CHAPTER I

INTRODUCTION TO THE THEORY OF QUADRIC SURFACES

Preliminary remark. We have, in previous volumes, given trouble to emphasizing the view that the use of the algebraic symbols is not necessary to the geometrical theory; and that the use of symbols of any particular system is equivalent to the adoption of definite geometrical restrictions.

It is, however, often conducive to clearness and brevity, to employ symbols; and it is usual to suppose that the symbols have the same laws of operation as the numbers of ordinary Analysis. Accordingly in the present volume we shall employ such symbols, whenever it seems desirable. The arithmetic notion of the *magnitude* of the symbols, and especially of *infinite* values, remains excluded—as indeed it is in a logical theory of Analysis; and, as heretofore, the *length* and *congruence* of geometrical lines are not employed, save in the conventional sense explained in Volume II (Chap. V).

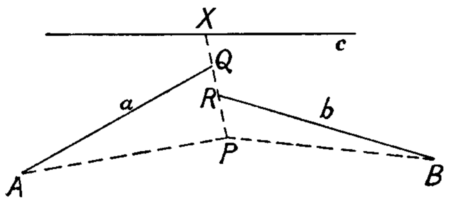
It will be seen that the utility of the symbols arises chiefly from the use of the *equation of a quadric* (or other) *surface*. We here deduce this from the geometrical definition of the surface. It becomes therefore an interesting problem to replace proofs depending on this equation by direct geometrical deductions from the definition.

Not much space is given in this Volume to the proof, which has already been given in Volume II, p. 191, that a plane is represented by a single equation which is linear in the coordinates; that this is so will be readily understood from the explanation given of the equation of a line, in a plane, in Volume II. A line, in this chapter, is generally given as the join of two points, or the intersection of two planes. The theory of the coordinates of a line is discussed at length in a subsequent section (pp. 56 ff.).

Definition of a quadric surface by means of its lines. We may define a quadric surface, or, as we shall often briefly say, a quadric, in several ways.

Take three arbitrary lines, in space of three dimensions, of which no two intersect. From any point of one of these can be drawn a transversal to meet both the other lines. Consider the aggregate of all the points on all the transversals so drawn. This does not include all points of space, since it is not in general possible to draw from an arbitrary point a transversal to meet three given lines.

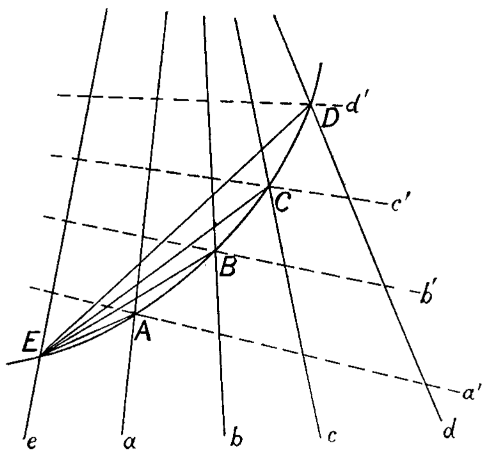
The points of the aggregate which lie in an arbitrary plane lie on a conic. For denote the given lines by a, b, c; let an arbitrary plane meet the lines a, b, respectively, in A and B, and let the transversal drawn from an arbitrary point, X, of the third line, c, meet the lines a, b, respectively, in Q and R, and meet the arbitrary plane in P.



Then the two axial pencils of planes, having, respectively, a and b as axes, given by the planes $a(X)$ and $b(X)$ as X varies on c, are related. On the arbitrary plane these give therefore the two related pencils of lines, $A(P)$ and $B(P)$. The locus of P is therefore a conic, passing through A and B; this conic equally passes through the point, say C, where the line c meets the arbitrary plane, for evidently a transversal of a and b can be drawn from this point.

