ELECTRICITY AND MAGNETISM.

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PRELIMINARY.

ON THE MEASUREMENT OF QUANTITIES.

ERRATA.

Vol. I. Page 335, *dele last* 14 *lines.*

„ 336, line 1, *dele* therefore.

„ 336, line 2, *for* the potential at C to exceed that at D by P, *read* a current, C, from X to Y.

„ 336, line 4, *for* C to D will cause the potential at A to exceed that at B by the same quantity P. *read* X to Y will cause an equal current C from A to B.

Sometimes, however, we find several units of the same kind founded on independent considerations. Thus the gallon, or the volume of ten pounds of water, is used as a unit of capacity as well as the cubic foot. The gallon may be a convenient measure in some cases, but it is not a systematic one, since its numerical relation to the cubic foot is not a round integral number.

2.] In framing a mathematical system we suppose the fundamental units of length, time, and mass to be given, and deduce all the derivative units from these by the simplest attainable definitions.

The formulae at which we arrive must be such that a person

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1.] EVERY expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity.

There must be as many different units as there are different kinds of quantities to be measured, but in all dynamical sciences it is possible to define these units in terms of the three fundamental units of Length, Time, and Mass. Thus the units of area and of volume are defined respectively as the square and the cube whose sides are the unit of length.

Sometimes, however, we find several units of the same kind founded on independent considerations. Thus the gallon, or the volume of ten pounds of water, is used as a unit of capacity as well as the cubic foot. The gallon may be a convenient measure in some cases, but it is not a systematic one, since its numerical relation to the cubic foot is not a round integral number.

2.] In framing a mathematical system we suppose the fundamental units of length, time, and mass to be given, and deduce all the derivative units from these by the simplest attainable definitions.

The formulae at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result.

Hence, in all scientific studies it is of the greatest importance to employ units belonging to a properly defined system, and to know the relations of these units to the fundamental units, so that we may be able at once to transform our results from one system to another.

This is most conveniently done by ascertaining the *dimensions* of every unit in terms of the three fundamental units. When a given unit varies as the *n*th power of one of these units, it is said to be of *n dimensions* as regards that unit.

For instance, the scientific unit of volume is always the cube whose side is the unit of length. If the unit of length varies, the unit of volume will vary as its third power, and the unit of volume is said to be of three dimensions with respect to the unit of length.

A knowledge of the dimensions of units furnishes a test which ought to be applied to the equations resulting from any lengthened investigation. The dimensions of every term of such an equation, with respect to each of the three fundamental units, must be the same. If not, the equation is absurd, and contains some error, as its interpretation would be different according to the arbitrary system of units which we adopt *.

The Three Fundamental Units.

3.] (1) *Length.* The standard of length for scientific purposes in this country is one foot, which is the third part of the standard yard preserved in the Exchequer Chambers.

In France, and other countries which have adopted the metric system, it is the mètre. The mètre is theoretically the ten millionth part of the length of a meridian of the earth measured from the pole to the equator; but practically it is the length of a standard preserved in Paris, which was constructed by Borda to correspond, when at the temperature of melting ice, with the value of the preceding length as measured by Delambre. The mètre has not been altered to correspond with new and more accurate measurements of the earth, but the arc of the meridian is estimated in terms of the original mètre.

* The theory of dimensions was first stated by Fourier, *Théorie de Chaleur*, § 160.

In astronomy the mean distance of the earth from the sun is sometimes taken as a unit of length.

In the present state of science the most universal standard of length which we could assume would be the wave length in vacuum of a particular kind of light, emitted by some widely diffused substance such as sodium, which has well-defined lines in its spectrum. Such a standard would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body.

In treating of the dimensions of units we shall call the unit of length $[L]$. If l is the numerical value of a length, it is understood to be expressed in terms of the concrete unit $[L]$, so that the actual length would be fully expressed by $l\,[L]$.

4.] (2) *Time.* The standard unit of time in all civilized countries is deduced from the time of rotation of the earth about its axis. The sidereal day, or the true period of rotation of the earth, can be ascertained with great exactness by the ordinary observations of astronomers; and the mean solar day can be deduced from this by our knowledge of the length of the year.

The unit of time adopted in all physical researches is one second of mean solar time.

In astronomy a year is sometimes used as a unit of time. A more universal unit of time might be found by taking the periodic time of vibration of the particular kind of light whose wave length is the unit of length.

We shall call the concrete unit of time $[T]$, and the numerical measure of time t.

5.] (3) *Mass.* The standard unit of mass is in this country the avoirdupois pound preserved in the Exchequer Chambers. The grain, which is often used as a unit, is defined to be the 7000th part of this pound.

In the metrical system it is the gramme, which is theoretically the mass of a cubic centimètre of distilled water at standard temperature and pressure, but practically it is the thousandth part of a standard kilogramme preserved in Paris.

