350.

NOTE ON BESSEL'S FUNCTIONS AS APPLIED TO THE VIBRATIONS OF A CIRCULAR MEMBRANE.

[*Philosophical Magazine*, Vol. XXI. pp. 53—58, 1911.]

IT often happens that physical considerations point to analytical conclusions not yet formulated. The pure mathematician will admit that arguments of this kind are suggestive, while the physicist may regard them as conclusive.

The first question here to be touched upon relates to the dependence of the roots of the function $J_n(z)$ upon the order n, regarded as susceptible of continuous variation. It will be shown that each root increases continually with n.

Let us contemplate the transverse vibrations of a membrane fixed along the radii $\theta = 0$ and $\theta = \beta$ and also along the circular arc $r = 1$. A typical simple vibration is expressed by*

$$w = J_n(z_n^{(s)} r) \,.\, \sin n\theta \,.\, \cos(z_n^{(s)} t), \qquad \text{..................(1)}$$

where $z_n^{(s)}$ is a finite root of $J_n(z) = 0$, and $n = \pi/\beta$. Of these finite roots the lowest $z_n^{(1)}$ gives the principal vibration, *i.e.* the one without internal circular nodes. For the vibration corresponding to $z_n^{(s)}$ the number of internal nodal circles is $s - 1$.

As prescribed, the vibration (1) has no internal nodal diameter. It might be generalized by taking $n = \nu\pi/\beta$, where ν is an integer; but for our purpose nothing would be gained, since β is at disposal, and a suitable reduction of β comes to the same as the introduction of ν.

In tracing the effect of a diminishing β it may suffice to commence at $\beta = \pi$, or $n = 1$. The frequencies of vibration are then proportional to the roots of the function J_1. The reduction of β is supposed to be effected by

* *Theory of Sound*, §§ 205, 207.

increasing without limit the potential energy of the displacement (w) at every point of the small sector to be cut off. We may imagine suitable springs to be introduced whose stiffness is gradually increased, and that without limit. During this process every frequency originally finite must increase*, finally by an amount proportional to $d\beta$; and, as we know, no zero root can become finite. Thus before and after the change the finite roots correspond each to each, and every member of the latter series exceeds the corresponding member of the former.

As β continues to diminish this process goes on until when β reaches $\frac{1}{2}\pi$, n again becomes integral and equal to 2. We infer that every finite root of J_2 exceeds the corresponding finite root of J_1. In like manner every finite root of J_3 exceeds the corresponding root of J_2, and so on†.

I was led to consider this question by a remark of Gray and Mathews‡—"It seems probable that between every pair of successive real roots of J_n there is exactly one real root of J_{n+1}. It does not appear that this has been strictly proved; there must in any case be an odd number of roots in the interval." The property just established seems to allow the proof to be completed.

As regards the latter part of the statement, it may be considered to be a consequence of the well-known relation

$$J_{n+1}(z) = \frac{n}{z} J_n(z) - J_n'(z). \qquad \text{.....................(2)}$$

When J_n vanishes, J_{n+1} has the opposite sign to J_n', both these quantities being finite§. But at consecutive roots of J_n, J_n' must assume opposite signs, and so therefore must J_{n+1}. Accordingly the number of roots of J_{n+1} in the interval must be *odd*.

The theorem required then follows readily. For the first root of J_{n+1} must lie between the first and second roots of J_n. We have proved that it exceeds the first root. If it also exceeded the second root, the interval would be destitute of roots, contrary to what we have just seen. In like manner the second root of J_{n+1} lies between the second and third roots of J_n, and so on. The roots of J_{n+1} *separate* those of J_n||.

* *Loc. cit.* §§ 88, 92*a*.

† [1915. Similar arguments may be applied to tesseral spherical harmonics, proportional to $\cos s\phi$, where ϕ denotes longitude, of fixed order n and continuously variable s.]

‡ *Bessel's Functions*, 1895, p. 50.

§ If J_n, J_{n+1} could vanish together, the sequence formula, (8) below, would require that every succeeding order vanish also. This of course is impossible, if only because when n is great the lowest root of J_n is of order of magnitude n.

|| I have since found in Whittaker's *Modern Analysis*, § 152, another proof of this proposition, attributed to Gegenbauer (1897).

