CHAPTER I

KINEMATICS OF RECTILINEAR MOTION

1. Velocity.

We begin with the elementary kinematical notions relating to motion in a straight line.

The position P of a moving point at any given instant t, i.e. at the instant when t units of time have elapsed from some particular epoch which is taken as the zero of reckoning, is specified by its distance x from some fixed point O on the line, this distance being reckoned positive or negative according to the side of O on which P lies. In any given case of motion x is then a definite and continuous function of t. When the form of this

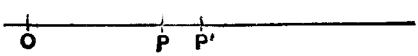

Fig. 1.

function is known it is often convenient to represent it graphically, in an auxiliary diagram, by means of a curve constructed with t as abscissa and x as ordinate. This may be called the 'space-time curve*.'

If equal spaces are described, in the same sense, in any two equal intervals of time the moving point is said to have a 'constant

* There are various classical experiments, especially in Acoustics, where such curves are produced by mechanical or optical contrivances, as e.g. in studying the nature of the vibration of a tuning-fork, or of a point of a piano-wire. On a different scale we have the graphical records of the oscillations of the barometer, &c.

velocity*,' the magnitude of this velocity being specified by the space described in the unit time. This space must of course have the proper sign attributed to it. Hence if the position change from x_0 to x in the time t, the velocity is $(x-x_0)/t$. Denoting this by u, we have

$$x = x_0 + ut. \qquad (1)$$

The space-time curve is therefore in this case a straight line.

If the velocity is not constant, the space described in any interval of time, divided by that interval, gives a result which may be called the 'mean velocity' in the interval. That is to say, a point having a constant velocity equal to this would describe an equal space in the same interval. Thus if, in Fig. 1, the points P, P' denote the positions at the instants t, t', respectively, the mean velocity in the interval $t'-t$ is

$$PP'/(t'-t), \quad \text{or} \quad (x'-x)/(t'-t),$$

where x, x' are the abscissae of P, P', respectively. If we write $x' = x + \delta x$, $t' = t + \delta t$, so that δx, δt denote corresponding increments of x and t, the mean velocity is denoted by

$$\frac{\delta x}{\delta t}.$$

In all cases which it is necessary to consider†, this fraction has a definite limiting value when the interval δt is indefinitely diminished; and this limit is adopted as the definition of the 'velocity at the instant t.' Denoting it by u, we have, in the notation of the Differential Calculus,

$$u = \frac{dx}{dt}. \qquad (2)$$

In the space-time curve, above referred to, the velocity at any instant is represented by the 'gradient' of the curve at the

* The phrase 'uniform velocity' is often used; but it seems preferable to use the word 'constant' when invariability in *time* is meant, the term 'uniform' being reserved to express invariability in *space*. Thus a constant field of force would be one which does not alter with the time; whilst a uniform field would be one which has the same properties at every point. Cf. Maxwell, *Matter and Motion*, London, 1876, pp. 24, 25.

† Except in the conventional treatment of problems of impact, where we may have different limits according as δt is positive or negative. See Chap. VI.

corresponding point, i.e. by the trigonometrical tangent of the angle which the tangent line to the curve, drawn in the direction of t increasing, makes with the positive direction of the axis of t, the differential coefficient dx/dt corresponding exactly to the dy/dx ordinarily employed in the Calculus.

The velocity u is in general a definite and continuous function of t, and may be represented graphically by a curve, called the 'velocity-time' curve, constructed with t as abscissa and u as ordinate. Since, by integration of (2), we have

$$x = \int u\,dt, \qquad \text{.........................(3)}$$

it appears that the area swept over by the ordinate of this curve in any interval of time gives the space described in that interval. The integral in (3) corresponds in fact to the ordinary $\int y\,dx$ of the Calculus.

If P_1, P_2 be two moving points, and x_1, x_2 their coordinates, then putting

$$\xi = P_1P_2 = x_2 - x_1, \qquad \text{....................(4)}$$

we have

$$\frac{d\xi}{dt} = \frac{dx_2}{dt} - \frac{dx_1}{dt}, \qquad \text{......................(5)}$$

i.e. the velocity of P_2 relative to P_1 is the difference of the velocities of P_1 and P_2.

