, as that through $(A_1, B_1, C_1, P_3, P_1)$ and a conic, S_3, similarly, it will be found that the three pairs of intersections (S_1, l_1), (S_2, l_2), (S_3, l_3) lie on a conic through P_1, P_2, P_3, this conic, through these nine points, being the conic triangularly circumscribed to the six primary conics.

The reader may also be reminded of Taylor's theorem (Vol. II, p. 61), that the poles of an arbitrary line in regard to the six primary conics lie on another conic.

Ex. 4. Any six lines of a plane can in fact be regarded as the projections of six chords of a cubic curve, from a point of this curve, these chords having the property that every five of them have a common transversal. For, taking the six conics touching the fives of the given lines, let the conic which is triangularly circumscribed to these have its equation put in the form $xz - y^2 = 0$, as is possible in an infinite number of ways. Then with an arbitrary point for the intersection of planes $x = 0$, $y = 0$, $z = 0$, this conic is on the cone projecting from this point the cubic curve whose points are given by $(\theta^3, \theta^2, \theta, 1)$. The tangent line of any conic given in the original plane, $t = 0$, may then be defined by a plane,

$$ux + vy + wz = 0,$$

through the point $(0, 0, 0, 1)$, and the tangential equation of the conic may be taken in the form

$$c'u^2 + a'v^2 - cw^2 - bvw - (a + a')\,wu - b'uv = 0,$$

in which u, v, w are tangential coordinates. The chord of the cubic curve joining the points, other than $(0, 0, 0, 1)$, in which the plane meets the curve, has for coordinates the six

$$wu - v^2,\ uv,\ -u^2,\ wu,\ vw,\ w^2;$$

thus the equation of the conic expresses that the tangent line of this is the projection of a chord of the cubic curve which belongs to the linear complex (a, b, c, a', b', c'). In particular, if

$$aa' + bb' + cc' = 0,$$

the conic is triangularly inscribed to the conic $xz - y^2 = 0$, as we easily verify (Cremona, *Giorn. d. Mat.*, x, 1872, p. 47). Whence, six conics triangularly inscribed to $xz - y^2 = 0$ can be regarded as having the equations $(r = 1, \ldots, 6)$

$$n_r'u^2 + l_r'v^2 - n_r w^2 - m_r vw - (l_r + l_r')\,wu - m_r'uv = 0,$$

each being obtained by projecting the chords of the cubic curve which meet a line of coordinates $(l_r, m_r, n_r, l_r', m_r', n_r')$. When the six conics touch the fives of six lines, so that every five of them have a common tangent, these lines are obtained by projection from six chords of the cubic, forming with the lines $(l_r, m_r, \ldots)$ a double six of lines. (Cf. Wakeford, *Proc. Lond. Math. Soc.*, xv, 1916, p. 340.)

It may be remarked, too, that the conic, $(l_r, m_r, \ldots)$, touching the projections of the chords of the cubic which meet the line $(l_r, m_r, \ldots)$, has, beside five of the originally given six lines, as a sixth tangent, the line $l_r'x + m_r'y + n_r'z = 0$. If we take the intersection of this with the remaining one of the six original lines, we obtain one of six points which lie on a conic. (Cf. Vol. III, p. 136, Ex. 8, and p. 201.)

Representation of a plane upon a quadric. Consider a quadric surface, Ω; and, upon this, a point, U, which is to be taken as centre of projection. Consider also a plane, ϖ. Let the generators at U, of the quadric Ω, meet the plane ϖ in the points I and J. Any plane meets the generators; and, thus, the conic, in which Ω is met by any plane which does not pass through U, is projected from U into a conic of the plane ϖ which passes through I and J. Regarding I and J as Absolute points, we may then speak of this conic as a circle. Conversely, any conic, in the plane ϖ, which passes through I and J, lies on a quadric cone of vertex U; this cone contains the generators UI, UJ; its remaining intersection with Ω is thus a curve of the second order, namely a plane section. Thus plane sections of the quadric Ω, not containing U, and circles of the plane ϖ, are transformable into one another by projection from U. Sections of Ω by planes through U project into lines of ϖ, and conversely.

Let α be any conic section of Ω not passing through U; and let A be the pole of the plane of α, in regard to Ω. Let α' be the circle of the plane ϖ which arises by projection of α from U. We prove that the line UA meets the plane ϖ in the pole, in regard to α', of the line IJ; that is, that the projection of A is the *centre* of the circle α'. Let the line in which the tangent plane at U, of the quadric Ω, is met by the plane of α, be denoted by u; and the point in which the line UA meets the plane of α be denoted by A_0. The line UA is the polar line of u, in regard to Ω; thus the polar plane of A_0 contains u, and A_0 is the pole of u in regard to the conic α; so that any line in the plane of α, through A_0, meets u in the harmonic conjugate of A_0, in regard to the two points in which this line meets the conic α. This relation persists after projection from U; and this proves the statement made. Conversely, however, in the correspondence between the points of the plane ϖ, and the points of the quadric Ω, the centre of the circle corresponds to the point, other than U, in which the line UA meets Ω.

