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Excerpt
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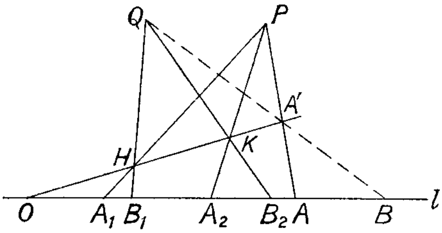
PRELIMINARY

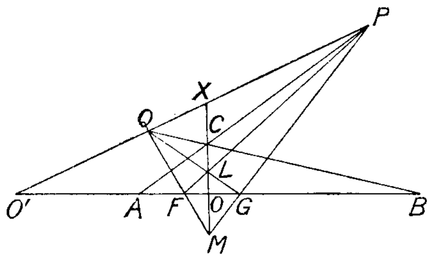
THERE are several matters, readily understood by the reader of Volume I, in regard to which we have not there entered into the detail which may be desirable for the purposes of the present volume.

Related ranges on the same line. With the purpose of avoiding the use of points whose existence could only be assumed after the consideration of the so-called imaginary points, we have (Vol. I, pp. 18, 25) defined two ranges on the same line as being *related* when, one of them is in perspective with a range on a second line which is related to the other range of the first line. From this definition we have shewn (Vol. I, p. 160) that in the abstract geometry two such related ranges on the same line have two corresponding points in common, though these may coincide. Assuming this, we may now formally prove that two such ranges also satisfy the general definition, namely that they are both in perspective with the same other range on another line, from different centres.



Let the ranges (a), (b), on the same line, l, be such that (a) is in perspective with a range (c), while (c) is related to (b). Let O be a point of the line l which corresponds to itself whether regarded as belonging to the range (a) or to the range (b); let A_1, A_2, A be other points of the range (a), respectively corresponding to the points B_1, B_2, B of the range (b). Let H, K be any two points in line with O; let A_1H, A_2K meet in P, and B_1H, B_2K meet in Q, and let PA meet the line OHK in A'. Then the range O, H, K, A' is in fact related to O, B_1, B_2, B. For the former is in perspective, from P, with the range O, A_1, A_2, A; this is, by hypothesis, in perspective with a range (c), which is itself related to O, B_1, B_2, B; so that the result follows from Vol. I, pp. 22–24. Thence, as the ranges, O, H, K, A' and O, B_1, B_2, B, have the point O in common, they are in perspective (Vol. I, p. 58, Ex. 2 (c)). Thus the line QB passes through A', and the two ranges (a), (b) are in perspective with the same range on the line OHK, respectively from P and Q. This is what we were to prove.

It is clear that the line PQ meets the line l in another common corresponding point of the two ranges (*a*), (*b*), which may however coincide with *O*.



Involution. Considering two related ranges upon the same line, it does not generally follow that, if *K* be the point of the second range corresponding to a point *H* of the first range, then to the point *K*, considered as belonging to the first range, there corresponds the point *H*, considered as belonging to the second. We proceed to shew, however, that, if this be true for one position, *F*, of *H*, and the corresponding position, *G*, of *K*, then it is true for every pair of corresponding points *H*, *K*.

For, in accordance with the preceding, let the ranges be in perspective, respectively from the points *P* and *Q*, with the same range; let *X* be the point of this range which lies on the line *PQ*, and *O'* the common corresponding point of the two ranges arising by perspective from *X*; let *L* be the point of this range which gives rise, respectively from *P* and *Q*, to the two particular corresponding points *F*, *G* of the two ranges, the points *P*, *L*, *F*, and also the points *Q*, *L*, *G*, being in line. Then, by hypothesis, the lines *PG*, *QF* meet in a point, *M*, of the line *XL*. Let *O* be the intersection of the line *XL* with the original line, so that *O* is also a common corresponding point of the two ranges.

