CHAPTER I

CONTINUITY

1. Continuous Variation.

In every problem of the Infinitesimal Calculus we have to deal with a number of magnitudes, or quantities, some of which may be constant, whilst others are regarded as variable, and (moreover) as admitting of *continuous* variation.

Thus in the applications to Geometry, the magnitudes in question may be lengths, angles, areas, volumes, &c.; in Dynamics they may be masses, times, velocities, forces, &c.

Algebraically, any such magnitude is represented by a letter, such as a or x, denoting the *ratio* which it bears to some standard or 'unit' magnitude of its own kind. This ratio may be integral, or fractional, or it may be 'incommensurable,' *i.e.* it may not admit of being exactly represented by any fraction whose numerator and denominator are finite integers. Its symbol will in any case be subject to the ordinary rules of Algebra.

A 'constant' magnitude, in any given process, is one which does not change its value. A magnitude to which, in the course of any given process, different values are assigned, is said to be 'variable.' The earlier letters $a, b, c, \ldots$ of the alphabet are generally used to denote constant, and the later letters $\ldots u, v, w, x, y, z$ to denote variable magnitudes.

Some kinds of magnitude, as for instance lengths, masses, densities, do not admit of variety of sign. Others, such as altitudes, rotations, velocities, may be either positive or negative. When we wish to designate the 'absolute' value of a magnitude of this latter class, without reference to sign, we enclose the representative symbol between two short vertical lines, thus

$$|x|, \ |\sin x|, \ \log|x|.$$

It is important to notice that, if a and b have the same sign,

$$|a+b| = |a| + |b|,$$

whilst, if they have opposite signs,

$$|a+b| < |a| + |b|.$$

The Infinitesimal Calculus had its origin in problems of Geometry, such as drawing tangents to curves, finding areas and lengths of curves, volumes of solids, and so on. It is therefore natural, and from the point of view of most applications even necessary, to adopt as a basis the geometrical notion of magnitude, with the various familiar assumptions, express or implied, which this involves.

A geometrical representation of any class of magnitudes is obtained by taking an unlimited straight line $X'X$, and in it a fixed origin O, and by measuring lengths OM proportional on any convenient scale to the various magnitudes considered. In the case of sign-less magnitudes (such as masses), these lengths are to be measured on one side only of O; in cases where there is a variety of sign, OM must be drawn to the right or left of O according as the magnitude to be represented is positive or negative. To each magnitude of the kind in question will then correspond a definite point M in the line $X'X$.

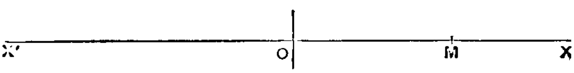

Fig. 1.

When we say that a magnitude admits of 'continuous' variation, we mean that the point M may occupy any position whatever in the line $X'X$ within (it may be) a certain range.

It will be observed that two things are postulated with respect to the magnitudes of the particular kind under consideration, viz. that every possible magnitude of the kind is represented by some point or other of the line $X'X$, and (conversely) that to every point on the line, within a certain range, there corresponds some magnitude of the kind. These conditions are fulfilled by all the kinds of magnitude with which we meet, either in Geometry, or in Mathematical Physics. It will be found on examination that these all involve in their specification a reference, direct or indirect, to linear magnitude. Thus an area may be represented by the altitude of an equivalent rectangle constructed on a given (unit) base; a velocity is represented by the length described in unit time, and so on.

2. Upper or Lower Limit of a Sequence.

The conception of a 'limit,' or 'limiting value,' occurs in various forms throughout the Calculus, and is of fundamental importance. In its primary form, now to be considered, it will be already more or less familiar to the student.

Suppose we have an endless ascending sequence of magnitudes of the same kind

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

i.e. each is greater than the preceding, so that the differences

$$x_2 - x_1,\ x_3 - x_2,\ \ldots,\ x_n - x_{n-1},\ \ldots$$

are all positive. Suppose, further, that the magnitudes (1) are known to be all less than some fixed finite quantity a. The sequence will in this case have an 'upper limit,' *i.e.* there will exist a certain quantity μ, greater than any one of the magnitudes (1), but such that if we proceed far enough in the sequence its members will ultimately exceed any assigned magnitude which is less than μ. In other words, it is impossible to interpose a barrier *between* the members of the sequence and the quantity μ.

