DIVISION I.

PRELIMINARY.

CHAPTER I.—KINEMATICS.

1. THE science which investigates the action of Force is called, by the most logical writers, DYNAMICS. It is commonly, but erroneously, called MECHANICS; a term employed by Newton in its true sense, the Science of Machines, and the art of making them.

2. Force is recognized as acting in two ways:

1° so as to compel rest or to prevent change of motion, and
2° so as to produce or to change motion.

Dynamics, therefore, is divided into two parts, which are conveniently called STATICS and KINETICS.

3. In Statics the action of force in maintaining rest, or preventing change of motion, the 'balancing of forces,' or Equilibrium, is investigated; in Kinetics, the action of force in producing or in changing motion.

4. In Kinetics it is not mere *motion* which is investigated, but the relation of *forces* to motion. The circumstances of mere motion, considered without reference to the bodies moved, or to the forces producing the motion, or to the forces called into action by the motion, constitute the subject of a branch of Pure Mathematics, which is called KINEMATICS, or, in its more practical branches, MECHANISM.

5. Observation and experiment have afforded us the means of translating, as it were, from Kinematics into Dynamics, and *vice versâ*. This is merely mentioned now in order to show the necessity for, and the value of, the preliminary matter we are about to introduce.

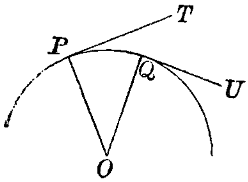
6. Thus it appears that there are many properties of motion, displacement, and deformation, which may be considered altogether independently of force, mass, chemical constitution, elasticity, temperature, magnetism, electricity; and that the preliminary consideration of such properties in the abstract is of very great use for Natural Philosophy. We devote to it, accordingly, the whole of this chapter;

which will form, as it were, the Geometry of the subject, embracing what can be observed or concluded with regard to actual motions, as long as the *cause* is not sought. In this category we shall first take up the free motion of a point, then the motion of a point attached to an inextensible cord, then the motions and displacements of rigid systems—and finally, the deformations of solid and fluid masses.

7. When a point moves from one position to another it must evidently describe a *continuous* line, which may be curved or straight, or even made up of portions of curved and straight lines meeting each other at any angles. If the motion be that of a *material particle*, however, there can be no abrupt change of velocity, nor of direction unless where the velocity is zero, since (as we shall afterwards see) such would imply the action of an *infinite* force. It is useful to consider at the outset various theorems connected with the geometrical notion of the path described by a moving point; and these we shall now take up, deferring the consideration of Velocity to a future section, as being more closely connected with physical ideas.

8. The *direction* of motion of a moving point is at each instant the tangent drawn to its path, if the path be a curve; or the path itself if a straight line. This is evident from the definition of the tangent to a curve.

9. If the path be not straight the direction of motion changes from point to point, and the *rate* of this change, per unit of length of the curve, is called the *Curvature*. To exemplify this, suppose two tangents PT, QU, drawn to a circle, and radii OP, OQ, to the points of contact. The angle between the tangents is the change of direction between P and Q, and the rate of change is to be measured by the relation between this angle and the length of the circular arc PQ. Now, if θ be the angle, s the arc, and r the radius, we see at once that (as the angle between the radii is equal to the angle between the tangents, and as the measure of an angle is the ratio of the arc to the radius, § 54)



$$r\theta = s,$$

and therefore $\frac{\theta}{s} = \frac{1}{r}$ is the measure of the curvature. Hence the curvature of a circle is inversely as its radius, and is measured, in terms of the proper unit of curvature, simply by the reciprocal of the radius.

10. Any small portion of a curve may be approximately taken as a circular arc, the approximation being closer and closer to the truth, as the assumed arc is smaller. The curvature at any point is the reciprocal of the radius of this circle for a small arc on each side of the point.

11. If all the points of the curve lie in one plane, it is called a *plane* *curve*, and if it be made up of portions of straight or curved lines it is called a *plane polygon*. If the line do not lie in one plane, we have in one case what is called a *curve of double curvature*, in the other a *gauche polygon*. The term 'curve of double curvature' is a very bad one, and, though in very general use, is, we hope, not ineradicable. The fact is, that there are not two curvatures, but only a curvature (as above defined) of which the plane is continuously changing, or twisting, round the tangent line. The course of such a curve is, in common language, well called 'tortuous;' and the measure of the corresponding property is conveniently called *Tortuosity*.

