INTRODUCTORY

THE present is the first volume of a brief attempt to place the reader in touch with the main ideas dominant in contemporary geometry, excluding the consideration of higher algebraic irrationalities. For this purpose an account of many of the preliminary facts of geometry is indispensable, but in later stages a good deal of variety was possible in the selection, and no complete recital of geometrical theorems is aimed at. In several respects the views taken are not those usual in current textbooks; but it is believed that the system here suggested is logically complete, and does not require that long preliminary study of elementary geometry to which at present so much time is devoted. In particular it is desired to enter a protest against the custom of regarding so-called projective geometry as based upon metrical geometry; in the present account *distance*, as a primary conception, does not enter at all. And an attempt is made to include as soon as possible the indispensable ideas of geometry of more than three dimensions, and of geometry of so-called imaginary points. While the view is taken that all geometrical deduction should finally be synthetic, it is also held that to exclude algebraic symbolism would be analogous to preventing a physicist from testing his theories by experiment; and it becomes part of the task to justify the use of this symbolism.

It was impossible on such a plan to avoid discussing the logical foundations—and to this the present volume is devoted. Here difficulties of exposition arise from the desire to be brief, and not to be pedantic. It is not so much aimed at to give a faithful transcript of views elsewhere stated to be final, as to suggest a position which may appeal to the reader's logical sense as likely to be satisfying with further analysis. Logically, Chapter II, headed *Real Geometry*, should have come first, and should have been developed in greater detail. To the writer, after much experience as a teacher, that seemed unpractical. He has thought it better to state at once, in Chapter I, rather as working hypotheses, the greater part of the foundations on which the whole is to be based, reserving however, for Chapter III, an account of the grounds on which imaginary elements are introduced, as well as a more formal *résumé* of the logical position adopted. Thus Chapter II is a less abstract, but so far as the writer can judge the reader's view, a more toilsome analysis of fundamental conceptions than Chapter I. Incidentally, the particular geometrical propositions of Chapter I furnish a good deal of practice in the habit of geometrical construction in three dimensions, which, to the writer, appears logically prior to a detailed consideration of plane figures.

A Science grows up from the desire to bring the results of observation, of the relations of a class of facts which appear to be connected, under as few general propositions as possible. Into these propositions it is generally found necessary, or convenient, when the science has reached a sufficient development, to introduce abstract entities, transcending actual observation, whose existence is only asserted by the postulation of their mutual relations. If the science is to be arranged as a body of thought developed deductively, it is necessary to begin by formulating fundamental relations connecting *all* the entities which are to be discussed, from which other properties are to follow as a logical consequence. If this is done we may in the first instance regard all the entities involved in these fundamental propositions as being abstract, even those to which we attach names agreeing with the names borne by entities which we regard as subject to actual observation. The usefulness of the science, for the purpose for which it was undertaken, will depend on the agreement of the relations obtained for these latter entities with those which we can observe. It would seem that this process of substituting conceived entities, limited by supposed interrelations, for those which are regarded as objects of experience, belongs to every science. But it is clear that the degree of abstractness which may usefully and safely be applied is a matter for judgment and choice, conditioned by knowledge of the matter in hand, even in dealing with the same experiences. In geometry, as applied to the external world, we cannot but be conscious, for instance, in dealing with the points of a line, of the difference between those points which we regard as accessible and those which we regard as the others; and then, given two points of the line, of the difference between those points which we think of as between these, and those not between these; and then, finally, of the order in which several points of the line, which we can think of simultaneously, are arranged. None of these considerations however is taken account of in Chapter I; nor indeed do they enter finally into the abstract system of geometry which we set up. But it is to be expected that, if we take account of them, we may be able to analyse further some of the conceptions which we have adopted without analysis in Chapter I. It is to shew how this can be carried through that Chapter II is written. In particular this Chapter gives grounds on which the fundamental theorem for the correspondence of points on two straight lines may be based. The theory is built up in this Chapter with the use only of points supposed accessible and lying in a limited (unclosed) region, and there are some striking differences between the propositions and those of Chapter I; for instance it is not assumed that two lines in a plane necessarily have a point of intersection; and a line in a plane divides the points of the plane into two distinct sets. It is therefore of importance to shew that there is no inconsistency in afterwards disregarding these special results, as we do; this is effected by shewing that we may consider the limited space as a portion of a closed space in which the more general, if less definite, propositions of Chapter I are true. This step is of interest too as shewing how the detailed considerations which make up the usual elementary expositions of the so-called non-Euclidian geometry are superseded by the more general point of view. To some extent the comparison between the two points of view is analogous to the comparison between the theory of quadratic equations in which there may or may not be two roots, and the theory in which there are always two roots.