In the geometrical representation the magnitudes (1) are represented by a sequence of points

$$M_1,\ M_2,\ M_3,\ \ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

each to the right of the preceding, but all lying to the left of some fixed point A. Hence every point on the line $X'X$, without exception, belongs to one or other of two mutually exclusive categories.

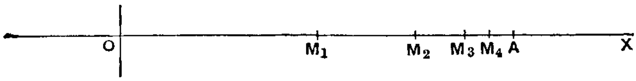

Fig. 2.

Either it has points of the sequence (2) to the right of it, or it has not. Moreover, every point in the former category lies to the left of every point of the latter. Hence there must be some point M, say, such that all points on the left of M belong to the former category and all points on the right of it to the latter. Hence if we put $\mu = OM$, μ fulfils the definition of an 'upper limit' above given.

In a similar manner we can shew that if we have an endless descending sequence of magnitudes

$$x_1,\ x_2,\ x_3,\ \ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

i.e. each is less than the preceding, so that the differences

$$x_1 - x_2,\ x_2 - x_3,\ \ldots,\ x_{n-1} - x_n$$

are all positive, whilst the magnitudes all exceed some finite quantity b, there will be a lower limit ν, such that every magnitude in the sequence is greater than ν, whilst the members of the sequence ultimately become less than any assigned magnitude which is greater than ν.

The above argument would evidently apply if, occasionally, two or more successive members of the sequence were equal. In that case the sequences are still usually styled 'ascending' or 'descending,' respectively, although the terms 'non-decreasing' and 'non-increasing' would be more accurately descriptive*.

Ex. 1. The sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \quad \text{......................(4)}$$

is ascending, with the upper limit 1. For

$$\frac{n}{n+1} = 1 - \frac{1}{n+1},$$

which can be made as nearly equal to 1 as we please by taking n great enough.

Ex. 2. If x be a positive quantity less than unity, the quantities

$$1, x, x^2, \ldots, x^n, \ldots \quad \text{........................(5)}$$

form a descending sequence, with the lower limit 0. For since $1/x$ is greater than unity we may write

$$1/x = 1 + y,$$

where y is positive. Then

$$(1/x)^n = (1+y)^n = 1 + ny + \ldots + y^n,$$

by the Binomial Theorem. Hence

$$1/x^n > 1 + ny,$$

and can therefore be made as great as we please by taking n great enough. It follows that x^n can be made as small as we please.

Ex. 3. Consider the sequence defined by

$$x_1 = 1, \quad x_{n+1} = \sqrt{(1 + x_n)}. \quad \text{....................(6)}$$

Since

$$x^2_{n+1} - x_n{}^2 = x_n - x_{n-1}, \quad \text{.......................(7)}$$

x_{n+1} will be greater than x_n if x_n is greater than x_{n-1}. But x_2 is obviously greater than x_1. The sequence is therefore an ascending one. Again

$$x_{n+1} = \frac{1 + x_n}{x_{n+1}} < \frac{1 + x_{n+1}}{x_{n+1}} < 1 + \frac{1}{x_{n+1}}. \quad \text{...............(8)}$$

Since $x_{n+1} > 1$ it follows that $x_{n+1} < 2$, for all values of n. The sequence has therefore an upper limit. Denoting this by μ, it appears from (6) that μ is the positive root of the equation

$$x^2 = x + 1. \quad \text{............................(9)}$$

* In recent times the term 'monotonic' has been invented to include both types of sequence.

By actual calculation from (6) the first few members are found to be, to four figures,

$$1,\ 1{\cdot}414,\ 1{\cdot}554,\ 1{\cdot}598,\ 1{\cdot}612,\ 1{\cdot}618.$$

The number last written is the accurate value of μ, to the degree of approximation aimed at.

The matter may be illustrated graphically by tracing the loci

$$y = x + 1, \quad y = x^2. \quad \ldots\ldots(10)$$

The figure shews how the successive values of x_n obtained from (6) converge towards the value of x at the intersection. A portion only of the graph is shewn.

Evidently, the same result is arrived at if we start with any positive value of x_1 instead of 1. Only, if x_1 is greater than the positive root of (9) the sequence would be a descending one.