The physical argument may easily be extended to show in like manner that all the finite roots of $J_n'(z)$ increase continually with n. For this purpose it is only necessary to alter the boundary condition at $r=1$ so as to make $dw/dr=0$ instead of $w=0$. The only difference in (1) is that $z_n^{(s)}$ now denotes a root of $J_n'(z)=0$. Mechanically the membrane is fixed as before along $\theta=0$, $\theta=\beta$, but all points on the circular boundary are free to slide transversely. The required conclusion follows by the same argument as was applied to J_n.

It is also true that there must be at least one root of J'_{n+1} between any two consecutive roots of J_n', but this is not so easily proved as for the original functions. If we differentiate (2) with respect to z and then eliminate J_n between the equation so obtained and the general differential equation, viz.

$$J_n'' + \frac{1}{z} J_n' + \left(1 - \frac{n^2}{z^2}\right) J_n = 0, \quad \text{.....................(3)}$$

we find

$$\left(1 - \frac{n^2}{z^2}\right) J'_{n+1} + \frac{n}{z^3}(n^2 - 1 - z^2) J_n' + \left(1 - \frac{n^2+n}{z^2}\right) J_n'' = 0. \quad \text{...(4)}$$

In (4) we suppose that z is a root of J_n', so that $J_n'=0$. The argument then proceeds as before if we can assume that $z^2 - n^2$ and $z^2 - n(n+1)$ are both positive. Passing over this question for the moment, we notice that J_n'' and J'_{n+1} have opposite signs, and that both functions are finite. In fact if J_n'' and J_n' could vanish together, so also by (3) would J_n, and again by (2) J_{n+1}; and this we have already seen to be impossible.

At consecutive roots of J_n', J_n'' must have opposite signs, and therefore also J'_{n+1}. Accordingly there must be at least one root of J'_{n+1} between consecutive roots of J_n'. It follows as before that the roots of J'_{n+1} separate those of J_n'.

It remains to prove that z^2 necessarily exceeds $n(n+1)$. That z^2 exceeds n^2 is well known*, but this does not suffice. We can obtain what we require from a formula given in *Theory of Sound*, 2nd ed. § 339. If the finite roots taken in order be $z_1, z_2, \ldots z_s \ldots$, we may write

$$\log J_n'(z) = \text{const.} + (n-1)\log z + \Sigma \log(1 - z^2/z_s^2),$$

the summation including all finite values of z_s; or on differentiation with respect to z

$$\frac{J_n''(z)}{J_n'(z)} = \frac{n-1}{z} - \Sigma \frac{2z}{z_s^2 - z^2}.$$

This holds for all values of z. If we put $z=n$, we get

$$\Sigma \frac{2n}{z_s^2 - n^2} = 1, \quad \text{..............................(5)}$$

* Riemann's *Partielle Differentialgleichungen*; *Theory of Sound*, § 210.

since by (3)

$$J_n''(n) \div J_n'(n) = -n^{-1}.$$

In (5) all the denominators are positive. We deduce

$$\frac{z_1^2 - n^2}{2n} = 1 + \frac{z_1^2 - n^2}{z_2^2 - n^2} + \frac{z_1^2 - n^2}{z_3^2 - n^2} + \ldots > 1; \quad \ldots\ldots\ldots\ldots\ldots(6)$$

and therefore

$$z_1^2 > n^2 + 2n > n(n+1).$$

Our theorems are therefore proved.

If a closer approximation to z_1^2 is desired, it may be obtained by substituting on the right of (6) $2n$ for $z_1^2 - n^2$ in the numerators and neglecting n^2 in the denominators. Thus

$$\frac{z_1^2 - n^2}{2n} > 1 + 2n\,(z_2^{-2} + z_3^{-2} + \ldots)$$

$$> 1 + 2n\left\{z_1^{-2} + z_2^{-2} + z_3^{-3} + \ldots - \frac{1}{n(n+2)}\right\}.$$

Now, as is easily proved from the ascending series for J_n',

$$z_1^{-2} + z_2^{-2} + z_3^{-2} + \ldots = \frac{n+2}{4n(n+1)};$$

so that finally

$$z_1^2 > n^2 + 2n + \frac{n^3}{(n+1)(n+2)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

When n is very great, it will follow from (7) that $z_1^2 > n^2 + 3n$. However the approximation is not close, for the ultimate form is*

$$z_1^2 = n^2 + [1{\cdot}6130]\,n^{4/3}.$$

As has been mentioned, the sequence formula

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

prohibits the simultaneous evanescence of J_{n-1} and J_n, or of J_{n-1} and J_{n+1}. The question arises—can Bessel's functions whose orders (supposed integral) differ by more than 2 vanish simultaneously? If we change n into $n+1$ in (8) and then eliminate J_n, we get

$$\left\{\frac{4n(n+1)}{z^2} - 1\right\} J_{n+1} = J_{n-1} + \frac{2n}{z} J_{n+2}, \quad \ldots\ldots\ldots\ldots(9)$$

from which it appears that if J_{n-1} and J_{n+2} vanish simultaneously, then either $J_{n+1} = 0$, which is impossible, or $z^2 = 4n(n+1)$. Any common root of J_{n-1} and J_{n+2} must therefore be such that its square is an integer.

