INTRODUCTION

THEORY OF VECTORS

1. Definition of a Vector.

The statement and the proof of many theorems in Mechanics are so much simplified by the terminology of the Theory of Vectors that it is worth while to begin with a brief account of the more elementary notions of this subject. The student who is already conversant with it may at once pass on.

The quantities with which we deal in mathematical physics may be classified into 'vectors' and 'scalars,' according as they do or do not involve the idea of direction.

A quantity which is completely specified by a numerical symbol, positive or negative, and has no intrinsic reference to direction in space, is called a 'scalar,' since it is defined by its position on the proper scale of measurement. Thus such quantities as mass, length, time, energy, hydrostatic pressure-intensity, belong to this category.

A 'vector' quantity, on the other hand, involves essentially the idea of direction as well as magnitude. To take a simple geometrical example, the position of a point B relative to another point A is specified by means of a straight line drawn from A to B. It may equally well be specified by any equal and parallel straight line drawn in the same sense from (say) C to D, since the position of D relative to C is the same as that of B relative to A. A straight line regarded in this way as having a definite magnitude and direction, but no definite location in space, is called a 'vector'*. Occasionally,

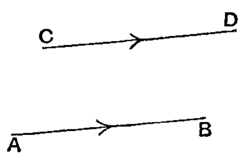

Fig. 1.

* Or 'carrier,' since (in the above instance) it indicates the operation by which a point is transferred from A to B. The terms 'vector' and 'scalar' are due to Sir W. R. Hamilton (1853).

when we wish to lay stress on the latter particular, it is called a 'free' vector. For example, if a rigid body be moved from one position to another without rotation, so that the lines joining the initial and final positions of the various points of the body are all equal and parallel, the displacement of the body as a whole is completely specified by a free vector, which may be any one of these lines.

As regards notation, a vector may be specified by means of the letters denoting the terminal points of a representative line, written in the proper order*. It is sometimes convenient, however, to denote vectors by single symbols. For this purpose what is called 'clarendon' type (**A**, **a**, ...) is often employed, whilst scalar quantities are denoted as usual by italic symbols.

For reasons which have already been indicated, two vectors **P** and **Q** which, like AB and CD in Fig. 1, have the same magnitude and direction, are regarded as equal, or rather identical, and the equation

$$\mathbf{P} = \mathbf{Q}$$

is used to express this complete identity. We have here the definition of the sign '=' as used in the present connection; and it is to be particularly noticed that there can be no question of equality between vectors whose directions are different. Since straight lines which are equal and parallel to the same straight line are equal and parallel to one another, it follows that if

$$\mathbf{P} = \mathbf{R} \text{ and } \mathbf{Q} = \mathbf{R},$$

then $$\mathbf{P} = \mathbf{Q}.$$

In words, vectors which are equal to the same vector are equal to one another.

2. Addition of Vectors.

There are certain modes of combination of vectors with one another, or with scalars, which have important geometrical and physical applications. As regards combinations of two or more vectors, the only kind which we need consider at present is that suggested by composition of displacements of pure translation of a

* Thus AB is to be distinguished from BA. In this book we shall use Roman letters when denoting a vector in this way, the italics AB or BA being used when the *length* only of the line is referred to. In manuscript work a bar may be drawn over two letters which are meant to denote a vector.

rigid body. Thus if such a body receives in succession two translations represented by AB and BC, the final result is equivalent to a translation represented by AC. It is therefore natural to speak of AC as in a sense the 'geometric sum,' or simply the 'sum,' of the vectors AB and BC, and to write

$$AB + BC = AC.$$

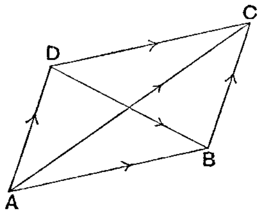

Fig. 2.

Hence to construct the sum of any two vectors **P**, **Q**, we draw a line AB to represent **P**, and then BC to represent **Q**; the sum **P** + **Q** is then represented by AC. This definition of vector addition is of course conventional and arbitrary, and it remains to be seen whether the process is subject to the same rules as those which govern ordinary algebraical addition.