2. Acceleration.

When the velocity increases by equal amounts (of the same sign) in any two equal intervals of time, the motion is said to be 'uniformly accelerated,' or (preferably) the moving point is said to have a 'constant acceleration'; and the amount of this acceleration is specified by the velocity gained per unit time*. Hence if the velocity change from u_0 to u in the time t, the acceleration is $(u - u_0)/t$. Denoting it by α, we have

$$u = u_0 + \alpha t. \qquad \text{..........................(1)}$$

The velocity-time curve is in this case a straight line.

* A negative acceleration is sometimes described as a 'retardation.'

In the general case, the increment of the velocity in any interval of time, divided by that interval, gives a result which may be called the 'mean rate of acceleration,' or briefly the 'mean acceleration' in that interval. That is to say, a point having a constant acceleration equal to this would have its velocity changed by the same amount in the same interval. Hence if u, u' denote the velocities at the instants t, t', respectively, the mean acceleration in the interval $t' - t$ is $(u' - u)/(t' - t)$, or

$$\frac{\delta u}{\delta t},$$

if we write $u' = u + \delta u$, $t' = t + \delta t$.

In all important cases, except those of impact, this fraction has a definite limiting value when δt is indefinitely diminished, and this limit is adopted as the definition of the 'acceleration at the instant t.' Hence, denoting the acceleration by α, we have

$$\alpha = \frac{du}{dt}. \quad \text{..........................(2)}$$

It appears that the acceleration is represented by the gradient, in the velocity-time curve.

Since $u = dx/dt$, we have

$$\alpha = \frac{d^2x}{dt^2}. \quad \text{..........................(3)}$$

It is often convenient to use the 'fluxional' notation, in which differentiations with respect to the time are denoted by dots placed over the symbol of the dependent variable. Thus the velocity may be denoted by $\dot{x}$, and the acceleration by $\dot{u}$ or $\ddot{x}$.

Another very important expression for the acceleration is obtained if we regard the velocity (u) as a function of the position (x). It is to be noted that, since the moving point may pass through a given position more than once, there may be more than one value of u corresponding to a given value of x. But if we fix our attention on one of these we have

$$\alpha = \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = u\frac{du}{dx}. \quad \text{.................(4)}$$

If x_1, x_2 be the coordinates of two moving points P_1, P_2 and if

$$\xi = P_1P_2 = x_2 - x_1, \quad \text{.....................(5)}$$

we have $$\frac{d^2\xi}{dt^2} = \frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2}; \quad \text{.....................(6)}$$

i.e. the acceleration of P_2 relative to P_1 is the difference of the accelerations of these points. In particular, if the velocity of P_1 be constant, so that $d^2x_1/dt^2 = 0$, we have

$$\frac{d^2\xi}{dt^2} = \frac{d^2x_2}{dt^2};$$

i.e. the acceleration of a moving point is the same whether it be referred to a fixed origin, or to an origin which is in motion with constant velocity.

Ex. 1. If x be a quadratic function of t, say

$$x = \tfrac{1}{2}(\alpha t^2 + 2\beta t + \gamma), \quad \text{.............................(7)}$$

we have $$\dot{x} = \alpha t + \beta, \quad \ddot{x} = \alpha. \quad \text{................................(8)}$$

The acceleration is therefore constant.

Ex. 2. If $$x = a\cos(nt + \epsilon), \quad \text{..............................(9)}$$

we have $$\dot{x} = -na\sin(nt + \epsilon), \quad \text{..........................(10)}$$

$$\ddot{x} = -n^2a\cos(nt + \epsilon) = -n^2x. \quad \text{.......................(11)}$$

The space-time curve is a curve of sines; and the velocity-time curve is a similar curve whose zero points synchronize with the maxima and minima of the former. See Fig. 4, p. 26.

Ex. 3. If $$x = Ae^{nt} + Be^{-nt}, \quad \text{..............................(12)}$$

we have $$\dot{x} = nAe^{nt} - nBe^{-nt}, \quad \text{..............................(13)}$$

$$\ddot{x} = n^2Ae^{nt} + n^2Be^{-nt} = n^2x. \quad \text{.......................(14)}$$

Ex. 4. If u^2 is a quadratic function of x, say

$$u^2 = Ax^2 + 2Bx + C, \quad \text{..............................(15)}$$

we have $$u\frac{du}{dx} = Ax + B. \quad \text{................................(16)}$$

The acceleration therefore varies as the distance from the point $x = -B/A$, unless $A = 0$, in which case it is constant.

3. Units and Dimensions.

The unit of *length* is generally defined by some material standard, or as some convenient multiple or submultiple thereof.