It is clear from the construction that the points *O*, *O'* are harmonic conjugates in regard to *F* and *G*, and therefore do not coincide with one another; unless, indeed, they both coincide either with *F*, or with *G* (I, pp. 14, 119), in which case, as *O'* is a self-corresponding point, *G* would coincide with *F*, which we suppose not to be the case. Thus, when the two common corresponding points of the two given ranges are coincident, the case of two such different reciprocally related points *F* and *G* as we are now considering does not arise. Further, the points *O'*, *X* are harmonic conjugates in regard to *P* and *Q*.

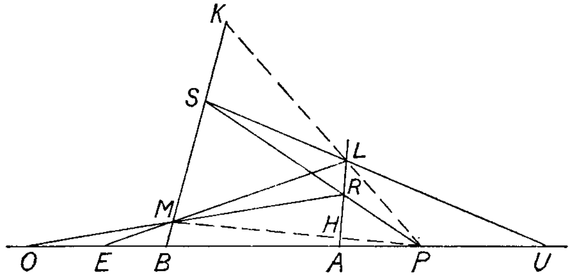
Now let *A*, *B* be any two other corresponding points, respectively of the two given ranges, arising from the point *C* of the line *XL*, by perspective from *P* and *Q*, respectively, so that *P*, *C*, *A*, and also *Q*, *C*, *B* are in line. Then, as *O'*, *X* are harmonic conjugates in regard to *P* and *Q*, it follows that *O'*, *O* are harmonic conjugates in regard to *A* and *B*. From this it follows that *PB* and *QA* meet on the line *XL*, and, therefore, that, to the point *B*, of the first range, corresponds the point *A*, of the second; as we desired to prove.

Conversely, any two points of the original line which are harmonic conjugates in regard to O' and O are a pair of reciprocally corresponding points of the two ranges. The aggregate of such pairs is called an *involution* of pairs of points; the points O', O are called the *double points* of the involution.

It is clear that if three pairs of points of a line, (A, B), (F, G) and (U, V), be pairs of an involution, then the range A, F, G, U, consisting of two points of one pair, and a point from each of the other two pairs, is related, point to point, to the range B, G, F, V, consisting of the respectively complementary points of the various pairs. For we can define two related ranges by the fact that the points A, F, U, of the one, correspond, respectively, to the points B, G, V, of the other; then, to the point G, of the former, corresponds the point F, of the latter. Conversely, if six points of a line be such that the range A, F, G, U is related, point to point, to the range B, G, F, V, then (A, B), (F, G) and (U, V) are three pairs of an involution. Also, an involution is established when two pairs are given; for, if these be (A, B) and (F, G), we have only to associate, to any point U, a point V for which the range B, G, F, V is related, point to point, to the range A, F, G, V.

Thus, further, a pair of points, O and O', exists, which are harmonic conjugates both in regard to one arbitrary pair of points, A, B, and also in regard to another arbitrary pair, F, G, which lie in the line AB. These points, O, O', are the double points of the involution determined by the pairs A, B and F, G.

From this it follows, also, that if there be two involutions of pairs of points upon the same line, there is a pair of points common to both involutions. This pair consists of the two points which are harmonic conjugates in regard to the double points of the first involution, and also harmonic conjugates in regard to the double points of the second involution.



The definition of an involution of pairs of points on a line may be approached differently, in connexion with a figure previously employed (Vol. I, pp. 76, 77).

Let A, B, O, U, E be arbitrary points of a line; in a plane through the line draw two lines AL, BM, met by a line through E respectively in L and M; let UL and BM meet in S, and OM meet AL in R. Let SR meet AB in P.

If, then, PL, PM, respectively, meet BM and AL in K and H, the range O, A, E, P is in perspective, from M, with the range R, A, L, H; this last is in perspective, from P, with the range S, B, K, M, and this, again, in perspective, from L, with the range U, B, P, E. As, then, the ranges O, A, E, P and U, B, P, E are proved to be related, it follows, from what is said above, that the pairs (O, U), (A, B) and (E, P) are in involution.

