CHAPTER I.

RATIONAL AND IRRATIONAL NUMBERS.

1. Introductory. The student is supposed to be familiar with the ordinary theory of the natural numbers, 1, 2,..., and its extension to the fractional and negative numbers, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$,..., -1, $-\frac{1}{2}$,.... Both classes are grouped together under the name of rational numbers. He is also supposed to have some acquaintance with the theory of irrational numbers. This latter theory occupies, however, a fundamental position in our present subject, and we propose to give a short account of it, sufficient for the purposes in hand.

2. Sets and Sequences. Any number of rational numbers, individually given, are said to form a finite *set*. For instance,

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{3}{4},\ 2,\ 3.$$

In this case, each being given singly, they must have a definite succession, and are said to have an *order*.

If numbers are given, not individually, but by means of some law, they are said to form a *set*, which is a *finite set*, if they can from the law be individually determined and assigned, without any numbers of the set being omitted; in the contrary case, they are said to form an *infinite set*, and to be infinite in number (more precisely, in cardinal number). If the law be such that from it the numbers are determined in a definite succession, one by one, the numbers are said to have an *order*.

Thus all the integers between 0 and 15 form a finite set, without order. The same integers in order of magnitude form a finite set in order, which is generally referred to as the *natural order*. All the rational numbers between 0 and 15 form an infinite set without order, and in order of magnitude they have an order, the natural order. All numbers satisfying the relation

$$u_n = u_{n-1} + u_{n-2},$$

where $$u_1 = 1, \quad u_2 = 1,$$

form an infinite set in order.

A set in order is also called a *series.*

An infinite number or series of rational numbers*

$$a_1, a_2, \ldots,$$

is said to form a sequence, if, given any small positive quantity ϵ, *a number* a_n *of the series can always be assigned, such that, if* a_p *and* a_q *be any numbers of the series subsequent to* a_n,

$$|a_p - a_q| < \epsilon.$$

The individual numbers a_1, a_2 are called the *constituents* of the sequence.

Thus, the numbers

$$1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16},\ \ldots\ \left(2-\frac{1}{2^m}\right),\ \left(2-\frac{1}{2^{m+1}}\right),\ \ldots \ldots (1)$$

for all integral values of m, form a sequence.

A series of rational numbers which constantly increase, or which constantly decrease, always defines a sequence, provided a finite number exists which in the former case is always greater, and in the latter case is always less than any number of the series.

Ex. $$1,\quad 1+1,\quad 1+1+\frac{1}{2\,!},\quad 1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots.$$

It may now happen that the sequence is such that a rational number b exists, to which the a's continually approximate to a degree closer than any assigned quantity. More precisely it may be such that, given any small positive quantity ϵ, a number a_n of the series can always be assigned, such that a_r being any number of the series subsequent to a_n,

$$|b - a_r| < \epsilon.$$

The sequence is then said to define the number b, which is clearly unique. Thus the number 2 is defined by the sequence (1).

It is evident that the same number b may be defined by sequences whose constituent numbers differ from one another. Thus each of the sequences

$$\frac{2}{3},\ \frac{10}{9},\ \frac{38}{27},\ \ldots\ldots\ \sum_1^n \left(\frac{2}{3}\right)^r,\ \ldots\ldots$$

$$\frac{8}{3},\ \frac{16}{9},\ \frac{56}{27},\ \ldots\ldots\ \sum_1^n (-)^{n+1}\frac{8}{3^n},\ \ldots\ldots$$

defines the number 2.

Sequences which define the same number are said to be equal.

* See concluding remarks on p. 5, where this restriction is removed.

3. Irrational numbers. The two most ready ways of setting up a sequence are:

(1) by means of the terminating decimal fractions which are the successive approximations to a non-terminating decimal fraction (or fraction expressed in some other scale), and

(2) by means of the successive convergents to a non-terminating "simple" continued fraction*.

The first of these forms a sequence since the constituent numbers continually increase and remain less than an assignable number: that the second forms a sequence follows from the known properties of continued fractions.

Since a rational number can be expressed in one, and only one, way (1) as a non-terminating decimal fraction, namely as a recurring decimal, and (2) as a simple continued fraction, namely a terminating one, it follows that a sequence set up in mode (1) does not define a rational number unless the decimal fraction recurs, and that (2) never defines a rational number.

In accordance with the usual law which holds in all extensions of mathematical reasoning, it is convenient still to use the word *number* when speaking of a sequence which does not define a rational number. The new numbers thus obtained are called irrational numbers; for instance the series of decimal fractions

$$\cdot 1,\ \cdot 12,\ \cdot 123,\ \cdot 1235,\ \cdot 12357,\ \cdot 1235711,\ \cdot 123571113, \ldots\ldots$$

where each is got from the last by appending the next prime number after the last appended, and the series of convergents

$$1,\ 2,\ \tfrac{3}{2},\ \tfrac{5}{3},\ \tfrac{8}{5},\ \tfrac{13}{8},\ \tfrac{21}{13},\ \tfrac{34}{21},\ \tfrac{55}{34}, \ldots$$

to the continued fraction

$$1 + \frac{1}{1+}\,\frac{1}{1+}\ \ldots\ldots,$$

are said to define irrational numbers.