The writer is not of those who hold that the process of analysing the fundamental conceptions and setting out the sufficient axioms can ever be final; any such formulation, it would seem, must be subject to the possibility that other logical alternatives of consecutive thought may be revealed. Indeed the progress of science appears to consist in the very gradual unfolding of such alternative, or more embracing, conceptions. The history of Mathematics furnishes many examples, none more instructive and interesting than in the case of non-Euclidian geometry. But the instinct to such analysis is not less imperative than the instinct to all scientific thinking, and in geometry as in other subjects the attempt has been fruitful in a better understanding of the subject-matter. From those who would base upon the confession of the incompleteness of the analysis a refusal to enter upon it the writer would appeal to this instinct, which is more fundamental than formal logic.

Such a volume as this would be impossible save for the work and insight of a long line of other writers. In the Bibliography at the end an attempt is made to give the most necessary acknowledgments. In the text the view is sometimes taken that, in Mathematics, an indication may be more acceptable than an exposition; one remembers the words of Desargues (1636, *Oeuvres* I, 420) "la démonstration est...en notes sans discours pour le soulagement de la veuë et de l'esprit."

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CHAPTER I

ABSTRACT GEOMETRY

SECTION I. PROPOSITIONS OF INCIDENCE

Propositions of incidence in three dimensions. The entities with which we deal in the first instance we call by the names, *point*, *line*, and *plane*. These are any objects which are subject to the following laws of combination, which we call the Propositions of Incidence, together with another law, explained below (Sect. II). It is provisionally assumed that these laws are self-consistent and, when properly explained, are sufficient to enable the reader to form a clear impression whether any statement made in regard to these entities is a consequence, or not, of the fundamental laws. It is also very frequently assumed provisionally, when, in the course of a geometrical construction, two points are obtained by certain rules and it is desired to continue the construction with the help of the line joining these points, that these points do not coincide. In many cases it may be easy to shew that the coincidence of the points would involve an undesired limitation in the given points of the figure. But there may be other cases in which it would be consistent with the assumed fundamental propositions to assume either that the two points always coincide, or that they do not always coincide. In taking the latter alternative we should then be neglecting possibilities which, even if special, may quite well be worth examination. We adopt words which are in common use, but it is not to be assumed that in effect the meanings which we attach to these are co-extensive with the usual meanings. As is customary we frequently employ diagrams to fix ideas and clarify statements, representing a line by a mark made on the paper and a point by a dot thereon. Notwithstanding this formal and abstract method of statement, the whole object of the theory is of course, as has been stated, to deal with the ideas suggested by our ordinary experience; *it is this experience itself which has gradually suggested the abstract statement.* The Propositions may be stated as follows:

Through an arbitrary point there pass an infinite number of lines, of which one passes through any other arbitrary point. Thus a line is determined uniquely by any two points. The line contains an infinite number of points beside the two determining ones, and is determined by any two of these. Through any given line there pass an infinite number of planes, of which one passes through any point not lying on the given line. Thus a plane is determined uniquely by any three points which do not lie in line. The plane contains an infinite number of points beside the three determining ones, and is determined by any three of these which do not lie in line. The plane entirely contains the line which is determined by any two of its points. Thus if two lines have a common point, and we take two other points, one on each of the lines, the plane determined by the three points contains both the lines; that is, two lines with a common point determine a plane, containing both the lines. It is also true that two lines which are in the same plane have a common point. We have said that there is an infinite number of planes all containing the points of any given line; it is also true that any two planes have common points, lying on a line, or that any two planes intersect in a line. It will be seen below that this statement ceases to be true when we are not limiting ourselves to space of three dimensions. Further it is true (still in space of three dimensions), that any plane contains a point of any line (and contains all the points of the line if it contains two of these), or that a line and a plane intersect in one point. Thus any three planes, which have not in common all the points of a line, have a point in common, that namely where the line of intersection of two of the planes meets the third plane. If now we compare this with the earlier statement that three points not in line determine a plane, and compare the other statements with one another in pairs, as for instance the statement that two points determine a line with the statement that two planes determine a line, the statement that a line and a point determine a plane with the statement that a line and a plane have a point in common, the statement that two lines in a plane have a point in common with the statement that two lines intersecting in a point lie in a plane, we see that there is a complete descriptive correspondence between the relations of points, lines and planes on the one hand, and the relations of planes, lines and points on the other hand. Two propositions obtained one from the other by this interchange will be said to be *reciprocal*, or *dual*, or *correlative*. If we limit our consideration to the points and lines of a single plane it is at once seen that there is a similar reciprocity between propositions relating to points and lines and other propositions relating to lines and points. And a similar duality will be found to subsist in any space of higher dimensions.