12. The nature of this will be best understood by considering the curve as a polygon whose sides are indefinitely small. Any two consecutive sides, of course, lie in a plane—and in that plane the curvature is measured as above; but in a curve which is not plane the third side of the polygon will not be in the same plane with the first two, and therefore the new plane in which the curvature is to be measured is different from the old one. The plane of the curvature on each side of any point of a tortuous curve is sometimes called the *Osculating Plane* of the curve at that point. As two successive positions of it contain the second side of the polygon above mentioned, it is evident that the osculating plane passes from one position to the next by revolving about the tangent to the curve.

13. Thus, as we proceed along such a curve, the curvature in general varies; and, at the same time, the plane in which the curvature lies is turning about the tangent to the curve. The rate of torsion, or the tortuosity, is therefore to be measured by the rate at which the osculating plane turns about the tangent, per unit length of the curve. The simplest illustration of a tortuous curve is the thread of a screw. Compare § 41 (*d*).

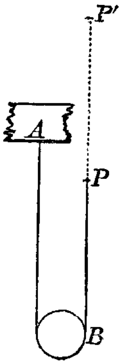
14. The *Integral Curvature*, or *whole change of direction*, of an arc of a plane curve, is the angle through which the tangent has turned as we pass from one extremity to the other. The *average curvature* of any portion is its whole curvature divided by its length. Suppose a line, drawn through any fixed point, to turn so as always to be parallel to the direction of motion of a point describing the curve: the angle through which this turns during the motion of the point exhibits what we have defined as the integral curvature. In estimating this, we must of course take the enlarged modern meaning of an angle, including angles greater than two right angles, and also negative angles. Thus the integral curvature of any closed curve or broken line, whether everywhere concave to the interior or not, is four right angles, provided it does not cut itself. That of a Lemniscate, 8, is *zero*. That of the Epicycloid  is eight right angles; and so on.

15. The definition in last section may evidently be extended to a plane polygon, and the integral change of direction, or the angle between the first and last sides, is then the sum of its exterior angles, all the sides being produced each in the direction in which the moving point describes it while passing round the figure. This is true whether the polygon be closed or not. If closed, then, as long as it is not crossed, this sum is four right angles,—an extension of the result in Euclid, where all *reëntrant* polygons are excluded. In the star-shaped figure , it is ten right angles, wanting the sum of the five acute angles of the figure; i. e. it is eight right angles.

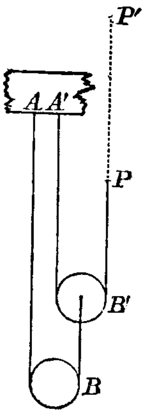
16. A chain, cord, or fine wire, or a fine fibre, filament, or hair, may suggest, what is not to be found among natural or artificial productions, a perfectly *flexible and inextensible line.* The elementary kinematics of this subject require no investigation. The mathematical condition to be expressed in any case of it is simply that the distance measured along the line from any one point to any other, remains constant, however the line be bent.

17. The use of a cord in mechanism presents us with many practical applications of this theory, which are in general extremely simple; although curious, and not always very easy, geometrical problems occur in connexion with it. We shall say nothing here about such cases as knots, knitting, weaving, etc., as being excessively difficult in their general development, and too simple in the ordinary cases to require explanation.

18. The simplest and most useful applications are to the *Pulley* and its combinations. In *theory* a pulley is simply a smooth body which *changes the direction* of a flexible and inextensible cord stretched across part of its surface; in *practice* (to escape as much as possible of the inevitable friction) it is a wheel, on part of whose circumference the cord is wrapped.



(1) Suppose we have a single pulley *B*, about which the flexible and inextensible cord *ABP* is wrapped, and suppose its free portions to be parallel. If (*A* being fixed) a point *P* of the cord be moved to *P′*, it is evident that each of the portions *AB* and *PB* will be shortened by one-half of *PP′*. Hence, when *P* moves through any space in the direction of the cord, the pulley *B* moves in the same direction, through half the space.



(2) If there be two cords and two pulleys, the ends *AA′* being fixed, and the other end of *AB* being attached to the pulley *B′*—then, if all free parts of the cord are parallel, when *P* is moved to *P′*, *B′* moves in the same direction through *half* the space, and carries with it one end of the cord *AB*. Hence *B* moves through half the space *B′* did, that is, *one fourth* of *PP′*.

(3) And so on for any number of pulleys, if they be arranged in the above manner. Similar considerations enable us to determine the relative motions of all parts of other systems of pulleys and cords as long as all the free parts of the cords are parallel.

Of course, if a pulley be *fixed*, the motion of a point of one end of the cord *to* or *from* it involves an equal motion of the other end *from* or *to* it.