If the points A, B, C are in line, the ray AC of the pencil $A(P)$ is AB, and coincides with the ray BC of the pencil $B(P)$. Then the locus of P consists, beside the line ABC, of a straight line. In other words, the aggregate now being considered, of points lying on all transversals of the three lines a, b, c, contains, in a plane which contains a transversal ABC of a, b, c, beside the points of this line, also the points of another line, say d, which does not meet a or b or c. Every point of this new line, d, is thus a point from which a transversal can be drawn to meet the given lines a, b and c. It is thus clear that the aggregate of points under consideration consists of the points of a system of lines a, b, c, d, ..., infinite in number, of which no two intersect, together with the points of another system of lines, every one of which meets all those of the first system, but of which no two intersect.

This aggregate of points constitutes a quadric. We see that through every point of the locus there pass two lines, consisting of points belonging to the locus, and the plane of these two lines contains no other points of the locus. But a general plane meets the quadric in the points of a conic. Through points $E, A, B, C, D, \ldots$ of this conic will pass lines e, a, b, c, d, ..., no two of which meet, of which the points are points of the quadric, and also lines e', a', b', c', d', ..., also skew to one another, of which every one meets all those of the first system. On the conic, the flat pencil of lines, $E(A, B, C, D)$, is a section of the axial pencil of planes ea', eb', ec', ed', which is met, by any particular one of the lines a, b, c, d, in the points where this line meets the lines a', b', c', d'. Thus we see that these lines a', b', c', d' meet the lines of the other system in related ranges, which are also related to the range, on the conic in a plane section, determined by the points where these lines a', b', c', d' meet the conic. Similarly the lines a, b, c, d determine, on any of the lines e', a', b', c', d', a range related to that of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ on the conic.



Conversely, if, on two lines, a, b, which do not intersect one another, we have two related ranges, the joins of corresponding points of these ranges are lines which are all met by an infinite number of other lines; the points of these lines are, therefore, those of a quadric surface. For, if $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ be points of the line a, to which correspond the points $\boldsymbol{P}'$, $\boldsymbol{Q}'$, $\boldsymbol{R}'$, respectively, of the line b, and a transversal, c, be drawn from any point of the line $\boldsymbol{PP}'$ to meet $\boldsymbol{QQ}'$ and $\boldsymbol{RR}'$, it follows at once, from what we have seen, that a line drawn from any point of c, to meet a and b, meets these in corresponding points of the two given related ranges thereon. The construction assumes, by hypothesis, that the lines, a, b, of the two related ranges, are not in one plane; if they are, the joins of corresponding points of the two ranges are, as we have seen in Volume II, the tangents of a conic lying in that plane, having the two given lines also as two tangents. The points of a plane, regarded as lying on the tangents of a conic in that plane, may thus be regarded as the points of a degenerate quadric, each tangent line of the conic being taken twice over as a line of the quadric, as a line of each of the systems lying on the quadric.

This account has been obtained by considering the aggregate of the points lying on all the lines which meet three given skew lines. But, in three dimensions, a line is self-dual, and two lines which have a point in common lie also in a common plane; it is proper to consider, then, also the aggregate of all the planes which pass through all the transversals of three given lines. As we have found that through any point of the quadric there pass two lines lying entirely thereon, so we have incidentally found that in any plane passing through a transversal of three given skew lines, a, b, c, there is another line, d, not meeting a, b, c, which meets all transversals of these.

To describe the dual correspondence in further detail it is necessary to speak first of what is meant by a quadric cone. We have seen that the quadric surface is the locus of points on lines joining corresponding points of two related ranges on skew lines, these joining lines degenerating into the tangents of a conic when the skew ranges are replaced by ranges on two intersecting lines. Similarly the aggregate of planes we are now considering, as the dual of the aggregate of points of a quadric, may be obtained by taking two skew lines as axes of two related axial pencils of planes; a line of intersection of a plane of one pencil, with the corresponding plane of the other pencil, is then a line meeting both the axes, and, as may easily be seen, in points of two related ranges thereon. The planes of the aggregate are the planes through such lines of intersection. But, now, if we take two intersecting lines, and have two related axial pencils of planes with these as axes, the line of intersection of a plane of one pencil with the corresponding plane of the other pencil, is a line through the point of intersection of the axes; the aggregate of these lines, passing through a point, constitutes what is called a quadric cone, and is the dual of the aggregate of lines in a plane which touch a conic. As two tangents of the conic meet in a point, so two of the lines forming the cone lie in a plane; as the point of intersection of two tangents of the conic becomes a point of the conic when the two tangents coincide (Vol. II, p. 25), so, if the two lines of the cone coincide, the plane containing them is replaced by a definite plane, called a tangent plane of the cone. The section of a cone by an arbitrary plane is evidently a conic, of which one point is determined by a line of the cone, and a tangent line by a tangent plane of the cone; conversely from any conic we obtain a quadric cone by projection from an arbitrary point not lying in the plane of the conic.