If we complete the parallelogram *ABCD*, as in Fig. 2, we have in virtue of our conventions,

$$DC = AB = \mathbf{P}, \quad AD = BC = \mathbf{Q},$$

and therefore $$\mathbf{Q} + \mathbf{P} = AD + DC = AC,$$

or $$\mathbf{Q} + \mathbf{P} = \mathbf{P} + \mathbf{Q}. \qquad (1)$$

This is the 'commutative law' of addition; it is not self-evident, but depends, as we see, on the Euclidean theory of parallels.

When we wish to indicate that a particular vector which occurs in a formula arises as the sum of two vectors **P** and **Q**, we enclose the sum in brackets, as (**P** + **Q**). There is accordingly a distinction of meaning in the first instance between, say, (**P** + **Q**) + **R** and **P** + (**Q** + **R**). Thus if (see Fig. 3) we make

$$AB = \mathbf{P}, \quad BC = \mathbf{Q}, \quad CD = \mathbf{R},$$

we have

$$(\mathbf{P} + \mathbf{Q}) + \mathbf{R} = AC + CD, \quad \mathbf{P} + (\mathbf{Q} + \mathbf{R}) = AB + BD,$$

but since each of these results is equal to AD, we have

$$(\mathbf{P} + \mathbf{Q}) + \mathbf{R} = \mathbf{P} + (\mathbf{Q} + \mathbf{R}). \qquad (2)$$

This is known as the 'associative law' of addition. It easily follows from this and from the commutative law that three or more vectors may be added in any order without affecting the result.

For this reason the brackets, which are in strictness necessary to define the succession of the operations, are in practice often omitted, either side of (2), for instance, being denoted by

$$\mathbf{P} + \mathbf{Q} + \mathbf{R}.$$

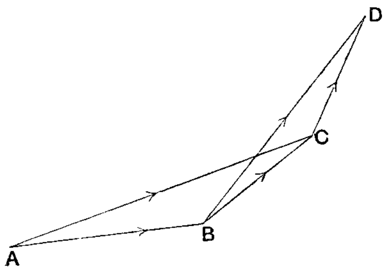

Fig. 3.

It is to be noticed that the points A, B, C, D need not be in the same plane, and consequently that the vectors $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ may have any directions whatever in space.

The symbol '$-$' prefixed to a vector is used to indicate that its direction is reversed, thus

$$\mathrm{BA} = -\mathrm{AB}. \qquad (3)$$

It is also usual to write for shortness

$$\mathbf{P} - \mathbf{Q} \text{ in place of } \mathbf{P} + (-\mathbf{Q}).$$

Thus in Fig. 2 we have

$$\mathbf{P} - \mathbf{Q} = \mathrm{AB} - \mathrm{BC} = \mathrm{AB} + \mathrm{CB} = \mathrm{DA} + \mathrm{AB} = \mathrm{DB}.$$

The difference of two vectors has a simple interpretation in the theory of displacements. Thus if $\mathbf{P}$, $\mathbf{Q}$ denote the absolute displacements (of pure translation) of two bodies, the vector $\mathbf{P} - \mathbf{Q}$ represents the displacement of the first body *relative* to the second.

A vector whose terminal points coincide is denoted by the symbol '$\mathbf{0}$,' and it is plain that all such evanescent vectors may be regarded as equivalent. Thus in Fig. 2 we have

$$\mathrm{AA} = \mathbf{0}, \quad \mathrm{AB} + \mathrm{BA} = \mathbf{0}, \quad \mathrm{AB} + \mathrm{BC} + \mathrm{CA} = \mathbf{0}. \quad \ldots(4)$$

Moreover

$$\mathrm{AB} + \mathrm{BB} = \mathrm{AB},$$

or

$$\mathbf{P} + \mathbf{0} = \mathbf{P}. \qquad (5)$$

Hence, also,

$$(\mathbf{P}-\mathbf{Q})+\mathbf{Q}=\mathbf{P}+(-\mathbf{Q}+\mathbf{Q})=\mathbf{P}+0=\mathbf{P}. \quad ......(6)$$

This will be recognized as the fundamental property of the sign '$-$' in formal algebra.