If the strings be not parallel, the relations of a single pulley or of a system of pulleys are a little complex, but present no difficulty.

19. In the mechanical tracing of curves, a flexible and inextensible cord is often supposed. Thus, in drawing an ellipse, the focal property of the curve shows us that if we fix the ends of such a cord to the foci and keep it stretched by a pencil, the pencil will trace the curve.

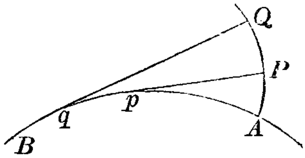
By a ruler moveable about one focus, and a string attached to a point in the ruler and to the other focus, and kept tight by a pencil sliding along the edge of the ruler, the hyperbola may be described by the help of its analogous focal property; and so on.

20. But the consideration of evolutes is of some importance in Natural Philosophy, especially in certain mechanical and optical questions, and we shall therefore devote a section or two to this application of Kinematics.

Def. If a flexible and inextensible string be fixed at one point of a plane curve, and stretched along the curve, and be then unwound in the plane of the curve, its extremity will describe an *Involute* of the curve. The original curve, considered with reference to the other, is called the *Evolute*.

21. It will be observed that we speak of *an* involute, and of *the* evolute, of a curve. In fact, as will be easily seen, a curve can have but one evolute, but it has an infinite number of involutes. For all that we have to do to vary an involute, is to change the point of the curve from which the tracing-point starts, or consider the involutes described by different points of the string; and these will, in general, be different curves. But the following section shows that there is but one evolute.

22. Let AB be any curve, PQ a portion of an involute, pP, qQ positions of the free part of the string. It will be seen at once that these must be tangents to the arc AB at p and q. Also the string at any stage, as pP, ultimately revolves about p. Hence pP is *normal* (or perpendicular to the tangent) to the curve PQ. And thus the evolute of PQ is a definite curve, viz. the envelop of (or line which is touched by) the normals drawn at every point of PQ, or, which is the same thing, the locus of the centres of the circles which have at each point the same tangent and curvature as the curve PQ. And we may merely mention, as an obvious result of the



mode of tracing, that the arc qp is equal to the difference of qQ and pP, or that the arc pA is equal to pP. Compare § 104.

23. The rate of motion of a point, or its *rate of change of position*, is called its *Velocity*. It is greater or less as the space passed over in a given time is greater or less: and it may be *uniform*, i.e. the same at every instant; or it may be *variable*.

Uniform velocity is measured by the space passed over in unit of time, and is, in general, expressed in feet or in metres per second; if very great, as in the case of light, it may be measured in miles per second. It is to be observed that *Time* is here used in the abstract sense of a uniformly-increasing quantity—what in the differential calculus is called an independent variable. Its physical definition is given in the next chapter.

24. Thus a point, which moves uniformly with velocity v, describes a space of v feet each second, and therefore vt feet in t seconds, t being any number whatever. Putting s for the space described in t seconds, we have

$$s = vt.$$

Thus with unit velocity a point describes unit of space in unit of time.

25. It is well to observe here, that since, by our formula, we have generally

$$v = \frac{s}{t},$$

and since nothing has been said as to the magnitudes of s and t, we may take these as small as we choose. Thus *we get the same result whether we derive v from the space described in a million seconds, or from that described in a millionth of a second.* This idea is very useful, as it makes our results intelligible when a variable velocity has to be measured, and we find ourselves obliged to approximate to its value (as in § 28) by considering the space described in an interval so short, that during its lapse the velocity does not sensibly alter in value.

26. When the point does not move uniformly, the velocity is variable, or different at different successive instants: but we define the *average* velocity during any time as the space described in that time, divided by the time; and, the less the interval is, the more nearly does the average velocity coincide with the actual velocity at any instant of the interval. Or again, we define the exact velocity at any instant as the space which the point would have described in one second, if for such a period it kept its velocity unchanged.

27. That there is at every instant a definite velocity for any moving point, is evident to all, and is matter of everyday conversation. Thus, a railway train, after starting, gradually increases its speed, and every one understands what is meant by saying that at a particular instant it moves at the rate of ten or of fifty miles an hour,—although, in the course of an hour, it may not have moved a mile altogether. We may suppose that, at any instant during the motion, the steam is so adjusted as to keep the train running for some time at a uniform velocity. This is the velocity which the train had at the instant in question. Without supposing any such definite adjustment of the driving-power to be made, we can evidently obtain an approximation to the velocity at a particular instant, by considering (§ 25) the motion for so short a time, that during that time the actual variation of speed may be small enough to be neglected.