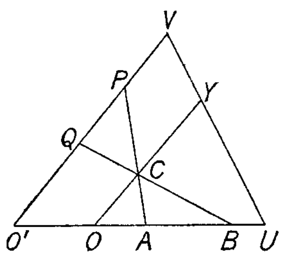
Thus the three pairs of joins of any four points of a plane (in this case M, R, L, S) meet an arbitrary line, of the plane, in three pairs of points of the same involution. As has been remarked in Vol. I (p. 181), the significance of the fact that five of these points determine the remaining one was noted by the Greeks.

Every result relating to pairs of points of a line corresponds, by the principle of duality, to a result relating to pairs of lines, in a plane, passing through a point. We may therefore consider pairs of lines, in a plane, through a point, forming a pencil in involution; these meet an arbitrary line of the plane, not passing through the centre of the pencil, in pairs of points of a range in involution. In particular, if four arbitrary lines be given in a plane, these, divided into two pairs in each of the three possible ways, determine, by the intersections of lines of a pair, three pairs of points. The lines which join an arbitrary point of the plane to these three pairs of points, are three pairs of lines belonging to the same involution.

Symbolical expression of the preceding results. We have in Vol. I (pp. 140, 154) reached the result that if a range of points represented by symbols $O + xU$, for different values of x, be related to a range of points represented by symbols $O + yU$, for different values of y, then there is a relation of the form

$$ayx + by + cx + d = 0,$$

wherein the symbols a, b, c, d are independent of x and y. We are, throughout, assuming Pappus' theorem, and there is no question of the commutativity of the multiplication of the symbols. It may be interesting, in the first place, to obtain this result by regarding the ranges as being in perspective, from different centres, with the same range on another line.



For this, let the ranges $(A\ldots)$, $(B\ldots)$, on a line $O'U$, be in perspective with a range $(C\ldots)$, on a line OY, respectively, from centres P and Q. Let the line PQ

meet $O'U$ in O'; let V be any point of this line, and let any line from V meet the line of the range $(C\ldots)$ in the point Y, and meet the line $O'U$ in the point U; let the line of the range $(C\ldots)$ meet the line $O'U$ in O. Then, regarding the points O', U, V as fundamental points of the plane, we may suppose, for the symbols of the points concerned, the following expressions

$$P = O' + pV, \quad Q = O' + qV, \quad O = O' + mU, \quad Y = U + V.$$

Thence, if we take for different points C,

$$C = O + cY,$$

with different values of c, that is

$$C = O' + mU + c(U + V), = O' + (m + c)U + cV,$$

we infer, for the points A, B, of the ranges on OU, which arise from C, respectively the symbols

$$(c^{-1} - p^{-1})O' + (c^{-1}m + 1)U, \quad (c^{-1} - q^{-1})O' + (c^{-1}m + 1)U;$$

if we write, then, $\quad A = O' + xU, \quad B = O' + yU,$

this gives
$$\frac{1 + mp^{-1}}{1 - mx^{-1}} = 1 + mc^{-1} = \frac{1 + mq^{-1}}{1 - my^{-1}}$$

and, hence, $\quad (p - q)xy + q(p + m)x - p(q + m)y = 0.$

This is of the form in question. It shews, putting $x = 0$, $y = 0$, that O' is a common corresponding point of the two ranges $(A\ldots)$ and $(B\ldots)$; and, putting $x = m$, $y = m$, that O is the other common corresponding point. It coincides with O' when $m = 0$.

