

Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

CAMBRIDGE LIBRARY COLLECTION

Books of enduring scholarly value

Mathematical Sciences

From its pre-historic roots in simple counting to the algorithms powering modern desktop computers, from the genius of Archimedes to the genius of Einstein, advances in mathematical understanding and numerical techniques have been directly responsible for creating the modern world as we know it. This series will provide a library of the most influential publications and writers on mathematics in its broadest sense. As such, it will show not only the deep roots from which modern science and technology have grown, but also the astonishing breadth of application of mathematical techniques in the humanities and social sciences, and in everyday life.

Principles of Geometry

Henry Frederick Baker (1866–1956) was a renowned British mathematician specialising in algebraic geometry. He was elected a Fellow of the Royal Society in 1898 and appointed the Lowndean Professor of Astronomy and Geometry in the University of Cambridge in 1914. First published between 1922 and 1925, the six-volume *Principles of Geometry* was a synthesis of Baker's lecture series on geometry and was the first British work on geometry to use axiomatic methods without the use of co-ordinates. The first four volumes describe the projective geometry of space of between two and five dimensions, with the last two volumes reflecting Baker's later research interests in the birational theory of surfaces. The work as a whole provides a detailed insight into the geometry which was developing at the time of publication. This, the second volume, describes the principal configurations of space of two dimensions.

Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

Cambridge University Press has long been a pioneer in the reissuing of out-of-print titles from its own backlist, producing digital reprints of books that are still sought after by scholars and students but could not be reprinted economically using traditional technology. The Cambridge Library Collection extends this activity to a wider range of books which are still of importance to researchers and professionals, either for the source material they contain, or as landmarks in the history of their academic discipline.

Drawing from the world-renowned collections in the Cambridge University Library, and guided by the advice of experts in each subject area, Cambridge University Press is using state-of-the-art scanning machines in its own Printing House to capture the content of each book selected for inclusion. The files are processed to give a consistently clear, crisp image, and the books finished to the high quality standard for which the Press is recognised around the world. The latest print-on-demand technology ensures that the books will remain available indefinitely, and that orders for single or multiple copies can quickly be supplied.

The Cambridge Library Collection will bring back to life books of enduring scholarly value (including out-of-copyright works originally issued by other publishers) across a wide range of disciplines in the humanities and social sciences and in science and technology.

Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

Principles of Geometry

VOLUME 2:
PLANE GEOMETRY

H.F. BAKER



Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9781108017787

© in this compilation Cambridge University Press 2010

This edition first published 1922

This digitally printed version 2010

ISBN 978-1-108-01778-7 Paperback

This book reproduces the text of the original edition. The content and language reflect the beliefs, practices and terminology of their time, and have not been updated.

Cambridge University Press wishes to make clear that the book, unless originally published by Cambridge, is not being republished by, in association or collaboration with, or with the endorsement or approval of, the original publisher or its successors in title.

Cambridge University Press

978-1-108-01778-7 - Principles of Geometry, Volume 2

H. F. Baker

Frontmatter

[More information](#)