28. In fact, if v be the velocity at either beginning or end, or at any instant, of an interval t, and s the space actually described in that interval; the equation $v = \frac{s}{t}$ (which expresses the definition of the average velocity, § 26) is more and more nearly true, as the velocity is more nearly uniform during the interval t; so that if we take the interval small enough the equation may be made as nearly exact as we choose. Thus the set of values—

Space described in one second,
Ten times the space described in the first tenth of a second,
A hundred „ „ „ hundredth „

and so on, give nearer and nearer approximations to the velocity at the beginning of the first second.

The whole foundation of Newton's differential calculus is, in fact, contained in the simple question, 'What is the rate at which the space described by a moving point increases?' i. e. What is the velocity of the moving point? Newton's notation for the velocity, i. e. the rate at which s increases, or the *fluxion* of s, is s. This notation is very convenient, as it saves the introduction of a second letter.

29. The preceding definition of velocity is equally applicable whether the point move in a straight or a curved line; but, since, in the latter case, the direction of motion continually changes, the mere amount of the velocity is not sufficient completely to describe the motion, and we must have in every such case additional data to thoroughly specify the motion.

In such cases as this the method most commonly employed, whether we deal with velocities, or (as we shall do farther on) with accelerations and forces, consists in studying, not the velocity, acceleration, or force, *directly*, but its resolved parts parallel to any three assumed directions at right angles to each other. Thus, for a train moving up an incline in a N.E. direction, we may have the whole velocity and the steepness of the incline given; or we may express the same ideas thus—the train is moving simultaneously northward, eastward, and upward—and the motion as to amount and direction will be completely known if we know separately the northward, eastward, and upward velocities—these being called the *components* of the whole velocity in the three mutually perpendicular directions N., E., and up.

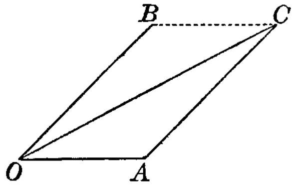
30. A velocity in any direction may be resolved in, and perpendicular to, any other direction. The first component is found by multiplying the velocity by the cosine of the angle between the two directions; the second by using as factor the sine of the same angle.

Thus a point moving with velocity V up an *Inclined Plane*, making an angle α with the horizon, has a vertical velocity $V \sin \alpha$ and a horizontal velocity $V \cos \alpha$.

Or it may be resolved into components in any three rectangular directions, each component being found by multiplying the whole velocity by the cosine of the angle between its direction and that of the component. The velocity resolved in any direction is the sum of the resolved parts (in that direction) of the three rectangular components of the whole velocity. And if we consider motion in one plane, this is still true, only we have but *two* rectangular components.

31. These propositions are virtually equivalent to the following obvious geometrical construction:—

To compound any two velocities as OA, OB in the figure; where OA, for instance, represents in magnitude and direction the space which would be described in one second by a point moving with the first of the given velocities—and similarly OB for the second; from A draw AC parallel and equal to OB. Join OC: then OC is the resultant velocity in magnitude and direction.



OC is evidently the diagonal of the parallelogram two of whose sides are OA, OB.

Hence the resultant of any two velocities as OA, AC, in the figure, is a velocity represented by the third side, OC, of the triangle OAC.

Hence if a point have, at the same time, velocities represented by OA, AC, and CO, the sides of a triangle *taken in the same order*, it is at rest.

Hence the resultant of velocities represented by the sides of any closed polygon whatever, whether in one plane or not, taken all in the same order, is zero.

Hence also the resultant of velocities represented by all the sides of a polygon but one, taken in order, is represented by that one taken in the opposite direction.

When there are two velocities, or three velocities, in two or in three rectangular directions, the resultant is the square root of the sum of their squares; and the cosines of its inclination to the given directions are the ratios of the components to the resultant.