This graphical method has a wide application to the numerical solution of equations, both algebraic and transcendental.

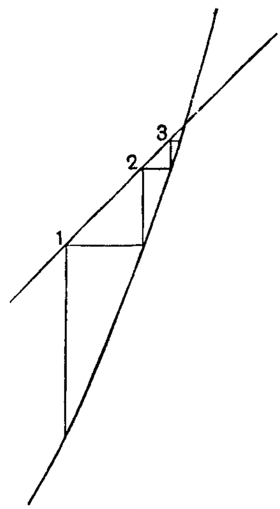

Fig. 3.

3. Application to Infinite Series. Series with positive terms.

The above has been called the fundamental theorem of the Calculus. An important illustration is furnished by the theory of infinite series whose terms are all of the same sign. In strictness, there is no such thing as the 'sum' of an infinite series of terms, since the operations indicated could never be completed, but under a certain condition the series may be taken as defining a particular magnitude.

Consider a series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

whose terms are all positive, and let

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \ \ldots\ldots\ s_n = u_1 + u_2 + \ldots + u_n. \ \ldots\ldots(2)$$

These quantities are called the 'partial sums.' If the sequence

$$s_1, \quad s_2, \ \ldots\ldots\ s_n, \ \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

has an upper limit S, the series (1) is said to be 'convergent,' and the quantity S is, by convention, called its 'sum.'

Again, if (1) be a series of positive terms which is known to be convergent, and if

$$u_1' + u_2' + u_3' + \ldots + u_n' + \ldots \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

be a series of positive terms which are respectively less than the corresponding terms in (1), *i.e.* $u_n' < u_n$ for all values of n, then (4) is also convergent. For if s_n' be the sum of the first n terms in (4), we have $s_n' < s_n$, and since the magnitudes s_n have by hypothesis an upper limit, the magnitudes s_n' will have one *à fortiori*.

Ex. 1. The series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

converges to the sum 2. For if in Fig. 2 (p 3) we make $OM_1 = 1$, $OA = 2$, and bisect M_1A in M_2, M_2A in M_3, and so on, the points M_1, M_2, M_3, ... will represent the magnitudes s_1, s_2, s_3, ... And since these points all lie to the left of A, whilst $M_nA = 1/2^{n-1}$ and can therefore be made as small as we please by taking n large enough, it appears that the sequence has the upper limit OA, $= 2$.

The case of any geometric progression whose common ratio is positive and less than unity may be illustrated in a similar manner.

Ex. 2. Consider the series

$$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \ldots + \frac{1}{n\,(n+1)} + \ldots.$$

If we write this in the form

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \ldots,$$

we see that

$$s_n = 1 - \frac{1}{n+1},$$

which has the upper limit 1.

Ex. 3. Further illustrations are supplied by every arithmetical process in which the digits of a non-terminating decimal are obtained in succession. For example, the ordinary process of extracting the square root of 2 gives the series

$$1{\cdot}414213\ldots$$

or

$$1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{3}{10^6} + \ldots.$$

Since s_n is always less than 1·5, there is an upper limit.

Ex. 4. The terms of the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

are (after the first three) respectively less than those of the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots.$$

The latter series is convergent and has the sum 3. Hence the former is also convergent, and its sum is less than 3.

4. Limiting Value in a Sequence.

Suppose that we have an endless series of magnitudes

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots \qquad \text{.....................(1)}$$

arranged in a definite order. Suppose, further, that whatever quantity ϵ we choose to fix upon, however small, there will always be a point in the sequence beyond which every member of it differs from some *fixed* quantity μ by a quantity less in absolute value than ϵ. The sequence is then said to be 'convergent,' and to have the 'limiting value' μ. Statements of this kind occur so frequently in the present subject that it is convenient to have a condensed expression for them. We write

$$\lim_{n\to\infty} x_n = \mu. \qquad \text{.......................(2)}$$

We have had particular cases of the above relation in the upper and lower limits discussed in Art. 2, but in the present wider definition it is not implied that the members of the sequence are arranged in order of magnitude, or that they are all greater or all less than the limiting value μ.