We now at once deduce several immediate consequences of these Propositions of Incidence.

The transversal to two lines from a point. Given any two lines a, b, which do not intersect, or, as we shall often say, are *skew* to one another, and a point, P, which does not lie on either of these lines, a single line can be drawn from P to meet both a and b. For

P and a determine a plane, as do P and b; and P is among the points common to these planes, that is, it is on their line of intersection. But this line, being on the plane Pa, intersects the line a, and, being on the plane Pb, intersects the line b; it is thus such a line as is desired. Conversely, any line through P which meets the line a and b must be on both the planes Pa and Pb, and must therefore be the line we have found. The correlative proposition is that there lies in any plane not containing either of two given skew lines a line which meets both; the join namely of the points where the plane intersects the respective lines.

Desargues' theorem. If two triads of points, A, B, C and A', B', C', be such that the three joining lines AA', BB', CC' meet in one point, say O, then the two lines $BC, B'C'$ meet, say in the point P, and similarly the lines $CA, C'A'$ meet, say in Q, and the lines $AB, A'B'$ meet, say in R, and the three points P, Q, R are in line. Conversely, given two triads of points, A, B, C and A', B', C', such that the lines $BC, B'C'$ intersect, say in P, that the lines $CA, C'A'$ intersect, say in Q, and that the lines $AB, A'B'$ intersect, say in R, while P, Q, R are in line, then the lines AA', BB', CC' cointersect.

It is supposed that the points A, B, C are not in line, so that they determine a plane, and similarly that A', B', C' determine a plane. Take first the case when these planes are different. Suppose further that no one of the joins BC, CA, AB is in the plane $A'B'C'$, and no one of the joins $B'C', C'A', A'B'$ is in the plane ABC. Then the fact that the lines BB', CC' intersect involves that the points B, B', C, C' are in one plane, and hence that the lines $BC, B'C'$ meet. Their point of intersection, say P, lies then on BC which is in the plane ABC, and on $B'C'$ which is in the plane $A'B'C'$, so that P lies on the line of intersection of these two planes. The point, Q, of intersection of the lines $CA, C'A'$, similarly found, as well as the point, R, of intersection of the lines $AB, A'B'$, are equally on the line of intersection of the planes $ABC, A'B'C'$. So that the former statement made is clearly true. Conversely if the lines $BC, B'C'$ meet in the point P, while $CA, C'A'$ meet in Q and $AB, A'B'$ meet in R, then, still assuming the planes $ABC, A'B'C'$ to be different, and assuming that no one of the lines $BC, B'C'$, $CA, C'A', AB, A'B'$ is common to both planes, the points P, Q, R are evidently in the line of intersection of these planes. The fact that the lines $BC, B'C'$ intersect, involves that the lines BB' and CC' intersect, while similarly the line AA' intersects both of these. Now the lines BB', CC' determine a plane, and the line AA' does not lie in this plane, since we have assumed that the planes $ABC, A'B'C'$ are different; wherefore the line AA' meets this plane in a point and can only intersect both BB' and CC' by passing through their point of intersection. Thus the lines AA', BB', CC' meet in a point, as we desired to shew.

We have explicitly excluded, for the sake of simplicity, the possibility of one of the six lines BC, ... lying in both planes, which is easily seen to be unimportant. Incidentally we see that if three lines be such that every two of them have a common point, then the three lines lie all in one plane, or pass through one point.