This being understood, consider the dual of the statement that a plane section of a quadric is a conic, through every point of which there passes a line of each of the two systems lying on the quadric. Let an arbitrary point of space be joined to every line of one system of lines lying on the quadric, by a plane. Each of these planes will, as we have seen, contain another line lying on the quadric, of the other system. The aggregate of these planes is then that of the tangent planes of a quadric cone. This is obvious from the fundamental duality of the figure. But it is clear, too, by considering that a variable line, a', of one system, of the quadric surface, meets two lines, a, b, of the other system, in related ranges of points. The plane Oa', joining the line a' to a fixed point O, thus meets the planes Oa, Ob in two related flat pencils of lines, with O as common centre. By taking a section by an arbitrary plane, the planes Oa' give rise to lines in this plane meeting two fixed lines, which are the sections of the arbitrary plane by the planes Oa and Ob, in two related ranges. These lines, therefore, are tangents of a conic in that plane; and the planes Oa' are tangent planes of a quadric cone of which all the lines pass through O. This point O is called the *vertex* of the cone, and its lines are called its *generators*; the lines of a quadric surface are also called the *generators* of the quadric.

A particular consequence is that, as there are two points of the quadric locus lying on an arbitrary line, so there are two of the aggregate of planes, passing through an arbitrary line. These may be obtained, directly from the definition, as the two common corresponding planes of two related axial pencils, just as in the case of the locus of points.

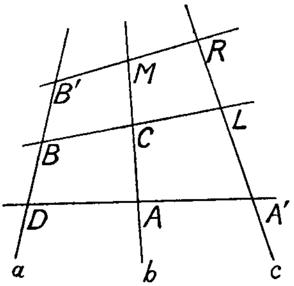
Ex. 1. If A, B, C, D, O be five general points in space, the pairs of planes joining O to the opposite pairs of joins of A, B, C, D, such as OAD, OBC, meet an arbitrary line in three pairs of points which are in involution.

Ex. 2. If an axial pencil of planes be drawn through a line, l, and a related axial pencil of planes be drawn through another line, m, which does not meet l, the lines of intersection, of a plane of the first pencil with the corresponding plane of the second pencil, define a quadric surface, upon which the lines l and m also lie.

Ex. 3. A line l meets the planes BCD, CAD, ABD, ABC, which contain the triads of four arbitrary points A, B, C, D, respectively in P, Q, R, S; another line, l', meets these planes, respectively, in P', Q', R', S'; and the four lines PP', QQ', RR', SS' all lie on the same quadric surface. Prove that the four transversals drawn from A, B, C, D, each to meet both the lines l and l', all lie on a quadric surface. (Cf. Vol. I, p. 30.)

Ex. 4. Let D, P, Q, R be four points in line, and A, B, C be three points whose plane does not contain the line. Let the transversal, l, be drawn from A to meet the lines BQ, CR; the transversal, m, be drawn from B to meet the lines CR, AP; and the transversal, n, be drawn from C to meet the lines AP, BQ. Prove that a line can be drawn from D to meet all of l, m, n.