Passing now to the condition for an involution: If, when A takes the present position of B, this takes the present position of A, we must have, beside the previous relation, also the relation

$$(p - q)xy + q(p + m)y - p(q + m)x = 0;$$

from these two relations we obtain, by subtraction,

$$[p(q + m) + q(p + m)][x - y] = 0,$$

and hence, A and B being any particular pair of corresponding points which do not coincide,

$$p(q + m) + q(p + m) = 0.$$

When this is satisfied the relation connecting the values of x and y, in general, is at once seen to reduce to

$$2xy - m(x + y) = 0.$$

We may, however, write

$$(x - y)O = (m - y)(O' + xU) - (m - x)(O' + yU),$$
$$= (m - y)A - (m - x)B;$$

thus the harmonic conjugate of O, in regard to A and B, is (Vol. I, p. 74) of symbol

$$(m - y)A + (m - x)B,$$

or
$$(2m - x - y)O' + [m(x + y) - 2xy]U,$$

and coincides with O' whatever pair of corresponding points A and B may be. Thus O and O' must be different, and the general pair of points of the involution is a pair of points harmonically conjugate in regard to these.

Examples of involution. *Ex.* 1. The condition that the pair of points represented by $O'+xU, O'+yU$, should be harmonic conjugates of one another in regard to the pair of points represented by $O'+aU$, $O'+bU$, being the condition that, for proper symbols p, q,

$$x=(pa+qb)/(p+q), \quad y=(pa-qb)/(p-q),$$

is
$$(x-a)(y-b)+(x-b)(y-a)=0,$$
or
$$xy-\tfrac{1}{2}(x+y)(a+b)+ab=0.$$

This remark, (a), gives the pair of points which are harmonic conjugates of one another in regard to each of two other given pairs of points. In particular, the pair of points harmonically conjugate both in regard to O', U, and in regard to $O'+aU$, $O'+bU$, is given by $O'+kU$, $O'-kU$, where $k^2=ab$. It also gives, (b), the relation for a pair of points belonging to a given involution, and, (c), shews that two given involutions, on the same line, have a common pair of points.

Ex. 2. When $O'+xU$, $O'+yU$, are, as above, corresponding points of two related ranges, of which O' is a common corresponding point, the relation connecting x and y,

$$(p-q)xy+q(p+m)x-p(q+m)y=0,$$

is capable of one of the two following forms:

$$\frac{y-m}{x-m}=\sigma\frac{y}{x}, \qquad \frac{1}{y}=\frac{1}{x}+\frac{1}{\lambda},$$

where
$$\sigma=\frac{p(q+m)}{q(p+m)}, \qquad \frac{1}{\lambda}=\frac{1}{p}-\frac{1}{q},$$

according as m is not zero, or m is zero. In general, the relation

$$ayx+by+cx+d=0,$$

if the equation
$$ax^2+(b+c)x+d=0,$$

for which $y=x$, have two different roots α, β, is capable of the form

$$\frac{y-\beta}{x-\beta}=\sigma\frac{y-\alpha}{x-\alpha},$$

where
$$\frac{(\sigma+1)^2}{\sigma}=\frac{(b-c)^2}{ad-bc};$$

but, if $\beta=\alpha$, or $(b+c)^2=4ad$, the relation is capable of the form

$$\frac{1}{y-\alpha}=\frac{1}{x-\alpha}+\frac{1}{\lambda},$$

where
$$\lambda=\tfrac{1}{2}(b-c)/a.$$

The general condition for a pair of points, represented by $O' + xU$, $O' + yU$, to belong to an involution is of the form

$$axy + b(x + y) + c = 0.$$

If we choose U for one of the two double points, we have $a = 0$. If then, also, O' be the other double point, we have also $c = 0$. If O' and U be any pair of the involution, we have $b = 0$.

In general, the double points are given by the values of x for which

$$ax^2 + 2bx + c = 0;$$

if these coincide ($b^2 = ac$), the involution reduces to one fixed point, for which $x = -b/a$, taken in turn with every other point of the line. This very degenerate case has been excluded from consideration in what has preceded.

Ex. 3. If P, Q and P', Q' be any two pairs of points of a line, and P'' be the harmonic conjugate of P in regard to P' and Q', while Q'' is the harmonic conjugate of Q in regard to P' and Q', the pairs (P, Q), (P', Q'), (P'', Q'') are in involution.