PRINCIPLES OF GEOMETRY

Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS

C. F. CLAY, MANAGER

LONDON : FETTER LANE, E.C. 4



LONDON : H. K. LEWIS AND CO., LTD.,
136, Gower Street, W.C. 1

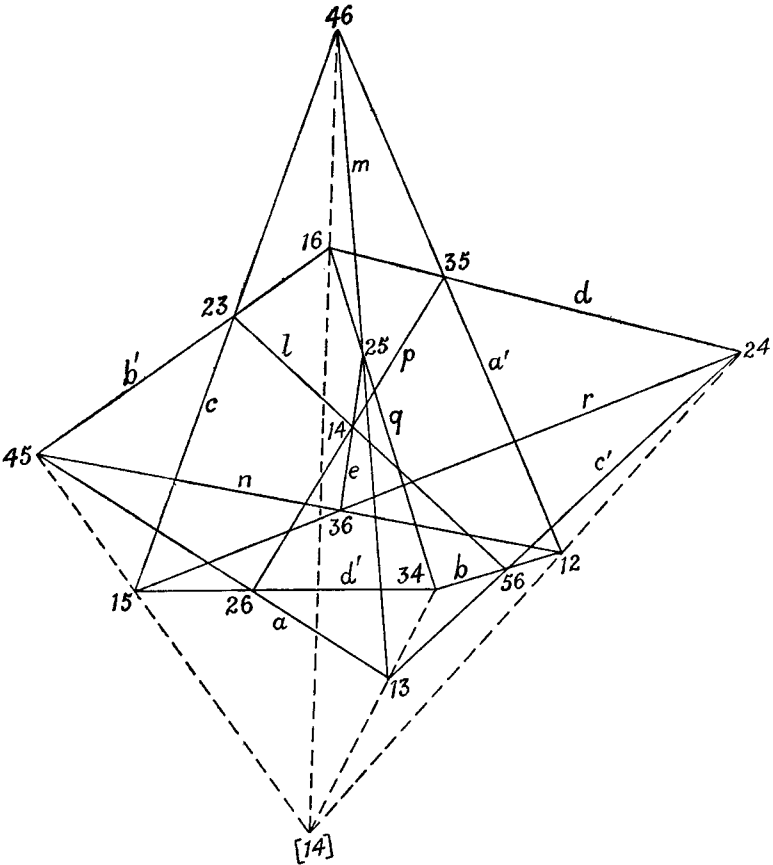
NEW YORK : THE MACMILLAN CO.

BOMBAY
CALCUTTA } MACMILLAN AND CO., LTD.
MADRAS }

TORONTO : THE MACMILLAN CO. OF
CANADA, LTD.

TOKYO : MARUZEN-KABUSHIKI-KAISHA

ALL RIGHTS RESERVED



HEXAGRAMMUM MYSTICUM
 (see p. 219)

Cambridge University Press
978-1-108-01778-7 - Principles of Geometry, Volume 2
H. F. Baker
Frontmatter
[More information](#)

PRINCIPLES OF GEOMETRY

BY

H. F. BAKER, Sc.D., F.R.S.,

LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY, AND FELLOW OF
ST JOHN'S COLLEGE, IN THE UNIVERSITY OF CAMBRIDGE

VOLUME II

PLANE GEOMETRY

CONICS, CIRCLES, NON-EUCLIDEAN GEOMETRY

In minimis maxima

CAMBRIDGE
AT THE UNIVERSITY PRESS

1922

Cambridge University Press

978-1-108-01778-7 - Principles of Geometry, Volume 2

H. F. Baker

Frontmatter

[More information](#)

PRINTED IN GREAT BRITAIN

Cambridge University Press

978-1-108-01778-7 - Principles of Geometry, Volume 2

H. F. Baker

Frontmatter

[More information](#)

PREFACE

THE present volume has in effect two aims: In the first place, in pursuance of the general purpose of the book, it seeks to put the reader in touch with the main preliminary theorems of plane geometry. Chapter I is devoted to a deduction, with synthetic methods, of the fundamental properties of conic sections; it is an introduction to what is usually called Projective Geometry, in the plane, in which, however, the notions of distance and congruence are not assumed. Chapter II, also without help of these notions, develops results that arise by considering conics in relation to two Absolute points, including, for instance, the properties of circles, and of confocal conics; the matter here contained is usually found in sequels to Euclid, books on Pure Geometry, and books on Geometrical Conics. Chapter III is designed to explain the application of the algebraic symbols to plane geometry; it contains methods and formulae found in works on Analytical Geometry of the Plane. Chapter IV is a brief consideration of some logical questions, and marks the recognition of a limitation in the symbols employed; it deals with the sense in which the words real and imaginary are used, and calls attention to the elements of Analysis assumed in the following chapter. Chapter V deals with the theory of measurement, of length and angle, with the help of an Absolute conic, shewing how the so-called non-Euclidean geometries may be regarded as included in our general formulation. It considers the metrical plane also as deduced from the geometry of a quadric surface, incidentally dealing with the fundamental properties of this surface and, in particular, with Spherical Trigonometry. As a corollary from this point of view, Riemann's space of constant curvature is seen not to require the assumption of absolute coordinates; and further, that form of the hyperbolic geometry in which lines are replaced by circles cutting a fixed circle at right angles (which, for instance, was an inspiration to Poincaré in his development of the theory of automorphic functions) is seen to arise naturally. Notes I and II deal with the theorems of incidence which were developed very gradually for the complete Pascal figure and appeared very intricate; from the point of view here explained they are natural, if particular, properties of a figure which arises otherwise, and will much concern us in a later volume. Note III gives some indications of the literature of non-Euclidean geometry. Note IV contains remarks and corrections for Volume I, for many of which I am indebted to friends. There is also an Index; but it is possible that the extensive Table of Contents

may be more useful. No attempt is made to give a general Bibliography for the contents of the volume.