3. Multiplication by Scalars.

Finally, we have to consider the multiplication of a vector $\mathbf{P}$ by a scalar m. We define $m\mathbf{P}$ to mean a vector whose length is to that of $\mathbf{P}$ in the ratio denoted by the absolute value of m, and whose direction is that of $\mathbf{P}$, or the reverse, according as m is positive or negative. It follows that

$$\text{if } \mathbf{P}=\mathbf{Q}, \text{ then } m\mathbf{P}=m\mathbf{Q}. \quad ....(1)$$

It only remains to examine whether the distributive law

$$m(\mathbf{P}+\mathbf{Q})=m\mathbf{P}+m\mathbf{Q}, \quad ....(2)$$

which is fundamental in ordinary algebra, holds on the above definition. The proof depends on the properties of similar triangles. If we make

$$\mathrm{OA}=\mathbf{P}, \quad \mathrm{OA}'=m\mathbf{P}, \quad \mathrm{AB}=\mathbf{Q}, \quad \mathrm{A'B'}=m\mathbf{Q},$$

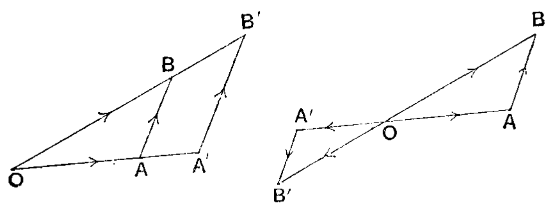

Fig. 4*.

then in the triangles OAB, $OA'B'$ we have

$$OA' : A'B' :: OA : AB,$$

whilst the angles $OA'B'$, OAB are equal. It follows that the points O, B, B' are collinear, and that

$$OB' : OB :: OA' : OA.$$

Hence

$$m\mathbf{P}+m\mathbf{Q}=\mathrm{OA}'+\mathrm{A'B'}=\mathrm{OB}'=m\,.\,\mathrm{OB}=m(\mathbf{P}+\mathbf{Q}),$$

and the theorem (2) is established.

* The two diagrams relate to the cases where m is positive and negative, respectively.

4. Geometrical Applications.

We have been at some pains to shew that although the literal symbols **P**, **Q**, ... no longer denote mere magnitudes, and although the signs '=,' '+,' '−,' '**0**' have received meanings different from, or rather more general* than, those which they bear in ordinary quantitative algebra, yet they are subject to precisely the same laws of operation as in that science. The conclusions which follow from the application of these laws will therefore possess the same validity. The theory of vectors furnishes us in this way with a convenient shorthand by which many interesting theorems of Geometry can be obtained in a concise manner. We shall see later that some of these theorems have important applications in Mechanics.

For example, if C be a point in a straight line AB such that

$$m_1 . \mathrm{CA} + m_2 . \mathrm{CB} = 0, \quad \text{.........(1)}$$

and O any point whatever, we have

$$m_1 . \mathrm{OA} + m_2 . \mathrm{OB} = (m_1 + m_2)\, \mathrm{OC}. \quad \text{...(2)}$$

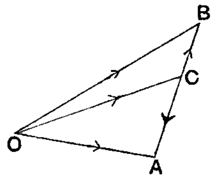

Fig. 5.

For

$$\begin{aligned} m_1 . \mathrm{OA} + m_2 . \mathrm{OB} &= m_1 (\mathrm{OC} + \mathrm{CA}) + m_2 (\mathrm{OC} + \mathrm{CB}) \\ &= (m_1 + m_2)\, \mathrm{OC} + (m_1 . \mathrm{CA} + m_2 . \mathrm{CB}) \\ &= (m_1 + m_2)\, \mathrm{OC}, \end{aligned}$$

in virtue of the commutative, associative, and distributive laws proved in Art. 2, and of the assumption (1).

In the particular case where $m_1 = m_2$, C is the middle point of AB, and the theorem becomes

$$\mathrm{OA} + \mathrm{OB} = 2.\mathrm{OC}. \quad \text{....................(3)}$$

This may be interpreted as expressing that the diagonal through O of the parallelogram constructed with OA, OB as adjacent sides has the same direction as OC and double the length; in other words, the diagonals of a parallelogram bisect one another.

It is to be noticed that if m_1, m_2 have opposite signs C will lie in the prolongation of AB, beyond A or beyond B according as

* The processes of ordinary algebra have their representation in the addition &c. of vectors *in the same line* (or of a system of *parallel* vectors).

m_1 or m_2 is the greater in absolute magnitude. The theorem fails when $m_1 + m_2 = 0$, since C is then at infinity; but in this case we have obviously

$$m_1.\mathrm{OA} + m_2.\mathrm{OB} = m_1(\mathrm{OA} - \mathrm{OB}) = m_1.\mathrm{BA}. \quad \ldots\ldots(4)$$