Next suppose the two triads A, B, C and A', B', C' to be in one plane, it being assumed as before that A, B, C are not in line and A', B', C' are not in line. Then, first, let the lines AA', BB', CC' meet in one point, O. Draw through O any line not lying in the plane of the two triads, and let P, P' be any two points on this line. Then the intersecting lines $AA'O$ and $PP'O$ determine a plane, and the lines AP, $A'P'$, lying therein, intersect in a point, say A''. Similarly the lines BP, $B'P'$ meet in a point, say B'', and the lines CP, $C'P'$ meet in a point, say C''. As P is not in the plane of ABC, $A'B'C'$, it is clear that the points A'', B'', C'' are not in this plane; further A'', B'', C'' are not in line, since otherwise, as the plane containing this line and P would contain the points A, B, C, these would be in line; finally the plane which is thus determined by the points A'', B'', C'' does not contain, for instance, the line AB, for otherwise the plane of AB and $A''B''$, which contains P, would contain C'' and C, and A, B, C would be in line. Hence it follows from what is proved above, as the lines AA'', BB'', CC'' meet in P, that there is, in the plane of ABC and $A'B'C'$, a line, where this plane is met by the plane $A''B''C''$, and upon this line three points, say L, M and N, such that the lines BC, $B''C''$ meet in L, the lines CA, $C''A''$ meet in M and the lines AB, $A''B''$ meet in N. The point L is then the point where the plane of ABC and $A'B'C'$ is met by the line $B''C''$, and M, N are similarly the points where this plane is met by $C''A''$ and $A''B''$. By considering that the three lines $A'A''$, $B'B''$, $C'C''$ meet in the point P', we prove however, in the same way, that $B'C'$ passes through L, and that $C'A'$, $A'B'$ respectively pass through M and N. We have thus proved that if two triads of points ABC, $A'B'C'$, in the same plane, be such that the joining lines AA', BB', CC' meet in a point, then the three points of intersection $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ are in line. Conversely, for two triads of points ABC, $A'B'C'$ in the same plane, assume that the three points $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ are in line. Denote these points respectively by L, M, N. Draw through the line LMN a plane, other than the original plane of ABC, $A'B'C'$; draw through the points L, M, N, in this new plane, respectively the lines $LB''C''$, $MC''A''$, $NA''B''$, giving by their intersections a further triad A'', B'', C'' in this new plane. This new plane does not contain any one of the lines BC, CA, AB, $B'C'$,

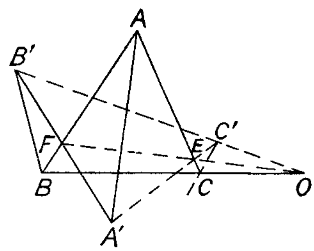
$C'A'$, $A'B'$, and it is supposed that the lines $LB''C''$, $MC''A''$, $NA''B''$ are all drawn so as not to lie in the original plane of ABC, $A'B'C'$. Then, as the lines BC, $B''C''$ meet in the point L, the lines CA, $C''A''$ meet in the point M, and the lines AB, $A''B''$ meet in the point N, it follows from what has preceded that the three lines AA'', BB'', CC'' meet in a point, say P, which does not lie in either the original plane or the new plane. It follows similarly, as the lines $B'C'$, $B''C''$ meet in the point L, and $C'A'$, $C''A''$ in M, and $A'B'$, $A''B''$ in N, that the lines $A'A''$, $B'B''$, $C'C''$ meet in a point, say P', not lying in either of the two planes described. Now let O be the point where the line PP' meets the original plane of ABC, $A'B'C'$. Then, since the lines AP, $A'P'$ intersect, in the point A'', and the points A, A', P, P' are not in line, these lines determine a plane, and therefore the line AA' intersects the line PP'. This intersection can only be at O, where the line PP' meets the original plane containing A and A'. Thus the line AA' passes through O, as, similarly, do also the lines BB' and CC'. We have thus proved that if two triads ABC, $A'B'C'$, of points in one plane, neither triad being in line, be such as to have three intersections (BC, $B'C'$), (CA, $C'A'$), (AB, $A'B'$) lying in line, then the three joining lines AA', BB', CC' meet in a point.

The aggregate of the four theorems now established will be referred to as Desargues' theorem. (See *Oeuvres de Desargues, par M. Poudra*, Paris, 1864, t. I, pp. 413, 430.)

Remarks in regard to Desargues' theorem. (*a*) The reader will notice that the theorem for two triads of points in one plane is proved by considerations involving the existence of points not lying in this plane. From the previously given Propositions of Incidence it is possible to select those which relate only to the geometry of points and lines lying in one plane. It is an interesting fact that the theorem of Desargues for two triads of points lying in this plane cannot be proved as a consequence only of the propositions of incidence relating to geometry in this plane—as we shall prove below (Chap. II). The propositions of incidence for one plane appear to furnish no criterion for deciding whether three constructed points lie in line; such criterion arises only when the line enters as the intersection of two planes. And they appear to furnish no criterion for deciding whether three constructed lines meet in one point; such criterion arises only when the point enters as the intersection of the plane with a line not lying therein.