32. The velocity of a point is said to be accelerated or retarded according as it increases or diminishes, but the word *acceleration* is generally used in either sense, on the understanding that we may regard its quantity as either positive or negative: and (§ 34) is farther generalized so as to include change of direction as well as change of speed. Acceleration of velocity may of course be either uniform or variable. It is said to be uniform when the point receives equal increments of velocity in equal times, and is then measured by the actual increase of velocity per unit of time. If we choose as the unit of acceleration that which adds a unit of velocity per unit of time to the velocity of a point, an acceleration measured by a will add a units of velocity in unit of time—and, therefore, at units of velocity in t units of time. Hence if v be the change in the velocity during the interval t,

$$v=at, \text{ or } a=\frac{v}{t}.$$

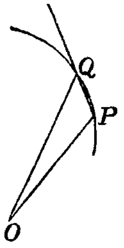
33. Acceleration is variable when the point's velocity does not receive equal increments in successive equal periods of time. It is then measured by the increment of velocity, which would have been generated in a unit of time had the acceleration remained throughout that unit the same as at its commencement. The *average* acceleration during any time is the whole velocity gained during that time, divided by the time. In Newton's notation $\dot{v}$ is used to express the acceleration in the direction of motion; and, if $v=\dot{s}$ as in § 28, we have $$a=\dot{v}=\ddot{s}.$$

34. But there is *another* form in which acceleration may manifest itself. Even if a point's velocity remain unchanged, yet if its *direction* of motion change, the resolved parts of its velocity in fixed directions will, in general, be accelerated. And as acceleration is merely a change of the component velocity in a stated direction, it is evident that its laws of composition and resolution are the same as those of velocity.

We therefore *expand* the definition just given, thus:—Acceleration is the *rate of change of velocity whether that change take place in the direction of motion or not.*

35. What is meant by change of velocity is evident from § 31. For if a velocity OA become OC, its change is AC, or OB.

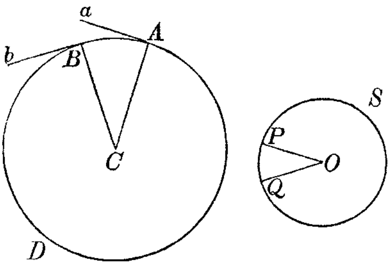
Hence, just as the direction of motion of a point is the tangent to its path, so the direction of acceleration of a moving point is to be found by the following construction:—



From any point O draw lines OP, OQ, etc., representing in magnitude and direction the velocity of the moving point at every instant. (Compare § 49.) The points, P, Q, etc., must form a continuous curve, for (§ 7) OP cannot change *abruptly* in direction. Now if Q be a point near to P, OP and OQ represent two successive values of the velocity. Hence PQ is the whole change of velocity during the interval. As the interval becomes smaller, the direction PQ more and more nearly becomes the tangent at P. Hence the direction of acceleration is that of the tangent to the curve thus described.

The amount of acceleration is the rate of change of velocity, and is therefore measured by the velocity of P in the curve PQ.

36. Let a point describe a circle, ABD, radius R, with uniform velocity V. Then, to determine the direction of acceleration, we must draw, as below, from a fixed point O, lines OP, OQ, etc., representing the velocity at A, B, etc., in direction and magnitude. Since the velocity in ABD is constant, all the lines OP, OQ, etc., will be equal (to V), and therefore PQS is a circle whose centre is O. The direction of acceleration at A is parallel to the tangent at P, that is, is perpendicular to OP, i.e. to Aa, and is therefore that of the radius AC.



Now P describes the circle PQS, while A describes ABD. Hence the velocity of P is to that of A as OP to CA, i.e. as V to R; and is therefore equal to

$$\frac{V}{R}.V \text{ or } \frac{V^2}{R},$$

and this (§ 35) is the amount of the acceleration in the circular path ABD.

37. The whole acceleration in any direction is the sum of the components (in that direction) of the accelerations parallel to any three rectangular axes—each component acceleration being found by the same rule as component velocities, that is, by multiplying by the cosine of the angle between the direction of the acceleration and the line along which it is to be resolved.

38. When a point moves in a curve the whole acceleration may be resolved into two parts, one in the direction of the motion and equal to the acceleration of the velocity; the other towards the centre of curvature (perpendicular therefore to the direction of motion), whose magnitude is proportional to the square of the velocity and also to the curvature of the path. The former of these changes the velocity, the other affects only the form of the path, or the direction of motion. Hence if a moving point be subject to an acceleration, constant or not, whose direction is continually perpendicular to the direction of motion, the velocity will not be altered—and the only effect of the acceleration will be to make the point move in a curve whose curvature is proportional to the acceleration at each instant, and inversely as the square of the velocity.

39. In other words, if a point move in a curve, whether with a uniform or a varying velocity, its change of direction is to be regarded as constituting an acceleration towards the centre of curvature, equal in amount to the square of the velocity divided by the radius of curvature. The whole acceleration will, in every case, be the resultant of the acceleration thus measuring change of direction and the acceleration of actual velocity along the curve.

40. If for any case of motion of a point we have given the whole velocity and its direction, or simply the components of the velocity in three rectangular directions, at any *time*, or, as is most commonly