The hypothesis is that a value of n can be found such that the members of the sequence which follow x_n, viz.

$$x_{n+1},\ x_{n+2},\ x_{n+3},\ \ldots,$$

all lie between the values $\mu - \epsilon$ and $\mu + \epsilon$. The value of n which is necessary to secure the fulfilment of this condition will be greater the smaller the value of ϵ, but it is implied that, however small ϵ be taken, such a value exists.

Ex. 1. The sequence

$$\frac{1}{2},\ \frac{3}{2},\ \frac{2}{3},\ \frac{4}{3},\ \ldots,\ 1-\frac{1}{n},\ 1+\frac{1}{n},\ \ldots \qquad \text{...........(3)}$$

has obviously the limiting value 1.

Ex. 2. In the sequence

$$\sin x,\ \frac{\sin 2x}{2},\ \frac{\sin 3x}{3},\ \ldots,\ \frac{\sin nx}{n},\ \ldots \qquad \text{...........(4)}$$

the numerator lies always between ± 1, whilst the denominator increases indefinitely. The sequence has therefore the limiting value 0.

It is sometimes possible, as in the examples just given, to shew that a given sequence has a certain known quantity as its limit, and is therefore convergent. The question to be resolved is, however, in general less simple, and a criterion is required as to whether a proposed sequence has or has not a definite limiting value. There are in fact many important mathematical quantities which can only be defined as limits, and it is therefore necessary in such a case to satisfy ourselves that the limit exists.

It is obvious in the first place that if the sequence (1) has a limit, a value of n can always be found such that the members of the sequence which follow x_n, viz. $x_{n+1}, x_{n+2}, \ldots, x_{n+p}, \ldots$, will all differ from x_n by quantities not exceeding ϵ, where ϵ may be any assigned quantity, however small. Conversely, if this condition is fulfilled, the sequence has a definite limit.

To shew this let us construct in the first place a descending sequence of positive quantities $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ whose limit is 0. Such a sequence may be formed, for instance, by making each member one-half of the preceding one. By hypothesis, a number n_1 can be found such that all the members of the sequence which follow x_{n_1} lie between the values

$$x_{n_1} - \epsilon_1 \quad \text{and} \quad x_{n_1} + \epsilon_1,$$

and will therefore have a lower limit (α_1) and an upper limit (β_1), such that

$$\beta_1 - \alpha_1 \not> 2\epsilon_1.$$

Similarly, a number $n_2 (> n_1)$ can be found such that the members which follow x_{n_2} have a lower limit (α_2) and an upper limit (β_2), such that

$$\beta_2 - \alpha_2 \not> 2\epsilon_2,$$

and so on. The quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ form an ascending sequence, and, since they are all less than β_1, they have an upper limit μ, say. Similarly, the quantities $\beta_1, \beta_2, \beta_3$ form a descending sequence, with a lower limit ν. Moreover, since

$$\nu - \mu \leq \beta_p - \alpha_p \leq 2\epsilon_p,$$

which may be as small as we please, these limits μ and ν cannot be different. Under the condition stated, the sequence (1) has the common value of μ and ν as its limit.

Ex. 3. An illustration is furnished by any arithmetical process in which successive approximations to a result are obtained, provided these are adjusted in the usual manner, the last significant figure being increased by unity whenever the next following digit is 5 or any greater number. Thus the operation of finding the square root of 7 gives

$$2{\cdot}6457513\ldots.$$

The successive approximations, adjusted as above, are

$$3,\ 2{\cdot}6,\ 2{\cdot}65,\ 2{\cdot}646,\ 2{\cdot}6458,\ 2{\cdot}64575,\ 2{\cdot}645751,\ \ldots,$$

forming a sequence of the kind now under discussion. The numbers which follow the first differ from it by less than ·5; those which follow the second differ from it by less than ·05; those which follow the third differ from it by less than ·005; and so on. The sequence has therefore a definite limit.