(*b*) We have said that two lines intersect one another when they lie in one plane; thus they intersect one another when there are two points A, A' on one of the lines, a, and two points B, B' on the other line, b, such that the cross joins AB, $A'B'$ meet; for then the two lines a, b lie in the plane of these lines AB, $A'B'$. Let

F be the intersection of AB and $A'B'$; and let C be any other point not lying on either of the lines a, b. Desargues' theorem furnishes a construction for the line joining C to the point of intersection of the lines, a, b; and this construction depends only on the points A, A' and B, B' by which these lines may be supposed to be determined. For, on the given line BC, in the plane ABC, wherein A is also given, take a point O and join it to the given point F of the line AB. Let the joining line meet AC in the point E. The plane determined by the points O, B', F contains the points A' on $B'F$, and E on FO; thus the line $A'E$ intersects OB', say in C'. We now have two triads of points, ABC and $A'B'C'$, such that the lines BC, $B'C'$ meet in O, the lines CA, $C'A'$ meet in E, and the lines AB, $A'B'$ meet in F, while the points O, E, F are in line. It therefore follows that the line CC' passes through the intersection of the given lines AA' and BB'.



That this construction should be useful when the point of intersection of the lines AA', BB' is not accessible depends on the possibility of taking O in such a position upon BC that the two points E, C' should be accessible (see Chapter II). The construction is valid when C is in the plane of the two lines AA', BB'; when C is not in this plane the line from C to the intersection of these lines may also be defined as the line of intersection of the two planes CAA' and CBB'.

(*c*) If we have two tetrads of points, A, B, C, D and A', B', C', D', with the property that the four lines AA', BB', CC', DD' meet in one point O, it being supposed that neither tetrad consists of four points lying on a plane, then it can be shewn that every two corresponding joining lines in the two figures, such as BC and $B'C'$, intersect one another, and that the six points of intersection so arising are in one plane, and are the intersections in pairs of four lines of that plane.

In fact, the lines DA, $D'A'$, in the plane of the lines ODD', OAA', must intersect, say in P; so the lines DB, $D'B'$ must intersect, say in Q, and the lines DC, $D'C'$ must intersect, say in R. The points P, Q, R are then not in line, since the points D, A, B, C are not in a plane; and thus the points P, Q, R define a plane. Again, as the joins of corresponding points of the two triads DBC and $D'B'C'$, namely DD', BB' and CC', meet, in O, it follows from Desargues' theorem that the lines BC, $B'C'$ intersect in a point of the line QR, where Q is the point $(DB, D'B')$ and R is the point $(DC, D'C')$, say in X. Similarly the lines CA, $C'A'$ meet in a point, say Y, of the line RP, and AB, $A'B'$ meet in a point, say Z, of the line PQ. The points X, Y, Z are on the plane ABC, and are thus in line, the line of intersection of the planes ABC and PQR. This proves the result stated.

(*d*) Two figures, such as the triads ABC, $A'B'C'$ of Desargues' theorem, or the two tetrads just considered, are said *to be in perspective*, when to any point P of one figure there corresponds a point P' of the other figure, such that all lines, like PP', which join two corresponding points, pass through the same point.

We may notice that the figure which arises in the proof of Desargues' theorem for the case of two triads in one plane, consists of three triads, of which every two are in perspective, the three centres of perspective being in line; while each set of three corresponding joining lines, such as BC, $B'C'$, $B''C''$, is formed of three lines which meet in a point. In all there are fifteen points and twenty lines; each line contains three of the points, and through each point there pass four of the lines. If we denote the points P, P', A'', B'', C'' respectively by the binary symbols, each formed with two numbers, 04, 05, 01, 02, 03, then it is appropriate to denote the points A, B, C, A', B', C' respectively by the symbols 14, 24, 34, 15, 25, 35, the point of meeting of the three lines BC, $B'C'$, $B''C''$ by 23, the point of meeting of the three lines CA, $C'A'$, $C''A''$ by 31, and that of AB, $A'B'$, $A''B''$ by 12, using 45 for the point of meeting of AA', BB', CC'. The points are then denoted by all the combinations of six symbols two at a time. Each line is met by nine others, lying in threes in three planes; the three triads so formed, one in each of these planes, are two and two in perspective, the centres of perspective being in line.

The figure arising in the proof of Desargues' theorem for two triads not lying in the same plane is simpler. Here there are five planes, ten lines, consisting of the lines of intersection of the planes in pairs, and ten points, consisting of the intersections of the planes in threes.

Example. If two triads of points, A, B, C and A', B', C', be such that the lines AB', $A'B$ intersect, say in R, also AC', $A'C$ intersect, say in Q, while AB, $A'B'$ intersect in H, and BC, $B'C'$ intersect, in F, then prove that BC', $B'C$ intersect, say in P, and that AC, $A'C'$ intersect, say in G, and that each of the four sets FGH, FQR, GRP, HPQ consists of three points in line.

The fourth harmonic point. We come now to an important construction by which we pass from three given points of a line to another unique point of this line. For the construction itself it is assumed that a plane can be drawn through the line, and for the proof that the fourth point of the line which is constructed is unique it is assumed that two planes can be drawn through the