The formula (2) has many applications. Thus if AA', BB', CC' be the median lines of a triangle ABC, and if in AA' we take G so that $\mathrm{AG} = 2.\mathrm{GA}'$, we have, by (3),

$$\mathrm{BB}' = \tfrac{1}{2}(\mathrm{BC} + \mathrm{BA}),$$

and, by (2),

$$\mathrm{BC} + \mathrm{BA} = 2.\mathrm{BA}' + \mathrm{BA} = 3.\mathrm{BG}.$$

Hence $$\mathrm{BG} = \tfrac{2}{3}\mathrm{BB}'. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

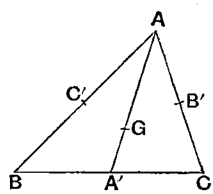

Fig. 6.

This, being a vector equation, implies that G lies in BB', and is a point of trisection on this line. In other words, we have proved that the three median lines of a triangle intersect in one point, which is a point of trisection on each.

It is also easily proved that

$$\mathrm{GA} + \mathrm{GB} + \mathrm{GC} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and that if O be any point whatever (not necessarily in the same plane with A, B, C),

$$\mathrm{OG} = \tfrac{1}{3}(\mathrm{OA} + \mathrm{OB} + \mathrm{OC}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The point G which possesses these properties is called the 'mean centre' of A, B, C.

In a subsequent chapter these relations will be greatly extended.

5. Parallel Projection of Vectors.

The particular kind of projection here contemplated is by means of systems of parallel lines or planes. Taking first the case of two dimensions, where all the points and lines considered lie in one plane, the 'projection' of a point A on a given straight line OX is defined as the point A' in which a straight line drawn through A in some *prescribed* direction meets OX. Again if AB represent any vector, and A', B' be the projections of the points A, B, the vector A′B′ is called the projection of AB.

A particular case of great importance is that of 'orthogonal' projection, where the projecting lines are perpendicular to OX.

The most important theorem in the present connection is that the projection of the sum of two or more vectors is equal to the sum of the projections of the several vectors. Thus, if AB, BC be drawn to represent any two vectors, and A', B', C' be the projections of A, B, C, respectively, we have obviously

$$\mathrm{A'B'} + \mathrm{B'C'} = \mathrm{A'C'}.$$

Now A′C′ is the projection of the vector AC, which is the geometric sum of AB and BC.

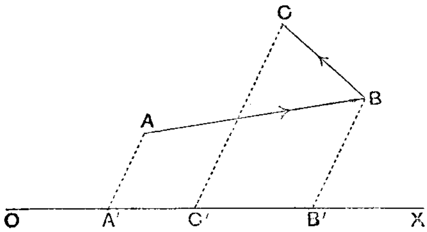

Fig. 7.

These projections of vectors on OX may evidently be specified by a series of scalar quantities, provided we fix on one direction along OX, say that from O to X, as the standard or positive direction. Thus if we specify the projection of AB by a, we mean that the length $A'B'$ is equal to the absolute value of a, and that the direction from A' to B' agrees with, or is opposed to, that from O to X, according as a is positive or negative. On this convention, the algebra of vectors in OX becomes identical in all respects with ordinary algebra.

In the particular case of orthogonal projection, the projection of a vector $\mathbf{P}$ is $P \cos \theta$, where P denotes the absolute value of $\mathbf{P}$, without regard to sign, and θ is the angle which the direction

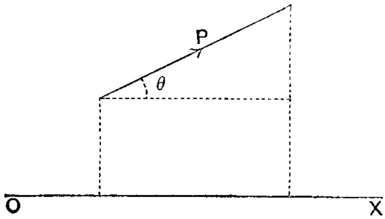

Fig. 8.

of $\mathbf{P}$ makes with the direction OX. This hardly needs proof, since the general definition of a cosine in Trigonometry is essentially that it is the projection of a unit vector on the initial line.

We here come in contact with the principles of Analytical Geometry. If we take two fixed lines of reference Ox, Oy, and project any point A on each of these by a line drawn parallel to the other, the projections of the vector OA are simply the ordinary Cartesian coordinates of A relative to the axes Ox, Oy. The vector OA, it may be added, is sometimes called the 'position vector' of A relative to the fixed origin O.

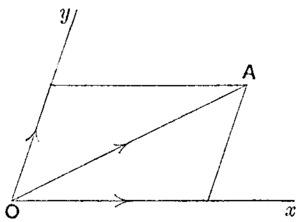

Fig. 9.