* *Phil. Mag.* Vol. xx. p. 1003, 1910, equation (8). [1913. A correction is here introduced. See Nicholson, *Phil. Mag.* Vol. xxv. p. 200, 1913.]

Pursuing the process, we find that if J_{n-1}, J_{n+3} have a common root z, then

$$(2n+1)\,z^2 = 4n\,(n+1)\,(n+2),$$

so that z^2 is rational. And however far we go, we find that the simultaneous evanescence of two Bessel's functions requires that the common root be such that z^2 satisfies an algebraic equation whose coefficients are integers, the degree of the equation rising with the difference in order of the functions. If, as seems probable, a root of a Bessel's function cannot satisfy an integral algebraic equation, it would follow that no two Bessel's functions have a common root. The question seems worthy of the attention of mathematicians.

351.

HYDRODYNAMICAL NOTES.

[*Philosophical Magazine*, Vol. XXI. pp. 177—195, 1911.]

Potential and Kinetic Energies of Wave Motion.—Waves moving into Shallower Water.—Concentrated Initial Disturbance with inclusion of Capillarity.—Periodic Waves in Deep Water advancing without change of Type.—Tide Races.—Rotational Fluid Motion in a Corner.—Steady Motion in a Corner of Viscous Fluid.

IN the problems here considered the fluid is regarded as incompressible, and the motion is supposed to take place in two dimensions.

Potential and Kinetic Energies of Wave Motion.

When there is no dispersion, the energy of a progressive wave of any form is half potential and half kinetic. Thus in the case of a long wave in shallow water, "if we suppose that initially the surface is displaced, but that the particles have no velocity, we shall evidently obtain (as in the case of sound) two equal waves travelling in opposite directions, whose total energies are equal, and together make up the potential energy of the original displacement. Now the elevation of the derived waves must be half of that of the original displacement, and accordingly the potential energies less in the ratio of 4 : 1. Since therefore the potential energy of each derived wave is one quarter, and the total energy one half that of the original displacement, it follows that in the derived wave the potential and kinetic energies are equal" *.

The assumption that the displacement in each derived wave, when separated, is similar to the original displacement fails when the medium is dispersive. The equality of the two kinds of energy in an infinite progressive train of simple waves may, however, be established as follows.

* "On Waves," *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 254.

Consider first an infinite series of simple stationary waves, of which the energy is at one moment wholly potential and [a quarter of] a period later wholly kinetic. If t denote the time and E the total energy, we may write

$$\text{K.E.} = E \sin^2 nt, \qquad \text{P.E.} = E \cos^2 nt.$$

Upon this superpose a similar system, displaced through a quarter wave-length in space and through a quarter period in time. For this, taken by itself, we should have

$$\text{K.E} = E \cos^2 nt, \qquad \text{P.E.} = E \sin^2 nt.$$

And, the vibrations being *conjugate*, the potential and kinetic energies of the combined motion may be found by simple addition of the components, and are accordingly independent of the time, and each equal to E. Now the resultant motion is a simple progressive train, of which the potential and kinetic energies are thus seen to be equal.

A similar argument is applicable to prove the equality of energies in the motion of a simple conical pendulum.

It is to be observed that the conclusion is in general limited to vibrations which are infinitely small.

Waves moving into Shallower Water.

The problem proposed is the passage of an infinite train of simple infinitesimal waves from deep water into water which shallows gradually in such a manner that there is no loss of energy by reflexion or otherwise. At any stage the whole energy, being the double of the potential energy, is proportional per unit length to the square of the height; and for motion in two dimensions the only remaining question for our purpose is what are to be regarded as corresponding lengths along the direction of propagation.

In the case of long waves, where the wave-length (λ) is long in comparison with the depth (l) of the water, corresponding parts are as the velocities of propagation (V), or since the periodic time (τ) is constant, as λ. Conservation of energy then requires that

$$(\text{height})^2 \times V = \text{constant}; \qquad \text{.......................(1)}$$

or since V varies as $l^{\frac{1}{2}}$, height varies as $l^{-\frac{1}{4}}$*.