It will be seen that the volume deals with a wide range of theory; in other conditions than the present, a less condensed treatment might have been desirable. The order in which the ideas are taken has been chosen largely in view of the second aim of the volume; it will not be difficult, with the help of the Table of Contents, for the reader to modify this order. It is believed, however, that a large amount of the time usually spent, at present, in learning geometry, could be saved by following, from the beginning, after an extensive study of diagrams and models, the order of development here adopted; and such a plan would make much less demand upon the memory.

But the second aim of the volume may, I hope, appeal to attentive readers. It is an attempt, tempered indeed by practical considerations, to test the application in detail of the logical principles explained in Volume I. It seeks to bring to light the assumptions which underlie an extensive literature in which coordinates are freely used without attempt at justification. It suggests the question whether, in the case of distance, as in many other cases, we may not have derived from familiarity with physical experiences, a confidence which a more careful scrutiny can only regard as an illusion. When this view, which seems sure, shall win acceptance, the change in scientific thought will be rapid and momentous. As the first step in this sense was made in the development of the theory of our geometrical conceptions, it is proper that the matter should be dealt with here. It will be of importance if the reader come to see how deep lying are the questions involved in the use of coordinates, and the assumption of distance as a fundamental idea.

As in the case of the first volume, I desire to express my thanks to the Staff of the University Press for their care and courtesy, and to Mr J. B. Peace, M.A., for the great trouble he has taken with the numerous diagrams.

H. F. B.

2 September 1922

TABLE OF CONTENTS

PRELIMINARY

	PAGES
Related ranges on the same line	1
Involution	2—4
Symbolical expression of the preceding results	4—6
Examples of involution	6—8
A general abbreviated argument for relating two ranges	8, 9
Of the distinction between the so-called real and imaginary points	9

CHAPTER I. GENERAL PROPERTIES OF CONICS

Definition of a conic	10—12
Involution on a line by conics having four common points	12, 13
MacClaurin's definition of a conic. Pascal's theorem	13, 14
Ranges of points on a conic	14
Involution of pairs of points of a conic	14, 15
Pascal's theorem and related ranges of points upon a conic	15, 16
Converse of Pascal's theorem	16
Introduction of the algebraic symbols	16—20
Geometrical theory resumed. Polar lines	20, 21
Duality in regard to a conic	21
Dual definition of a conic	22, 23
Examples of the application of the foregoing theorems (Exx. 1—36)	23—62
Ex. 1. <i>The self-polar triad of points for conics through four points ; the self-polar triad of lines for conics touching four lines</i>	23—25
Ex. 2. <i>Dual of Pascal's theorem</i>	25, 26
Ex. 3. <i>The director conic of a given conic in respect of two arbitrary points</i>	26, 27
Ex. 4. <i>The polars of three points determine a triad of points in per- spective with the original triad</i>	27
Ex. 5. <i>A construction for two points conjugate in regard to a conic</i>	27, 28
Ex. 6. <i>Range determined by a conic, and by the joins of three points of the conic, upon any line drawn through a fixed point of the conic</i>	28, 29
Ex. 7. <i>The six joins of two triads of points of a conic touch another conic</i>	29
Ex. 8. <i>The pedal line of three points of a conic in regard to three other points of the conic, taken in a particular order</i>	29, 30
Ex. 9. <i>The Hessian line of three points of a conic. The Hessian point of three tangents of a conic</i>	30, 31