Ex. 4. Consider the sequence in which

$$x_1 = 1, \quad x_{n+1} = \frac{1}{1+x_n}. \quad \text{.........................(5)}$$

The members are all positive, and (after the first) less than unity. It follows that all members after the second are greater than $\frac{1}{2}$. Again, we have

$$x_{n+2} - x_{n+1} = \frac{1}{1+x_{n+1}} - \frac{1}{1+x_n},$$

or

$$\frac{x_{n+2} - x_{n+1}}{x_n - x_{n+1}} = \frac{1}{(1+x_n)(1+x_{n+1})}. \quad \text{...............(6)}$$

Each member of the sequence is therefore alternately greater and less than the one preceding it. Moreover, since the above ratio is, for $n > 1$, less than $\frac{4}{9}$, the intervals between successive members diminish indefinitely. It easily follows that the sequence must converge to a definite limit, which is obviously the positive root of

$$x^2 + x = 1. \quad \text{.............................(7)}$$

By actual calculation from (5) we find in succession

1, ·5, ·6667, ·6, ·625, ·6154, ·6190, ·6176, ·6182,

the latter number being the correct value of the root in question, to four figures.

The character of the sequence may be illustrated graphically by means of the loci

$$y = x, \quad y = \frac{1}{1+x}. \quad \text{..........................(8)}$$

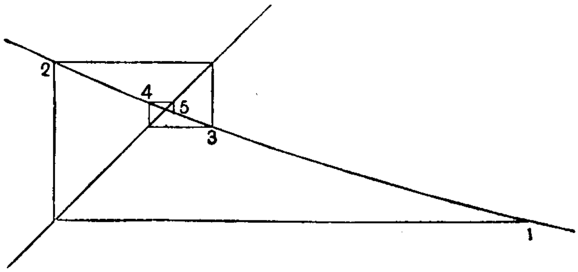

Fig. 4.

The figure shews the essential part of the graph.

In this Example, and in Art. 2, Ex. 3, we have simple illustrations of a method of approximating to the intersection of two curves which is often useful. The convergence is however slow if the curves have nearly the same inclination (in the same or in opposite senses) to the axis of x.

5. Application to Infinite Series.

If in the infinite series

$$u_1 + u_2 + \dots + u_n + \dots, \quad \dots\dots\dots\dots(1)$$

whose terms are no longer restricted to be all of the same sign, we write

$$s_1 = u_1, \; s_2 = u_1 + u_2, \dots\dots \; s_n = u_1 + u_2 + \dots + u_n, \dots \quad \dots(2)$$

and if the sequence

$$s_1, \; s_2, \dots\dots \; s_n, \dots \quad \dots\dots\dots\dots(3)$$

has a limiting value S, the series is said to be 'convergent,' and S is called its 'sum.'

It follows from Art. 4 that the necessary and sufficient condition for the convergence of (1) is that it should be possible to find a number n such that the partial sums $s_{n+1}, s_{n+2}, \dots, s_{n+p}, \dots$ all differ from s_n by less than ϵ, where ϵ may be any assigned quantity, however small.

An important theorem in the present connection is that if the series

$$|u_1| + |u_2| + \dots + |u_n| + \dots, \quad \dots\dots\dots\dots(4)$$

formed by taking the absolute values of the several terms of (1), be convergent, the series (1) will be convergent.

For if (4) be convergent, the positive terms of (1) must *à fortiori* form a convergent series, and so also must the negative terms. Let the sum of the positive terms be p and that of the negative terms be $-q$. Also, let s_{m+n}, the sum of the first $m+n$ terms of (1), consist of m positive terms whose sum is p_m, and n negative terms whose sum is $-q_n$. We have, then,

$$\begin{aligned}(p-q) - s_{m+n} &= (p-q) - (p_m - q_n) \\ &= (p - p_m) - (q - q_n). \quad \dots\dots\dots\dots(5)\end{aligned}$$

If $m+n$ be sufficiently great, $p - p_m$ and $q - q_n$ will both be less than ϵ, where ϵ is any assigned magnitude, however small; and the difference of these positive quantities will be *à fortiori* less than ϵ in absolute value. Hence s_{m+n} has the limiting value $p - q$.

When the series (4), composed of the absolute values of the several terms of (1), is convergent, the series (1) is said to be 'absolutely,' or 'essentially,' or 'unconditionally' convergent.

It is possible, however, for a series to be convergent, whilst the series formed by taking the absolute values of the terms has no upper limit. In this case, the convergence of the given series is said to be 'accidental,' or 'conditional.'