The preceding conventions are easily extended to the case of three dimensions. The only modification is that we project by a system of parallel *planes*. The points A', B', C', ... in which the planes of the system which pass through A, B, C, ... meet OX are called the projections of A, B, C, ..., respectively; the vector A′B′ is the projection of AB, and so on. Again, if we project on each of a system of three fixed axes Ox, Oy, Oz by planes parallel to the other two, the projections of a position vector OA are identical with the Cartesian coordinates of A.

EXAMPLES. I.

1. Illustrate geometrically the formulæ

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{A}+\mathbf{B}) + \tfrac{1}{2}(\mathbf{A}-\mathbf{B}),$$
$$\mathbf{B} = \tfrac{1}{2}(\mathbf{A}+\mathbf{B}) - \tfrac{1}{2}(\mathbf{A}-\mathbf{B}).$$

2. Find a point O in the plane of a quadrilateral $ABCD$ such that

$$\mathrm{OA} + \mathrm{OB} + \mathrm{OC} + \mathrm{OD} = 0.$$

3. If O, O' be the middle points of any two straight lines AB, $A'B'$, prove that

$$\mathrm{AA'} + \mathrm{BB'} = 2 . \mathrm{OO'}.$$

4. If AB, $A'B'$ be any two parallel straight lines, the line joining the middle points of AA', BB' is parallel to AB and $A'B'$, and equal to

$$\tfrac{1}{2}(AB + A'B').$$

What is the corresponding result for the line joining the middle points of AB', $A'B$?

5. If A, B, C, D be any four points, prove that

$$\mathrm{AB} + \mathrm{AD} + \mathrm{CB} + \mathrm{CD} = 4 . \mathrm{PQ},$$

where P, Q are the middle points of AC and BD, respectively.

6. The middle points of the sides of any quadrilateral (plane or skew) are corners of a parallelogram.

7. $ABCD$ is a parallelogram, and H, K are the middle points of AB, CD. Prove that if DH, BK be drawn, they trisect the diagonal AC.

8. If G be the mean centre of A, B, C, and G' that of A', B', C', prove that

$$\mathrm{AA'} + \mathrm{BB'} + \mathrm{CC'} = 3 . \mathrm{GG'}.$$

9. If points P, Q, R be taken in the sides of a triangle ABC such that

$$\mathrm{BP} = m . \mathrm{BC}, \quad \mathrm{CQ} = m . \mathrm{CA}, \quad \mathrm{AR} = m . \mathrm{AB},$$

the mean centre of P, Q, R will coincide with that of A, B, C.

10. If points P, Q, R, S be taken in the sides AB, BC, CD, DA of a parallelogram, so that

$$\mathrm{AP} = m . \mathrm{AB}, \quad \mathrm{BQ} = n . \mathrm{BC}, \quad \mathrm{CR} = m . \mathrm{CD}, \quad \mathrm{DS} = n . \mathrm{DA},$$

then $PQRS$ will be a parallelogram having the same centre as $ABCD$.

11. If I be the centre of the circle inscribed in the triangle ABC, prove that

$$a . \mathrm{IA} + b . \mathrm{IB} + c . \mathrm{IC} = 0,$$

where a, b, c denote the lengths of the sides.

What is the corresponding statement when I is the centre of an escribed circle?

12. If OA, OB, OC be concurrent edges of a parallepiped, and

$$\mathrm{OA} = \mathbf{P}, \quad \mathrm{OB} = \mathbf{Q}, \quad \mathrm{OC} = \mathbf{R},$$

interpret the vectors

$$\mathbf{P} + \mathbf{Q} + \mathbf{R}, \quad \mathbf{Q} + \mathbf{R} - \mathbf{P}, \quad \mathbf{R} + \mathbf{P} - \mathbf{Q}, \quad \mathbf{P} + \mathbf{Q} - \mathbf{R}.$$

13. Prove that the four diagonals of a parallelepiped meet in a point and bisect one another.

14. If OA, OB, OC be three concurrent edges of a parallelepiped, prove that the point G where the line joining O to the opposite corner D meets the plane ABC is the mean centre of A, B, C. Also that

$$\mathrm{OG} = \tfrac{1}{3}\mathrm{OD}.$$

15. Prove that the three straight lines which join the middle points of opposite edges of a tetrahedron all meet, and bisect one another.