The accuracy with which the masses of bodies can be compared by weighing is far greater than that hitherto attained in the measurement of lengths, so that all masses ought, if possible, to be compared directly with the standard, and not deduced from experiments on water.

In descriptive astronomy the mass of the sun or that of the earth is sometimes taken as a unit, but in the dynamical theory of astronomy the unit of mass is deduced from the units of time and length, combined with the fact of universal gravitation. The astronomical unit of mass is that mass which attracts another body placed at the unit of distance so as to produce in that body the unit of acceleration.

In framing a universal system of units we may either deduce the unit of mass in this way from those of length and time already defined, and this we can do to a rough approximation in the present state of science; or, if we expect * soon to be able to determine the mass of a single molecule of a standard substance, we may wait for this determination before fixing a universal standard of mass.

We shall denote the concrete unit of mass by the symbol $[M]$ in treating of the dimensions of other units. The unit of mass will be taken as one of the three fundamental units. When, as in the French system, a particular substance, water, is taken as a standard of density, then the unit of mass is no longer independent, but varies as the unit of volume, or as $[L^3]$.

If, as in the astronomical system, the unit of mass is defined with respect to its attractive power, the dimensions of $[M]$ are $[L^3 T^{-2}]$.

For the acceleration due to the attraction of a mass m at a distance r is by the Newtonian Law $\frac{m}{r^2}$. Suppose this attraction to act for a very small time t on a body originally at rest, and to cause it to describe a space s, then by the formula of Galileo,

$$s = \tfrac{1}{2} f t^2 = \tfrac{1}{2} \frac{m}{r^2} t^2;$$

whence $m = 2\frac{r^2 s}{t^2}$. Since r and s are both lengths, and t is a time, this equation cannot be true unless the dimensions of m are $[L^3 T^{-2}]$. The same can be shewn from any astronomical equation in which the mass of a body appears in some but not in all of the terms †.

* See Prof. J. Loschmidt, 'Zur Grösse der Luftmolecule,' *Academy of Vienna*, Oct. 12, 1865; G. J. Stoney on 'The Internal Motions of Gases,' *Phil. Mag.*, Aug. 1868; and Sir W. Thomson on 'The Size of Atoms,' *Nature*, March 31, 1870.

† If a foot and a second are taken as units, the astronomical unit of mass would be about 932,000,000 pounds.

Derived Units.

6.] The unit of Velocity is that velocity in which unit of length is described in unit of time. Its dimensions are $[LT^{-1}]$.

If we adopt the units of length and time derived from the vibrations of light, then the unit of velocity is the velocity of light.

The unit of Acceleration is that acceleration in which the velocity increases by unity in unit of time. Its dimensions are $[LT^{-2}]$.

The unit of Density is the density of a substance which contains unit of mass in unit of volume. Its dimensions are $[ML^{-3}]$.

The unit of Momentum is the momentum of unit of mass moving with unit of velocity. Its dimensions are $[MLT^{-1}]$.

The unit of Force is the force which produces unit of momentum in unit of time. Its dimensions are $[MLT^{-2}]$.

This is the absolute unit of force, and this definition of it is implied in every equation in Dynamics. Nevertheless, in many text books in which these equations are given, a different unit of force is adopted, namely, the weight of the national unit of mass; and then, in order to satisfy the equations, the national unit of mass is itself abandoned, and an artificial unit is adopted as the dynamical unit, equal to the national unit divided by the numerical value of the force of gravity at the place. In this way both the unit of force and the unit of mass are made to depend on the value of the force of gravity, which varies from place to place, so that statements involving these quantities are not complete without a knowledge of the force of gravity in the places where these statements were found to be true.

The abolition, for all scientific purposes, of this method of measuring forces is mainly due to the introduction of a general system of making observations of magnetic force in countries in which the force of gravity is different. All such forces are now measured according to the strictly dynamical method deduced from our definitions, and the numerical results are the same in whatever country the experiments are made.

The unit of Work is the work done by the unit of force acting through the unit of length measured in its own direction. Its dimensions are $[ML^2T^{-2}]$.

The Energy of a system, being its capacity of performing work, is measured by the work which the system is capable of performing by the expenditure of its whole energy.

The definitions of other quantities, and of the units to which they are referred, will be given when we require them.

In transforming the values of physical quantities determined in terms of one unit, so as to express them in terms of any other unit of the same kind, we have only to remember that every expression for the quantity consists of two factors, the unit and the numerical part which expresses how often the unit is to be taken. Hence the numerical part of the expression varies inversely as the magnitude of the unit, that is, inversely as the various powers of the fundamental units which are indicated by the dimensions of the derived unit.

On Physical Continuity and Discontinuity.

7.] A quantity is said to vary continuously when, if it passes from one value to another, it assumes all the intermediate values.