But for a dispersive medium corresponding parts are not proportional to V, and the argument requires modification. A uniform regime being established, what we are to equate at two separated places where the waves are of different character is the *rate of propagation of energy* through these places. It is a general proposition that in any kind of waves the ratio of the energy propagated past a fixed point in unit time to that resident in unit

* *Loc. cit.* p. 255.

length is U, where U is the *group-velocity*, equal to $d\sigma/dk$, where $\sigma = 2\pi/\tau$, $k = 2\pi/\lambda$*. Hence in our problem we must take

$$\text{height varies as } U^{-\frac{1}{2}}, \quad \text{.........................(2)}$$

which includes the former result, since in a non-dispersive medium $U = V$.

For waves in water of depth l,

$$\sigma^2 = gk \tanh kl, \quad \text{.........................(3)}$$

whence $$2\sigma U/g = \tanh kl + kl\,(1 - \tanh^2 kl). \quad \text{.................(4)}$$

As the wave progresses, σ remains constant, (3) determines k in terms of l, and U follows from (4). If we write

$$\sigma^2 l/g = l', \quad \text{.........................(5)}$$

(3) becomes $$kl \,.\, \tanh kl = l', \quad \text{.........................(6)}$$

and (4) may be written

$$2\sigma U/g = kl + (l' - l'^2)/kl. \quad \text{.....................(7)}$$

By (6), (7) U is determined as a function of l' or by (5) of l.

If kl, and therefore l', is very great, $kl = l'$, and then by (7) if U_0 be the corresponding value of U,

$$2\sigma U_0/g = 1, \quad \text{.........................(8)}$$

and in general

$$U/U_0 = kl + (l' - l'^2)/kl. \quad \text{.........................(9)}$$

Equations (2), (5), (6), (9) may be regarded as giving the solution of the problem in terms of a known σ. It is perhaps more practical to replace σ in (5) by λ_0, the corresponding wave-length in a great depth. The relation between σ and λ_0 being $\sigma^2 = 2\pi g/\lambda_0$, we find in place of (5)

$$l' = 2\pi l/\lambda_0 = k_0 l. \quad \text{.........................(10)}$$

Starting in (10) from λ_0 and l we may obtain l', whence (6) gives kl, and (9) gives U/U_0. But in calculating results by means of tables of the hyperbolic functions it is more convenient to start from kl. We find

kl	l'	U/U_0	kl	l'	U/U_0
∞	kl	1·000	·6	·322	·964
10	kl	1·000	·5	·231	·855
5	4·999	1·001	·4	·152	·722
2	1·928	1·105	·3	·087	·566
1·5	1·358	1·176	·2	·039	·390
1·0	·762	1·182	·1	·010	·200
·8	·531	1·110	kl	$(kl)^2$	$2kl$
·7	·423	1·048	—	—	—

* *Proc. Lond. Math. Soc.* Vol. IX. 1877; *Scientific Papers*, Vol. I. p. 326.

It appears that U/U_0 does not differ much from unity between $l' = \cdot 23$ and $l' = \infty$, so that the shallowing of the water does not at first produce much effect upon the height of the waves. It must be remembered, however, that the wave-length is diminishing, so that waves, even though they do no more than maintain their height, grow *steeper*.

Concentrated Initial Disturbance with inclusion of Capillarity.

A simple approximate treatment of the general problem of initial linear disturbance is due to Kelvin*. We have for the elevation η at any point x and at any time t

$$\eta = \frac{1}{\pi}\int_0^\infty \cos kx \cos \sigma t \, dk$$

$$= \frac{1}{2\pi}\int_0^\infty \cos(kx - \sigma t)\, dk + \frac{1}{2\pi}\int_0^\infty \cos(kx + \sigma t)\, dk, \quad \text{......(1)}$$

in which σ is a function of k, determined by the character of the dispersive medium—expressing that the initial elevation ($t = 0$) is concentrated at the origin of x. When t is great, the angles whose cosines are to be integrated will in general vary rapidly with k, and the corresponding parts of the integral contribute little to the total result. The most important part of the range of integration is the neighbourhood of places where $kx \pm \sigma t$ is stationary with respect to k, *i.e.* where

$$x \pm t\frac{d\sigma}{dk} = 0. \quad \text{......(2)}$$

In the vast majority of practical applications $d\sigma/dk$ is positive, so that if x and t are also positive the second integral in (1) makes no sensible contribution. The result then depends upon the first integral, and only upon such parts of that as lie in the neighbourhood of the value, or values, of k which satisfy (2) taken with the lower sign. If k_1 be such a value, Kelvin shows that the corresponding term in η has an expression equivalent to

$$\eta = \frac{\cos(\sigma_1 t - k_1 x - \frac{1}{4}\pi)}{\sqrt{\{-2\pi t\, d^2\sigma/dk_1^2\}}}, \quad \text{......(3)}$$

σ_1 being the value of σ corresponding to k_1.