For, by construction, P, P'' and Q, Q'' are pairs of an involution with P', Q' as double points, and the harmonic ranges P, P'', P', Q' and Q'', Q, P', Q' are related; the latter, and, therefore, also the former, is related to the range Q, Q'', Q', P' (Vol. I, p. 25, Ex. 1). This shews that (P, Q), (P'', Q''), (P', Q') are pairs of an involution.

If the points P, Q correspond to the roots of $ax^2 + 2hx + b = 0$, or, say, $F = 0$, being given by $O + x_1 U$, $O + x_2 U$, where x_1, x_2 are the roots of this equation, and, similarly, with the same points of reference O, U, the points P', Q' correspond to the roots of $a'x^2 + 2h'x + b' = 0$, or, say, $F' = 0$, it may be shewn that P'', Q'' correspond, similarly, to the roots of the equation

$$(a'b' - h'^2)F - (ab' + a'b - 2hh')F' = 0.$$

Ex. 4. If A, A'; B, B'; ... be pairs of points of a line, which are in involution, and we take the harmonic conjugate of every one of these in regard to two points, U and V, of the line, so obtaining the pairs P, P'; Q, Q'; ..., then these are also pairs in involution.

For any range P, Q ... is then related to the corresponding range A, B ...; this in turn is related to the range A', B' ...; and this to P', Q' ....

Ex. 5. Given any two pairs of points of a line A, B and A', B', let the pair of points which are harmonic both in regard to A, B and in regard to A', B', be denoted by $(AB, A'B')$; and, therefore, if A'' and B'' be two other points of the line, let $\{(AB, A'B'), A''B''\}$ denote the pair harmonic both in regard to the pair $(AB, A'B')$ and in regard to the pair A'', B''. Shew that the three pairs of points

$$\{(AB, A'B'), A''B''\}, \quad \{(A'B', A''B''), AB\}, \quad \{(A''B'', AB), A'B'\}$$

are in involution. If A, B be given by $f=0$, where

$$f = ax^2 + 2hxy + by^2,$$

and A', B' be given by $f'=0$, where $f' = a'x^2 + 2h'xy + b'y^2$, shew that $(AB, A'B')$ are given by

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a, & h, & b \\ a', & h', & b' \end{vmatrix} = 0,$$

and $\{(AB, A'B'), A''B''\}$ are given by $Hf - H'f' = 0$, where, if $A''B''$ be similarly given by $f''=0$, the function H is $a'b'' - 2h'h'' + b'a''$, while $H' = ab'' - 2hh'' + ba''$.

A general abbreviated argument for relating two ranges. Consider two lines, which may coincide, or, more generally, may lie in space of any number of dimensions. Let O, U be two points of the former line, and O', U' be two points of the latter line; if the lines coincide O' and U' will each be in syzygy with O and U, the symbols of O' and U' being each expressible by O and U; if the lines intersect there will be one syzygy connecting the four points. Now, suppose that we have a geometrical construction whereby there is determined a definite point P', of the second line, corresponding to every point P of the first line, and a construction whereby we may conversely pass back from P' to the same point P. The construction may be such as gives, when we start from P, other points $A', B', \ldots$, of the second line, beside P', provided these remain the same for every position of P; and similarly for the construction by which we pass back from P' to P.