We have explained (Vol. II, p. 166) how to measure the angle between two lines of a plane, say of equations $P+\lambda Q=0$, $P+\mu Q=0$, with respect to two other lines $P=0$, $Q=0$, by means of the ratio λ/μ. In particular, the angle between two lines lying in the tangent plane of the quadric Ω, at a point O, and meeting at this point, may be measured with respect to the generators of the quadric at this point. As each of these generators meets one of the generators at U, if O project from U into O' on the plane ϖ, the generators at O project into the lines $O'I$, $O'J$. Thus the angle in question is that between the lines in the plane ϖ, into which the two original tangents of Ω at O project, when measured with respect to the two Absolute points I, J of the plane ϖ. If O be one of the intersections of two plane sections of the quadric Ω, the other being Q, and the lines taken at O be the two tangents of these sections at O, the flat pencil in the tangent plane at O, consisting of these lines and the generators at O, is the section by the tangent plane at O of the axial pencil consisting of the two planes, which meet in OQ, and the two tangent planes of Ω drawn from the line OQ; for each of these tangent planes contains a generator at O and a generator at Q. Thus the angle in question is that of the planes of the two sections measured in regard to the two tangent planes of Ω drawn from OQ. If, in particular, the planes of the two sections are conjugate to one another, each containing the pole of the other, then they are harmonic in regard to the two tangent planes of Ω drawn through their line of intersection. Thus we reach the results that two circles in the plane ϖ intersect at the same angle at both their common points, this being the angle between the corresponding planes of the quadric Ω when suitably measured, and that the circles cut at right angles when these planes are conjugate to one another (cf. Vol. II, p. 193). By considering sections of Ω through U, all conjugate to a chosen plane section, we obtain the result that all lines through the centre of a circle cut this at right angles; by considering plane sections all passing through a point not on Ω, this being the pole of a certain section, we obtain the aggregate of all circles in the plane ϖ which cut a certain circle at right angles; by considering sections of Ω by planes through a line, we obtain a system of coaxial circles, whose limiting points are the projections of the two points in which Ω is met by the polar line of the given line, while sections by planes through this polar line give circles cutting at right angles those of the original system; thence, two points of Ω, lying on a line through the pole of a chosen section of Ω, project into two points which are conjugate in regard to the circle into which the chosen section projects; and so on, all the familiar properties of circles being easily interpretable. It may be remarked that the angle between two plane sections of Ω, which we have identified with the angle between two circles, is also the same as the interval between the points which are the poles of these plane sections, measured with respect to the intersections with Ω of the join of these poles.

The generalised Miquel theorem. We have given (Vol. II, p. 70) the theorem that if D, E, F be any points respectively on the joins, BC, CA, AB, of three points of a plane, then the three circles AEF, BFD, CDE meet in a point. We consider this result in particular from our present point of view. There is a corresponding theorem for any number of dimensions, capable of similar proof.

(Roberts, *Proc. Lond. Math. Soc.*, XII (1881), p. 117; *ibid.*, XXV (1893), p. 306; Grace, *Trans. Camb. Phil. Soc.*, XVI (1897), p. 168; Haskell, *Arch. d. Math. u. Phys.*, V (1903), p. 278.)

Suppose we have, in space of three dimensions, any three planes passing through a point, O, say $x=0$, $y=0$, $z=0$, and take three arbitrary points: A on the line of intersection of $y=0$, $z=0$; B on the line of intersection of $z=0$, $x=0$, and C on the line $x=0$, $y=0$; and then take, on the plane OBC, the point D; on the plane OCA, the point E, and on the plane OAB, the point F; the planes AEF, BFD, CDE will meet in a point, say Q. If O is taken on an arbitrary given quadric, and A, B, C are the intersections with this quadric of the lines $y=0$, $z=0$, etc., while D, E, F are on the sections of this quadric by the planes $x=0$, $y=0$, $z=0$, respectively, then the point Q also lies on this quadric. For, all quadrics through the seven points O, A, B, C, D, E, F have in common a further point; of such quadrics a degenerate one consists of the two planes $OBCD$, AEF; another of the planes $OCAE$, BFD, and a third of the planes $OABF$, CDE. As these degenerate quadrics all contain the point Q, so does the original. When, now, we project the points of the given quadric from O, on to an arbitrary plane, the plane $x=0$ gives a line containing the projections of the points B, C, D, and so on, and the section of the quadric by the plane AEF gives a circle; the three such circles meet in the point which is the projection of Q. This proves the theorem of Miquel.