Thus in the metric system we have the metre, with its subdivisions the decimetre, centimetre, etc., and its multiple the kilometre. The standard metre was originally intended to represent the ten-millionth part of a quadrant of the earth's meridian as closely as possible. The agreement, though since found not to be exact*, is very close; but the practical and legal definition of the metre is of course by reference to the material standard, and not to the earth's dimensions. The reason for this particular choice of the standard was that on the decimal division of the quadrant a minute of latitude on the earth's surface corresponds to a kilometre.

In the British system of measurement we have the standard yard, with its subdivisions of foot and inch, and its multiple the mile. There is here no simple relation to the earth's dimensions, and the sea-mile, which corresponds to a (sexagesimal) minute of latitude, differs considerably from the statute mile of 1760 yards.

The relations between the two systems of length measurement are shewn by the following table†, which gives the factors required to reduce the various British units to centimetres, with their reciprocals, to four significant figures.

	Cm.	Reciprocals
Inch	2·540	·3937
Foot	30·48	·03281
Yard	91·44	·01094
Mile	$1·609 \times 10^5$	$6·214 \times 10^{-6}$
Sea-mile	$1·852 \times 10^5$	$5·398 \times 10^{-6}$

Owing to the decimal basis of the metric system the relations between other units can be read off at once. Thus a mile is 1609 metres; and a kilometre ($= 10^5$ cm.) is 1094 yards.

* The most authoritative value for the length of the earth-quadrant is 10,001,869 metres (Clarke, *Geodesy*, London, 1880).

† Taken from Everett's *Units and Physical Constants*.

In this book we shall employ mainly the foot or the centimetre, the latter being the unit now generally adopted in scientific measurements.

For the measurement of *time* some system based on the earth's rotation is universally adopted, all clocks and watches being regulated ultimately by reference to this. From a purely scientific standpoint the simplest standard would be the sidereal day, i.e. the period of a complete rotation of the earth relatively to the fixed stars; but this would have the serious inconvenience that ordinary time-keepers would have to be discarded for scientific purposes. The units commonly employed are based on the 'mean solar day,' i.e. the average interval between two successive transits of the sun across any given meridian. This bears to the sidereal day the ratio 1·00274. In scientific measurements the unit is generally the mean solar second, i.e. the $\frac{1}{86400}$th part of the mean solar day, whilst for practical purposes the hour, or the day, or year, are of course often employed.

The units of length and time are in the first instance arbitrary and independent. Those of velocity and acceleration depend upon them, and are therefore classed as 'derived' units. The unit velocity, i.e. the velocity which is represented by the number 1 according to the definition of Art. 1, is such that a unit of length is described in the unit time. Its magnitude therefore varies directly as that of the unit length, and inversely as that of the unit time. It is therefore said to be of one 'dimension' in length, and *minus* one dimension in time. If we introduce symbols L and T to represent the magnitudes of the units of length and time respectively, this may be expressed concisely by saying that the unit velocity is L/T, or LT^{-1}. The number which expresses any actual velocity will of course vary *inversely* as the magnitude of this unit. When it is necessary to specify the particular unit adopted we may do this by the addition of words such as 'feet per second,' or 'miles per hour,' or more briefly 'ft./sec.,' or 'mile/hr.'

The unit acceleration is such that a unit of velocity is acquired in the unit time. Its dimensions are therefore indicated by $\mathsf{LT}^{-1}/\mathsf{T}$, or L/T^2, or LT^{-2}. An actual acceleration may be specified by a number followed by the words 'feet per second per second,' or 'miles per hour per hour,' &c., as the case may be. These indications are conveniently abbreviated into 'ft./sec.^{2},' or 'mile/hr.^{2},' &c. The *double* reference to time in the specification of an acceleration is insisted upon sufficiently in elementary works on Mechanics; but the student may notice that it is indicated again by the form of the differential coefficient d^2x/dt^2.

Ex. To translate from the mile and hour to the foot and second as fundamental units we may write

$$\mathsf{L}'=5280\,\mathsf{L},\quad \mathsf{T}'=3600\,\mathsf{T}.$$

Hence $$\mathsf{L}'/\mathsf{T}'=\tfrac{22}{15}\,\mathsf{L}/\mathsf{T}.$$

The units of velocity on the two systems are therefore as 22 to 15, and the numerical values of any given velocity as 15 to 22. Thus a speed of 60 miles an hour is equivalent to 88 feet per second.