	PAGES
Ex. 10. <i>Polar triads in regard to a conic. Hesse's theorem</i>	31—33
Ex. 11. <i>Outpolar conics</i>	33—35
Ex. 12. <i>Two self-polar triads of one conic are six points of another conic</i>	35, 36
Ex. 13. <i>Particular cases of outpolar, or of inpolar, conics</i>	36
Ex. 14. <i>The joins of four points of a conic and the intersections of the tangents at these points. The common points, and the common tangents, of two conics</i>	36, 37
Ex. 15. <i>The polar lines of any point in regard to the conics through four given points all pass through another point</i>	37, 38
Ex. 16. <i>A point to point correspondence in regard to four given points</i>	38, 39
Ex. 17. <i>Four conics can be drawn through two given points to touch three given lines</i>	39—41
Ex. 18. <i>The poles of a line, in regard to conics with four common points, lie on a conic</i>	41, 42
Ex. 19. <i>Two conics of which the tangents at their common points meet, in fours, at two points</i>	42—44
Ex. 20. <i>A conic derived from two given conics</i>	44, 45
Ex. 21. <i>To construct a conic in regard to which two triads of points of a given conic shall both be self-polar</i>	45, 46
Ex. 22. <i>To construct a conic for which a given triad of points shall be a self-polar triad, and a given point and line shall be pole and polar</i>	46
Ex. 23. <i>To construct a conic for which a given triad of points shall be self-polar, to pass through two given points</i>	46, 47
Ex. 24. <i>The polar reciprocal of one conic in regard to another. A particular case</i>	47, 48
Ex. 25. <i>Gaskin's theorem for conics through two fixed points outpolar to a given conic</i>	48—50
Ex. 26. <i>Two related ranges upon a conic obtained by composition of two involutions</i>	50—52
Ex. 27. <i>Three triads of points of a conic are all in perspective with another triad of points of the conic</i>	52
Ex. 28. <i>The envelope of the line joining two corresponding points of two related ranges on a conic</i>	52, 53
Ex. 29. <i>The tangents from three points, A, B, C, to a conic, meet BC, CA, AB in six points of a conic</i>	53, 54
Ex. 30. <i>Two tangents of one conic meet two tangents of another in four points lying on a conic through the common points of the two conics, if the four points of contact are in line</i>	54, 55
Ex. 31. <i>Poncelet's theorem, for sets of points A, B, C, ... of one conic, of which the joins AB, BC, ... separately touch other fixed conics, all the conics having four points in common</i>	55—58
Ex. 32. <i>Construction of a fourth tangent, with its point of contact, of a conic touching three lines at given points</i>	58, 59

Contents

xi

	PAGES
Ex. 33. <i>Hamilton's extension of Feuerbach's theorem</i>	59, 60
Ex. 34. <i>Theorem in regard to the remaining common tangents of three conics, which touch three given lines</i>	60, 61
Ex. 35. <i>The four conics through two points of which each contains three of four other given points</i>	61
Ex. 36. <i>The polars of any point in regard to the six conics which can be drawn through the sets of five, from six arbitrary points of a plane, touch a conic</i>	61, 62

CHAPTER II. PROPERTIES RELATIVE TO TWO POINTS OF REFERENCE

Introductory	63
Middle points, Perpendicular lines, Centroid, Orthocentre	63, 64
Circles. Circles at right angles	65, 66
Coaxial circles	66, 67
Inversion in regard to a circle	67, 68
Examples of inversion in regard to a given circle	68—70
Examples in regard to circles (Exx. 1—9)	70—75
<i>Miquel's theorem, pedal (Wallace) line, property of four circles</i>	70—72
<i>Proof of Feuerbach's theorem by inversion</i>	73—75
Angle properties for a circle	75—77
Foci and axes of a conic	77—79
Confocal conics	79
Common chords of a circle with a conic	79, 80
The director circle of a conic. The director circles of conics touching four lines are coaxial	80, 81
Parabolas. The directrix of a parabola	81
Examples in regard to a parabola (Exx. 1—5)	81—83
Definition of a rectangular hyperbola	83, 84
Examples in regard to a rectangular hyperbola (Exx. 1—3)	84, 85
Auxiliary circle of a conic	85, 86
Examples in regard to auxiliary circle, and pedal circle (Exx. 1—5)	87, 88
The locus of the centres of rectangular hyperbolas passing through three given points	88—90
The polars of a point in regard to a system of confocal conics touch a parabola. Four normals can be drawn from a point to a conic	90, 91
A particular theorem for intersection of a conic with a rectangular hyperbola	91, 92
The normals of a conic which pass through a given point. The feet of three of these, and the opposite of the other foot, lie on a circle. Case of the parabola also. Examples (Exx. 1, 2)	92—94
The dual of the director circle of a conic. Sets of four points of one conic whose joins in order touch another conic	94—96