For greater definiteness we must also add that the construction must be analogous to those considered in the Third Section of Chap. I of Vol. I (p. 74); it must be such that, if the point P have the symbol $O + xU$, and the point P' the symbol $O' + x'U'$, then x' is determined from x by those laws of combination of the symbols which do not involve the solution of any equation of the second or higher order, the solution of such an equation not being without ambiguity; and x must be determined from x' in a similar way. There is therefore a single algebraic equation, connecting x and x', of the form $\Sigma a x'^m x^n = 0$, containing only a finite number of terms, in which every coefficient a is quite definite, m and n being positive integers. Then, as, to every value of x, there belongs only one value of x', other than those independent of x, the equation may be expected to be such that the highest value of the exponent m is 1; and, similarly, such that the highest value of n is also 1. (Cf. Chap. IV, below.) The relation would then be of the form

$$ax'x + bx' + cx + d = 0.$$

When we have such a determinate construction we shall, sometimes, assume, for the sake of brevity, that it is possible to find a third range with which both the given ranges are in perspective, from appropriate centres; so that the given ranges are related.

Of the distinction between the so-called real and imaginary points. In the general discussion introductory to Chap. III of Vol. I (p. 141), and in some other cases, we have spoken of the distinction between real and imaginary points in a way which, if definite when we approach the matter from the point of view of the Real Geometry, is not so clear from the point of view to which we desire to reach. It might be proper then to enter now into more detail. But we do in fact regard the distinction as arising in connexion with an arbitrary limitation of the possibilities of the points which can exist in the space considered; for this reason we postpone this discussion until (in Chap. IV, below) we definitely agree to make this limitation. This limitation is represented by a restriction in the system of symbols appropriate to the geometrical results obtained. While we wish to leave the logical possibilities as open as we can, we desire to expound a system of geometry in harmony with what is commonly accepted. For this purpose we have suggested, in Vol. I, that we may take as a geometrical postulate the possibility of what we have called Steiner's construction (Vol. I, pp. 155 ff.). In a similar way we shall in the first three chapters of the present volume adopt as a postulate the theorem that *two* curves in a plane which are *conics*, in the sense to be immediately explained, *have four common points, of which two, or more, may coincide.* It will be seen below (in Chap. IV) how this would be proved when the limitation referred to is adopted. Cf. pp. 19, 157 below.

CHAPTER I

GENERAL PROPERTIES OF CONICS

Definition of a Conic. Consider two pencils of lines, in the same plane, of centres A and C, and suppose these are related to one another, in the sense explained in Vol. I; to any line, AP, of one pencil, there corresponds, then, a definite line, CP, of the other, and, conversely, to any line, CP, there corresponds one line, AP. The curve which is the locus of the intersection, P, of such corresponding rays AP, CP, is called a conic section, or, briefly, a conic. Conversely, it will be seen that any conic can be so obtained, from any two points, A, C, of itself.

By its definition the curve contains one point, P, beside A, upon any line drawn through A; though, when the line drawn through A is that which corresponds to the line CA drawn through C, the point P coincides with A. Similarly for a line drawn through C. Consider however any line not passing through A nor C. This line contains two points of the conic. For the related pencils, of centres A and C, determine upon this line two related ranges. These have two common corresponding points. If these be P_1 and P_2, the ray AP_1, of the one pencil, corresponds to the ray CP_1, of the other, and P_1 is on the locus; and it is the same for P_2.

There is however one case, which we regard as exceptional, to which reference should be made: It may be that, to the ray AC of the pencil (A), there corresponds the ray CA of the pencil (C). When this is so, the locus consists of a line of the plane, together with the line AC itself. For if P_1 and P_2 be, then, any two points of the locus, not lying on the line AC, and the line P_1P_2 meet the line AC in B, the two related pencils (A), (C) determine, on the line P_1P_2, two related ranges having three common corresponding points, namely P_1, P_2 and B. These two ranges thus coincide, and every point of the line P_1P_2 is a point of the locus. If there were any point, P, of the plane, not lying on the line P_1P_2 or the line AC, which was upon the locus, then the join of P to any point of P_1P_2, would, by what has been said, be a line of which every point was a point of the locus; and in that case every point of the plane would be a point of the locus. The complete locus thus consists of the line P_1P_2, and of the line AC, of which every point evidently satisfies the definition. Conversely, any two lines of the plane may be regarded as constituting a degenerate conic, determined by