Again $$\mathsf{L}'/\mathsf{T}'^2=\tfrac{11}{27000}\,\mathsf{L}/\mathsf{T}^2,$$

so that the units of acceleration are as 11 to 27000.

4. The Acceleration of Gravity.

It may be taken as a result of experiment, although the best experimental evidence is indirect (Art. 11), that a particle falling freely at any given place near the earth's surface has a definite acceleration g, the same for all bodies.

The precise value of g varies however with the locality, increasing from the equator towards the poles, and diminishing slightly with altitude above the sea-level. There are also local irregularities of comparatively small amount. According to recent investigations*, the value of g at sea-level is represented to a high degree of accuracy by the formula

$$g=978{\cdot}03\,(1+{\cdot}0053\sin^2\phi), \qquad \text{(1)}$$

where ϕ is the latitude, the units being the centimetre and the second. In terms of the foot and the second this makes

$$g=32{\cdot}088\,(1+{\cdot}0053\sin^2\phi). \qquad \text{(2)}$$

The total variation from equator to pole is therefore a little more than one-half per cent. In latitude 45° we have $g=980{\cdot}62$, or $32{\cdot}173$, according to the units chosen.

* F. R. Helmert, *Encycl. d. math. Wiss.*, Bd. vi.

The variation with altitude is given by the formula

$$g' = g\,(1 - {\cdot}0000003h), \quad \text{.....................(3)}$$

where g is the value at the sea-level, and g' that at a height h (in metres) above this level. This variation is accordingly for most purposes quite unimportant.

For illustrative purposes it is in general sufficiently accurate to assume $g = 980$ cm./sec.², or $= 32$ ft./sec.², the latter number being specially convenient for mental calculations, on account of its divisibility.

5. Differential Equations.

The formulæ (3) and (4) of Art. 2 enable us to find at once expressions for the acceleration when the position (x) is given as a function of the time, or the velocity as a function of the position. But in dynamical questions we have more usually to deal with the inverse problem, where the acceleration is given as a function of the time or the position, or both, or possibly of the velocity as well, and it is required to find the velocity and the position at any assigned instant. We notice here one or two of the more important types of differential equation which thus present themselves, and the corresponding methods of solution.

1. The acceleration may be given as a function of the time; thus

$$\frac{d^2x}{dt^2} = f(t). \quad \text{..........................(1)}$$

This can be integrated at once with respect to t. We have

$$\frac{dx}{dt} = \int f(t)\,dt + A = f_1(t) + A, \quad \text{..............(2)}$$

say, where $f_1(t)$ stands for *any* indefinite integral of $f(t)$, and the additive constant A is arbitrary. Integrating again we have

$$x = \int f_1(t)\,dt + At + B, \quad \text{....................(3)}$$

where B is a second arbitrary constant.

The reason why two arbitrary constants appear in this solution is that a point may be supposed to start at a given instant from any arbitrary position with any arbitrary velocity, and to be governed as to its subsequent motion by the law expressed in (1). A solution, to be general, must therefore be capable of adjustment to these arbitrary initial conditions. See Ex. 1 below. The reason why the arbitrary element in the solution occurs in the particular form $At+B$ is that the superposition of any constant velocity does not affect the acceleration.

2. The acceleration may be given as a function of the position; thus

$$\frac{d^2x}{dt^2}=f(x). \quad \text{.........................(4)}$$

If we multiply both sides of this equation by dx/dt it becomes integrable with respect to t; thus

$$\frac{dx}{dt}\frac{d^2x}{dt^2}=f(x)\frac{dx}{dt},$$

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2=\int f(x)\frac{dx}{dt}\,dt+A$$

$$=\int f(x)\,dx+A, \quad \text{...................(5)}$$

by the ordinary formula for change of variable in an indefinite integral.

This process will occur over and over again in our subject, and the result (5) has usually an important interpretation as the 'equation of energy.' It may be obtained in a slightly different manner as follows. Taking x as the independent variable we have, in place of (4),

$$u\frac{du}{dx}=f(x), \quad \text{.........................(6)}$$

which is integrable with respect to x; thus

$$\tfrac{1}{2}u^2=\int f(x)\,dx+A. \quad \text{...................(7)}$$

In either way we obtain u^2 or $\dot{x}^2$ as a function of x, say

$$\left(\frac{dx}{dt}\right)^2=F(x), \quad \text{.......................(8)}$$

whence

$$\frac{dx}{dt}=\pm\sqrt{\{F(x)\}}. \quad \text{.......................(9)}$$