It is equally the case that, in space of three dimensions, if A, B, C, D be any four points, and points, P, Q, R, P', Q', R' be taken arbitrarily, respectively on the lines DA, DB, DC, BC, CA, AB, then the four spheres, each containing one of the four points A, B, C, D and also the three points on the joins of this to the other points, meet in a point. These are the spheres $APQ'R'$, $BQR'P'$, $CRP'Q'$, $DPQR$. A proof is as follows. In space of four dimensions, consider four threefolds, $x=0$, $y=0$, $z=0$, $t=0$, meeting in a point O; any three of these threefolds will meet in a line through O, and upon each of the four lines so obtained a point may be taken; let the point on the line $y=0$, $z=0$, $t=0$ be A, the others being B, C, D, of which, for example, D is on $x=0$, $y=0$, $z=0$. Upon the plane $y=0$, $z=0$, which contains the lines OA, OD, let the point P be taken, arbitrarily; and, similarly, the points Q and R be taken on the planes OBD, OCD, respectively, as also P', Q', R' on the respective planes OBC, OCA, OAB. Then consider the threefolds $APQ'R'$, $BQR'P'$, $CRP'Q'$, $DPQR$, which we denote, respectively, by $\xi=0$, $\eta=0$, $\zeta=0$, $\tau=0$; and let T be their point of intersection. It can then be shewn that T lies on any quadric threefold constructed to contain the points $O, A, B, C, D, P, Q, R, P', Q', R'$. In fact the equation of a quadric threefold, in fourfold space, contains fifteen terms; thus a quadric through the eleven specified points will be of the form $\lambda_1 U_1 + \ldots + \lambda_4 U_4 = 0$, where $\lambda_1, \ldots, \lambda_4$ are arbitrary, and $U_1=0, \ldots, U_4=0$ are four such quadrics, which we suppose to be linearly independent. It follows that any quadric through the eleven specified points will also pass through the remaining 2^4-11, or five, common points of these four quadrics. This assumes that these four quadrics have no common curve, or surface, and intersect in sixteen points. We can, however, at once specify four degenerate quadrics through the eleven points, namely $x\xi=0$, $y\eta=0$, $z\zeta=0$, $t\tau=0$; for instance $x=0$ contains O, B, C, D, P', Q, R, and $\xi=0$ contains A, P, Q', R'. Thus, every quadric through the eleven points contains the point, T, common to $\xi=0$, $\eta=0$, $\zeta=0$, $\tau=0$. This is the result desired. The four other common points of these four quadrics, which equally lie on any quadric through the eleven points, are the points such as $\eta=0$, $\zeta=0$, $\tau=0$, $x=0$. It can be shewn now that, by projection from O upon any threefold space, the theorem enunciated in regard to the four spheres is obtained. But, for convenience, the discussion of the derivation of a sphere by projection from space of four dimensions is deferred to a later section of this chapter (p. 36). A similar proof holds in higher space; this depends on a theorem that, in space of n dimensions, all quadric $(n-1)$-folds through $\frac{1}{2}n(n+1)+1$ general points are expressible by n such quadrics, and pass through $2^n-1-\frac{1}{2}n(n+1)$ other points.

From the Miquel theorem in a plane we were able to infer (Vol. II, p. 71) that, if four arbitrary lines be given, the circles each containing the intersections of three of these lines, four in all, would meet in a point; namely, by supposing the points D, E, F, in the enunciation given above, to be in line. This theorem, ascribed to Wallace (Scoticus, *Leybourn's Math. Repos.*, I, 1806, p. 170), will be considered below (p. 18). If, in the corresponding theorem above considered, for four spheres in space of three dimensions, we suppose the six points P, Q, R, P', Q', R' to be in a plane, the four spheres there taken will still meet in a point, but this point will be on the plane containing the six points. This is obvious by considering the Wallace theorem for the four lines in this plane which are obtained by the intersections of this plane with the four planes BCD, CAD, ABD, ABC; the four spheres of the theorem meet this plane in circles. We cannot, therefore, go on to infer that the sphere through A, B, C, D passes through the point of intersection of the first four spheres, as in the plane case; the five spheres meet in fours in points lying one on each of the five planes involved. It will be seen below (p. 59) that the generalisation of the Wallace theorem, which thus does not hold in space of three dimensions, holds nevertheless in space of four dimensions, and in space of any even number of dimensions (see Grace, *as above*, p. 163).

Ex. 1. Through five points, A, B, C, D, E, in three dimensions, can be drawn five linearly independent quadrics. The conics in which these meet an arbitrary plane, ϖ, are also linearly independent, and determine a definite conic, σ, inpolar to all of these. Thus every quadric through A, B, C, D, E meets the plane ϖ in a conic which is outpolar to σ. Such a quadric is formed by the plane ABC taken with any plane through the line DE; hence the point in which ϖ is met by DE is the pole, in regard to σ, of the line in which ϖ is met by the plane ABC. Thus the system of ten points and lines, in which the joining lines and planes of A, B, C,