The representation of this definition by means of the algebraic symbols. We may represent the matter very simply by means of the algebraic symbols. Let a, b, c be three skew lines, of which DAA', BCL, $B'MR$ are any three transversals. Taking D, A, C, B, which do not lie in a plane, as fundamental points, we can choose the symbols of the points A, A', relatively to that of D, so that $A' = D + A$, and, similarly, the symbols of C, L, relatively to that of B, so that $L = B + C$. Then the symbol of R may be supposed to be



$$R = A' + \mu L = D + A + \mu(B + C)$$
$$= D + \mu B + A + \mu C,$$

where $D + \mu B$ is the symbol of some point on the line a, and $A + \mu C$ of some point on the line b. Hence we infer, for the symbols of B' and M, which lie on a line through R, respectively,

$$B' = D + \mu B, \quad M = A + \mu C.$$

Any other point of the transversal $B'M$ has thus a symbol $B' + \lambda M$, namely

$$\lambda A + \mu B + \lambda\mu C + D,$$

which, then, for different symbols λ, μ, is the general point of the general transversal of the fundamental lines a, b, c. It is thus the general point of the quadric surface under consideration. The symbol can also be written

$$D + \lambda A + \mu(B + \lambda C),$$

and the point lies on a transversal of the lines DA, BC, this transversal meeting these, respectively, in the points $D + \lambda A$, $B + \lambda C$.

We see that every point of the quadric is characterised by two particular algebraic symbols, λ and μ; and further that all points of the quadric for which λ is the same are on a line, not intersecting a, or b, or c; and all points for which μ is the same are on a line, meeting a, b, c but not meeting DA, BC, $B'M$.

The two points of the quadric which lie on an arbitrary line, say on the line joining the points whose symbols are

$$a_1 A + b_1 B + c_1 C + D, \quad a_2 A + b_2 B + c_2 C + D,$$

would then be found by choosing σ, so that the point of this line given by

$$(a_1 + \sigma a_2) A + (b_1 + \sigma b_2) B + (c_1 + \sigma c_2) C + (1 + \sigma) D$$

is the same, for proper values of λ and μ, as the point

$$\lambda A + \mu B + \lambda\mu C + D;$$

this requires

$$(1 + \sigma)(c_1 + \sigma c_2) = (a_1 + \sigma a_2)(b_1 + \sigma b_2).$$

We can express the character of the quadric also by representing any point of it by a symbol

$$XA + YB + ZC + TD,$$

so that X, Y, Z, T are the *coordinates* of this point relatively to the points A, B, C, D; then the sole condition for X, Y, Z, T, is that

$$XY = ZT;$$

this is called the *equation of the quadric*. If we take new points of reference A_1, B_1, C_1, D_1, such that

$$A_1 = C - D, \quad B_1 = A + B, \quad C_1 = A - B, \quad D_1 = C + D,$$

any point of space given by

$$xA_1 + yB_1 + zC_1 + tD_1,$$

is $$x(C - D) + y(A + B) + z(A - B) + t(C + D),$$

and is a point of the quadric if

$$X = y + z, \quad Y = y - z, \quad Z = t + x, \quad T = t - x,$$

that is, if $$x^2 + y^2 - z^2 = t^2.$$

This, then, is another form for the equation of the quadric. Whatever θ may be, this equation is satisfied by

$$y + z = \theta^{-1}(t + x), \quad y - z = \theta(t - x);$$

also, whatever ϕ may be, it is satisfied by

$$y + z = -\phi^{-1}(t - x), \quad y - z = -\phi(t + x).$$

It is, however, easy to see that a single *linear* equation connecting the coordinates, (x, y, z, t), of a point, implies that this point lies on a certain plane; for instance, an equation $t = lx + my + nz$ shews that the point, $xA_1 + yB_1 + zC_1 + (lx + my + nz)D_1$ is on the plane through the three points $A_1 + lD_1$, $B_1 + mD_1$, $C_1 + nD_1$. Thus the points for which $y + z = \theta^{-1}(t + x)$, $y - z = \theta(t - x)$ are those of a line; and, by taking different values of θ, we thus obtain a system of lines lying on the quadric surface; it is easy to see that no two of these have a point in common. Another system of mutually non-intersecting lines is given by the equations $y + z = -\phi^{-1}(t - x)$, $y - z = -\phi(t + x)$, for different values of ϕ. It is easily verified that, whatever θ and ϕ may be, the θ-line, of the first system, has a point in common with the ϕ-line, of the second system, this being, in fact, that given by

$$x = \theta + \phi, \quad y = 1 - \theta\phi, \quad z = 1 + \theta\phi, \quad t = \theta - \phi.$$