We may obtain the conception of continuity from a consideration of the continuous existence of a particle of matter in time and space. Such a particle cannot pass from one position to another without describing a continuous line in space, and the coordinates of its position must be continuous functions of the time.

In the so-called 'equation of continuity,' as given in treatises on Hydrodynamics, the fact expressed is that matter cannot appear in or disappear from an element of volume without passing in or out through the sides of that element.

A quantity is said to be a continuous function of its variables when, if the variables alter continuously, the quantity itself alters continuously.

Thus, if u is a function of x, and if, while x passes continuously from x_0 to x_1, u passes continuously from u_0 to u_1, but when x passes from x_1 to x_2, u passes from u_1' to u_2, u_1' being different from u_1, then u is said to have a discontinuity in its variation with respect to x for the value $x = x_1$, because it passes abruptly from u_1 to u_1' while x passes continuously through x_1.

If we consider the differential coefficient of u with respect to x for the value $x = x_1$ as the limit of the fraction

$$\frac{u_2 - u_0}{x_2 - x_0},$$

when x_2 and x_0 are both made to approach x_1 without limit, then, if x_0 and x_2 are always on opposite sides of x_1, the ultimate value of the numerator will be $u_1' - u_1$, and that of the denominator will be zero. If u is a quantity physically continuous, the discontinuity can exist only with respect to the particular variable x. We must in this case admit that it has an infinite differential coefficient when $x = x_1$. If u is not physically continuous, it cannot be differentiated at all.

It is possible in physical questions to get rid of the idea of discontinuity without sensibly altering the conditions of the case. If x_0 is a very little less than x_1, and x_2 a very little greater than x_1, then u_0 will be very nearly equal to u_1 and u_2 to u_1'. We may now suppose u to vary in any arbitrary but continuous manner from u_0 to u_2 between the limits x_0 and x_2. In many physical questions we may begin with a hypothesis of this kind, and then investigate the result when the values of x_0 and x_2 are made to approach that of x_1 and ultimately to reach it. The result will in most cases be independent of the arbitrary manner in which we have supposed u to vary between the limits.

Discontinuity of a Function of more than One Variable.

8.] If we suppose the values of all the variables except x to be constant, the discontinuity of the function will occur for particular values of x, and these will be connected with the values of the other variables by an equation which we may write

$$\phi = \phi(x, y, z, \&c.) = 0.$$

The discontinuity will occur when $\phi = 0$. When ϕ is positive the function will have the form $F_2(x, y, z, \&c.)$. When ϕ is negative it will have the form $F_1(x, y, z, \&c.)$. There need be no necessary relation between the forms F_1 and F_2.

To express this discontinuity in a mathematical form, let one of the variables, say x, be expressed as a function of ϕ and the other variables, and let F_1 and F_2 be expressed as functions of ϕ, y, z, &c. We may now express the general form of the function by any formula which is sensibly equal to F_2 when ϕ is positive, and to F_1 when ϕ is negative. Such a formula is the following—

$$F = \frac{F_1 + e^{n\phi} F_2}{1 + e^{n\phi}}.$$

As long as n is a finite quantity, however great, F will be a continuous function, but if we make n infinite F will be equal to F_2 when ϕ is positive, and equal to F_1 when ϕ is negative.

Discontinuity of the Derivatives of a Continuous Function.

The first derivatives of a continuous function may be discontinuous. Let the values of the variables for which the discontinuity of the derivatives occurs be connected by the equation

$$\phi = \phi(x, y, z \ldots) = 0,$$

and let F_1 and F_2 be expressed in terms of ϕ and $n-1$ other variables, say $(y, z \ldots)$.

Then, when ϕ is negative, F_1 is to be taken, and when ϕ is positive F_2 is to be taken, and, since F is itself continuous, when ϕ is zero, $F_1 = F_2$.

Hence, when ϕ is zero, the derivatives $\frac{dF_1}{d\phi}$ and $\frac{dF_2}{d\phi}$ may be different, but the derivatives with respect to any of the other variables, such as $\frac{dF_1}{dy}$ and $\frac{dF_2}{dy}$, must be the same. The discontinuity is therefore confined to the derivative with respect to ϕ, all the other derivatives being continuous.

Periodic and Multiple Functions.

9.] If u is a function of x such that its value is the same for x, $x+a$, $x+na$, and all values of x differing by a, u is called a periodic function of x, and a is called its period.

If x is considered as a function of u, then, for a given value of u, there must be an infinite series of values of x differing by multiples of a. In this case x is called a multiple function of u, and a is called its cyclic constant.

The differential coefficient $\frac{dx}{du}$ has only a finite number of values corresponding to a given value of u.

On the Relation of Physical Quantities to Directions in Space.

10.] In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of those coordinate axes which we usually employ in defining the positions of things. The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions, and the line joining two points is in like manner considered as the resultant of three lines.

But for many purposes in physical reasoning, as distinguished