CHAPTER III. THE EQUATION OF A LINE,
 AND OF A CONIC

	PAGES
Preliminary remarks	97
The coordinates of a point in a plane	97, 98
The equation of a line	98, 99
The coordinates of a line. Symbol of a line	99, 100
The equation of a point	100
The equation of a conic. The expression by a parameter	101—104
A particular form of the equation of a conic. Equation of tangent. Tangential equation of a conic. Conic referred to self-polar triad	104—107
Equation of a circle. Of circles of coaxial system. Condition that two lines or two circles should cut at right angles	107—110
Examples of the algebraic treatment of the theory of conics (Exx. 1—29)	110—152
Ex. 1. <i>Condition that a pair of points should be harmonic conjugates in regard to another pair</i>	110, 111
Ex. 2. <i>A result in regard to involutions</i>	111
Ex. 3. <i>Representation of all the pairs of an involution</i>	111
Ex. 4. <i>A general result for two pairs of points</i>	111
Ex. 5. <i>Condition that two circles, in general form, should cut at right angles</i>	111
Ex. 6. <i>The radical axis of two circles; the radical centre of three circles</i>	111, 112
Ex. 7. <i>The common tangents and centres of similitude of two circles</i>	112
Ex. 8. <i>Feuerbach's theorem. Apolar triads</i>	112—114
Ex. 9. <i>Continuation of the preceding example</i>	114—116
Ex. 10. <i>Hamilton's modification of Feuerbach's theorem. Introductory results</i>	116, 117
Ex. 11. <i>Hamilton's modification of Feuerbach's theorem</i>	117, 118
Ex. 12. <i>Conic referred to its centre and axes</i>	118, 119
Ex. 13. <i>Confocal conics</i>	119—121
Ex. 14. <i>Conics derived from a conic intersecting the joins of three points. Bipunctual conics</i>	121—123
Ex. 15. <i>Two triads reciprocal in regard to a conic are in perspective</i>	123, 124
Ex. 16. <i>The Hessian line of three points of a conic</i>	124
Ex. 17. <i>The director circle of a conic, and generalisations</i>	124—127.
Ex. 18. <i>Envelope of a line joining points of two conics which are conjugate in regard to a third, with particular applications</i>	127—134
Ex. 19. <i>Indication of a generalisation of the preceding results</i>	134
Ex. 20. <i>Of correspondences, in particular of general involutions, upon a conic</i>	134—139
Ex. 21. <i>Reciprocation of one conic into another when they have a common self-polar triad</i>	139, 140

Contents

xiii

	PAGES
Ex. 22. <i>The determination of the common self-polar triad of two conics in general</i>	140, 141
Ex. 23. <i>Fundamental invariants for two conics</i>	141, 142
Ex. 24. <i>The algebraical condition for one conic to be outpolar to another, or to be triangularly circumscribed to another</i>	142—146
Ex. 25. <i>Enumeration of reduced forms for the equations of any two conics</i>	146, 147
Ex. 26. <i>A particular involution of sets of three points. Apolar triads</i>	147—149
Ex. 27. <i>Three conics related in a particular way</i>	149—151
Ex. 28. <i>The director circle of a conic touching three lines is cut at right angles by a circle, for which the three lines form a self-polar triad</i>	151
Ex. 29. <i>A transformation of conics through four points into conics having double contact</i>	152

CHAPTER IV. RESTRICTION OF THE ALGEBRAIC SYMBOLS. THE DISTINCTION OF REAL AND IMAGINARY ELEMENTS

Review of the development of the argument. Real and imaginary symbols	153—155
The distinction of positive and negative for the real symbols	155, 156
Order of magnitude for the real symbols	156, 157
The roots of a polynomial	157
The exponential and logarithmic functions	157
A simple application to the double points of an involution. Reality of double points of involution defined by two pairs	157—160
General statement of the meaning of real and imaginary points in a plane	160, 161
Some applications of the definition. Imaginary points and lines	161, 162
Real and imaginary conics	162—164
The interior and exterior points of a real conic	164
The reality of the common points, and of the common self-polar triad, of two conics	164, 165
Example. Construction of a conic given three points and two pairs of conjugate points	165