These expressions then satisfy identically the equation

$$x^2 + y^2 - z^2 - t^2 = 0.$$

Quadric surface defined by two conics in space having two points in common, and a line which meets these conics. We have seen that the points of a quadric which are on a plane lie on a conic. If two points, A, B, of the quadric be taken, and two planes be drawn through these two points, we thus obtain two conics lying on the quadric. Considering only the lines of the quadric belonging to one system of generators, there will be one line of this system through an arbitrary point, P, of one of these conics; this line will meet the plane of the other conic in a point, say P', which, as the line lies entirely on the quadric, will be on the other conic. There is thus a (1, 1) correspondence between the points of the two conics, whereby to every point, P, of one of these conics, there corresponds a definite point, P', of the other, while P' equally determines P. By what has been proved above, the range of points, P', on one conic, is related to the range of points, P, on the other.

Conversely, let two arbitrary conics be given, lying in different planes, but having two points, A, B, in common; let a further arbitrary fixed point, C, be taken on one conic, and a point, C', on the other; then, we can, to any variable point, P, of the former conic, make correspond a definite point, P', of the latter conic, by the condition that the range of points, A, B, C, P, of the former conic is related to the range of points, A, B, C', P', of the latter. The aggregate of all the points lying on all the lines PP' is then a quadric surface. And when the conics only are given, there is an infinite number of quadrics so obtainable, since we can choose the point C', which is to correspond to C, in an infinite number of ways. A definite quadric is determined by the two conics together with the line CC'. In particular, if the tangents of one conic at the points A, B meet in the point T, and the tangents of the other conic at the points A, B meet in the point U, and the line CC' intersect the line TU, say in O, it can be shewn that the conics are in perspective from the point O. For if the lines which join the point O to the points, P', of one of the two given conics, σ', be allowed to meet the plane of the other conic, σ, in points, Q, it follows, because the pencils of lines, $A(P')$, $B(P')$, are related, that the pencils of lines, $A(Q)$, $B(Q)$, are also related, so that Q describes a conic, τ, in the plane of the conic σ. As O, A, T, U are in one plane, the conic τ will touch the conic σ at A; similarly it will touch it at B; and the point C, lying on OC', will be common to the conics τ and σ. Two conics which touch at two points, and have another point in common, are however coincident. Thus the lines, PP', joining corresponding points of the two conics, all pass through O. The quadric determined by this particular correspondence of C' to C thus reduces to a quadric cone, of vertex O.

In symbols, referred to A, B, T, U, the corresponding points P, P' of the two conics may be taken to be

$$P = \theta^2 A + \theta T + B, \quad P' = \phi^2 A + \phi U + B,$$

where θ, ϕ, by their variation, are to give two related ranges on the two conics, with the condition that the values $\theta^{-1} = 0$, $\theta = 0$, at

A, B, are to correspond to the values $\phi^{-1}=0$, $\phi=0$; thus ϕ is of the form θa^{-1}, where a is the same for all points P, P', but depends upon the initial correspondence adopted, of C' to C. A general point $\rho P+aP'$, of the line PP', is then

$$\theta^2(\rho+a^{-1})A+\theta\rho T+\theta U+(\rho+a)B,$$

or, say, $xA+tT+zU+yB$, so that

$$x=\theta^2(\rho+a^{-1}),\quad t=\theta\rho,\quad z=\theta,\quad y=\rho+a;$$

eliminating ρ we have, for the equation satisfied by the coordinates of any point of the line PP',

$$xy=z^2+(a+a^{-1})zt+t^2;$$

if we put

$$\xi=x+y,\quad \eta=x-y,\quad \zeta=(a^{\frac{1}{2}}+a^{-\frac{1}{2}})(z+t),\quad \tau=(a^{\frac{1}{2}}-a^{-\frac{1}{2}})(z-t),$$

this equation is the same as

$$\xi^2-\eta^2-\zeta^2+\tau^2=0,$$

which is another form of the equation of the quadric.