In the case of deep-water waves where $\sigma = \sqrt{(gk)}$, there is only one predominant value of k for given values of x and t, and (2) gives

$$k_1 = gt^2/4x^2, \qquad \sigma_1 = gt/2x, \quad \text{......(4)}$$

making

$$\sigma_1 t - k_1 x - \tfrac{1}{4}\pi = gt^2/4x - \tfrac{1}{4}\pi, \quad \text{......(5)}$$

and finally

$$\eta = \frac{g^{\frac{1}{2}} t}{2\pi^{\frac{1}{2}} x^{\frac{3}{2}}} \cos\left\{\frac{gt^2}{4x} - \frac{\pi}{4}\right\}, \quad \text{......(6)}$$

the well-known formula of Cauchy and Poisson.

* *Proc. Roy. Soc.* Vol. XLII. p. 80 (1887); *Math. and Phys. Papers*, Vol. IV. p. 303.

In the numerator of (3) σ_1 and k_1 are functions of x and t. If we inquire what change (Λ) in x with t constant alters the angle by 2π, we find

$$\Lambda\left\{k_1 + \left(x - t\frac{d\sigma}{dk_1}\right)\frac{dk_1}{dx}\right\} = 2\pi,$$

so that by (2) $\Lambda = 2\pi/k_1$, *i.e.* the effective wave-length Λ coincides with that of the predominant component in the original integral (1), and a like result holds for the periodic time*. Again, it follows from (2) that $k_1x - \sigma_1 t$ in (3) may be replaced by $\int k_1 dx$, as is exemplified in (4) and (6).

When the waves move under the influence of a capillary tension T in addition to gravity,

$$\sigma^2 = gk + Tk^3/\rho, \quad \text{..............................(7)}$$

ρ being the density, and for the wave-velocity (V)

$$V^2 = \sigma^2/k^2 = g/k + Tk/\rho, \quad \text{..........................(8)}$$

as first found by Kelvin. Under these circumstances V has a minimum value when

$$k^2 = g\rho/T. \quad \text{..................................(9)}$$

The group-velocity U is equal to $d\sigma/dk$, or to $d(kV)/dk$; so that when V has a minimum value, U and V coincide. Referring to this, Kelvin towards the close of his paper remarks "The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interesting, but time does not permit of its being included in the present communication."

A glance at the simplified form (3) shows, however, that the special case arises, not when V is a minimum (or maximum), but when U is so, since then $d^2\sigma/dk_1^2$ vanishes. As given by (3), η would become infinite—an indication that the approximation must be pursued. If $k = k_1 + \xi$, we have in general in the neighbourhood of k_1,

$$kx - \sigma t = k_1 x - \sigma_1 t + \left(x - t\frac{d\sigma}{dk_1}\right)\xi - \frac{t}{1\,.\,2}\frac{d^2\sigma}{dk_1^2}\xi^2 - \frac{t}{1\,.\,2\,.\,3}\frac{d^3\sigma}{dk_1^3}\xi^3. \quad \text{...(10)}$$

In the present case where the term in ξ^2 disappears, as well as that in ξ, we get in place of (3) when t is great

$$\eta = \frac{\cos(k_1x - \sigma_1 t)}{2\pi\,\{\tfrac{1}{6}t\,d^3\sigma/dk_1^3\}^{\frac{1}{3}}}\int_{-\infty}^{+\infty}\cos\alpha^3 . \, d\alpha, \quad \text{..............(11)}$$

varying as $t^{-\frac{1}{3}}$ instead of as $t^{-\frac{1}{2}}$.

The definite integral is included in the general form

$$\int_{-\infty}^{+\infty}\cos\alpha^m . \, d\alpha = \frac{2}{m}\Gamma\left(\frac{1}{m}\right)\cos\frac{\pi}{2m}, \quad \text{..................(12)}$$

* Cf. Green, *Proc. Roy. Soc. Ed.* Vol. XXIX. p. 445 (1909).