CHAPTER V. PROPERTIES RELATIVE TO AN ABSOLUTE CONIC. THE NOTION OF DISTANCE. NON-EUCLIDEAN GEOMETRY

The interval of two points of a line relatively to two given points of the line	166, 167
Angular interval in regard to two absolute points	167
Interval in regard to an absolute conic. Meaning of movement	167—170

	PAGES
Case of an imaginary conic as Absolute	170—172
Application to a triangle, in particular a right-angled triangle	172—177
The sum of the angular intervals for a triangle	177
Formulae for a general triangle	177, 178
The case of a real conic as Absolute	178—180
The sum of the angles of a triangle and a quadrangle of three right angles	180, 181
Comparison with the Lobatschewsky-Bolyai geometry resumed	181—183
Case of degenerate conic as Absolute	183—186
The arbitrariness and comparison of the several methods of measurement. Extent between two lines, of a triangle, of a circle, in the elliptic geometry	186—189
Plane metrical geometry by projection from a quadric surface. Spherical geometry. Elementary properties of a quadric surface	189—195
Metrical geometry in regard to an absolute conic. Cayley's measure of distance deduced from Laguerre's	195—197
Representation of the original plane upon another plane. Riemann's space of constant curvature. Lobatschewsky lines as circles cutting a fixed line at right angles	197—204
Remark, comparison of the two cases considered	204
Examples of the preceding considerations (Exx. 1—9)	204—211
Ex. 1. <i>Delambre's formulae in spherical trigonometry</i>	204—206
Ex. 2. <i>Napier's analogies</i>	206
Ex. 3. <i>A particular triangle corresponding to the division of a sphere into four congruent triangles</i>	206—209
Ex. 4. <i>General form of the theorem for the sum of the focal distances of a point of a conic. Circular sections and focal lines of a cone</i>	209, 210
Exx. 5, 6. <i>Particular theorems of importance</i>	210
Exx. 7, 8. <i>Generalisation of result of Ex. 4</i>	210, 211
Ex. 9. <i>Elliptic integral for porism of two conics</i>	211

NOTE I. ON CERTAIN ELEMENTARY CONFIGURATIONS, AND
 ON THE COMPLETE FIGURE FOR PAPPUS' THEOREM

The simpler Desargues' figure	212, 213
The figure of three desmic tetrads of points in space of three dimensions	213, 214
The figure formed from three triads of points which are in perspective in pairs, with centres of perspective in line	214, 215
The complete figure for Pappus' theorem	215, 216
The figure formed from six points in four dimensions	216—218
Exx. 1—4. <i>The configuration in space of n dimensions, formed from $(n+2)$ points, or from $(n+2)$ spaces of $(n-1)$ dimensions; it is self-polar</i>	218

Contents

XV

NOTE II. ON THE *HEXAGRAMMUM MYSTICUM*
 OF PASCAL

	PAGES
Introduction, and arrangement of note	219, 220
Sylvester's syntheses from six elements. The combinatorial identity with the Pascal figure	220—224
The geometrical interpretation of the relations, in space of four dimensions (see also the Frontispiece of the volume). Statement and proof of the Pascal incidences	224—229
Direct deduction of Pascal's figure from a cubic surface with a node	229—231
Examples of particular properties of the figure (Exx. 1—6)	231—235
Ex. 1. <i>The pairs of conjugate Steiner planes</i>	231, 232
Ex. 2. <i>The Steiner planes and Steiner-Plücker lines</i>	232
Ex. 3. <i>The separation of the complete figure into six figures</i>	232
Ex. 4. <i>Tetrads of Steiner points each in threefold perspective with a tetrad of Kirkman points</i>	232, 233
Ex. 5. <i>Relation of the Cayley-Salmon lines and the Salmon planes</i>	233
Ex. 6. <i>Proof of the relations by plane geometry</i>	233—235
References to the literature of the matter	235, 236
 NOTE III. IN REGARD TO THE LITERATURE FOR NON-EUCLIDEAN GEOMETRY	 237
 NOTE IV. REMARKS AND CORRECTIONS OF VOLUME I	 238
 INDEX	 239—243