But, without assigning the correspondence of the two points C, C', we may suppose the two conics to be given, respectively, with the coordinates here used, by the equations

$$z=0,\ xy-t^2=0,\quad \text{and}\quad t=0,\ xy-z^2=0;$$

then, whatever k may be, the equation

$$xy=z^2+2kzt+t^2,$$

is evidently that of a locus containing both the given conics. And k can be chosen so that the locus contains an arbitrary point of space which does not lie in the plane of either of the two conics. A plane drawn through such an arbitrary point, say O, will meet each of the two conics in two points; through the four points so found, and the point O, a definite conic can be drawn. The points of all such conics, obtained by drawing different planes through the same point O, do in fact constitute a quadric determined by the two conics and the point O. To prove this, it is sufficient, after what has preceded, to shew that a quadric can be found to contain two conics which have two points in common, and to contain also an arbitrary point not lying on the plane of either of the two conics; and this follows from the equation just put down. But we can deduce the result at once without the symbols, from the preceding theory, by remarking that a line (two lines, indeed) can be drawn from the arbitrary point to meet the given conics, say in C and C' respectively, and these points C, C' can then be used to establish a (1, 1) relation of points of the two conics. That such a transversal of the two conics can be drawn from the arbitrary point, is clear by

Cambridge University Press
978-1-108-01779-4 - Principles of Geometry, Volume 3
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projecting one of the conics, from the point, on to the plane of the other conic; thereby a conic is obtained having, beside A and B, two points of intersection with the given conic on that plane. Thereby two such transversals are obtained; but these lead to the same quadric, being the generators of the two systems of the quadric which pass through the arbitrary point.

When k is real, in the quadric given by the equation

$$xy = z^2 + 2kzt + t^2,$$

there is a difference according to the value of k, that is, according to the position of the arbitrary point, which is given in addition to the two conics. When $k^2 > 1$, there exists a *real* number a such that $2k = a + a^{-1}$; then the quadric contains real lines, as we have seen. When $k^2 < 1$, the lines are imaginary. When $k^2 = 1$, the equation of the quadric surface is one of the two represented by

$$xy = (z+t)^2, \quad xy = (z-t)^2,$$

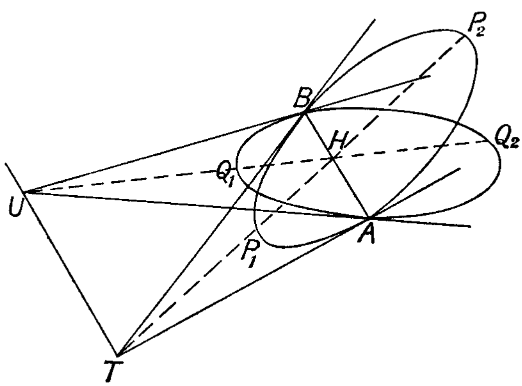
and the surface is, in fact, a cone. In this case the corresponding points, P, P', of the two conics, are

$$P = \theta^2 A + \theta T + B, \quad P' = \theta^2 A \pm \theta U + B,$$

and the line PP' contains one of the two points $T \mp U$, whatever P and P' may be.

An interesting simple result follows from what we have said; *if we have three conics in space of which every two have two points in common, then a quadric surface exists containing all these conics.* This remark is made by Poncelet, *Traité des propriétés projectives des figures*, 1865, I, p. 378, § 606.

Ex. 1. There are in fact just two quadric cones which can be constructed to contain two conics in space which have two points



in common, as will appear also in another way. We may construct these, if T, U be, as above, the intersections of the tangents of the