4. Magnitude and Equality. In § 2 we stated that two sequences which defined the same rational number might be regarded as equal. We now give a definition of the equality of two sequences which applies when the two sequences do not define rational numbers: this definition will include the preceding as a particular case.

* That is to say a continued fraction of the form

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\ \ldots\ldots\ldots$$

where the a's are positive integers; Chrystal's *Algebra*, Vol. II. p. 397.

The definition is as follows:—

Two sequences $a_1, a_2, \ldots\ldots$

$$b_1, b_2, \ldots\ldots$$

are said to be equal, when, given any small positive quantity ϵ, *however small, an integer* m *can be found such that, for all values of* $n \geqslant m$,

$$|a_n - b_n| < \epsilon.$$

We may put this definition in an apparently more general, but actually equivalent, form. The two sequences are said to be equal, when, choosing out from each a partial sequence,

$$a_1', a_2', \ldots\ldots, \qquad b_1', b_2', \ldots\ldots,$$

these partial sequences satisfy the above condition of equality.

That this definition is equivalent to the former is evident from the fact that the a's and b's form sequences.

Taking any two sequences at random, and forming any partial sequences from them as above, it is easily proved that the quantities

$$a_1' - b_1', \quad a_2' - b_2', \ldots\ldots$$

form a sequence, and that all sequences so formed by means of partial sequences from two given sequences are equal. The number defined by any one of these sequences is called the *excess* of the number a, defined by the a-sequence, over the number b, defined by the b-sequence, and when taken positively is called the *difference* of a and b.

If the sequences be equal, the difference will be zero; otherwise the excess will be positive or negative, and the number a is, in the former case, said to be *greater* than the number b, $a > b$, and, in the latter case, a is said to be *less* than b, $a < b$.

An irrational number is said to be *positive or negative*, according as it is greater or less than 0.

It is evident that, (1) if $a > b$, $b < a$, and that, (2) if $a > b$ and $b > c$, $a > c$; finally that every number a is either greater than, equal to or less than any given number b. These facts are summed up by saying that *numbers may be compared as to magnitude.*

It is now easily seen that the rational numbers

$$a_1' + b_1', \quad a_2' + b_2', \ldots$$

form a sequence, whose magnitude is independent of the choice of the partial sequences. The number defined by any one of these sequences is called the *sum of the two numbers* a *and* b.

Similarly, $$a_1' b_1', \; a_2' b_2', \ldots,$$
and, if the b-sequence do not define the number 0,

$$\frac{a_1'}{b_1'}, \; \frac{a_2'}{b_2'}, \ldots$$

form sequences, whose magnitude is independent of the choice of the partial sequences. The former of these is said to define the *product*, and the latter the *quotient* of a and b.

Hence it follows that we can attach a definite meaning to the symbol $R(a, b, \ldots k)$, where R is a rational function of the finite set of irrational numbers $a, b, \ldots k$, provided that, in the process of calculating R by approximation, no quotient occurs whose value is, actually or in the limit, zero.

In particular, since the difference of two irrational numbers is now clearly defined, we can, in the definition of a sequence in § 2, insert the words "or irrational" after "rational," and so obtain a general definition of sequence, independent, not only of the fact whether or no the number defined be rational or not, but also of the rationality or irrationality of the constituents themselves.

5. The number ∞. We have now attached a definite number to every sequence of numbers, rational or irrational, and we saw that if

$$a_1, \; a_2, \ldots$$

be a given sequence, then, provided that a_r does not become less in absolute magnitude than any assigned number,

$$\frac{1}{a_1}, \; \frac{1}{a_2}, \ldots$$

is also a sequence. Also

$$(a_1)\frac{1}{a_1}, \; (a_2)\frac{1}{a_2}, \ldots$$

forms a sequence defining the number 1, so that, by our definitions, the product of the two former numbers is unity; we therefore denote them by a and $\frac{1}{a}$, and say that they are the *inverses* of one another.

We shall now add another number to our list of numbers, and so remove the last restriction as to the nature of the given sequence. If the a's become less in absolute magnitude than any assigned number, the constituents of the inverse series become greater in absolute magnitude than any assigned number, and do not form a sequence as we have defined it. In other words, the number 0 is the only number which at present has no inverse. We now introduce the number ∞ as the inverse of 0. That is to say, *given a series of numbers which become and remain greater in absolute magnitude than any assigned number, we still regard this series as a sequence, and introduce the symbol* ∞ *(infinity), to denote the "number" defined by such a sequence.*

We see that this agrees so far with the preliminary definition of the term "infinite number" given in § 2.

With this convention we may add that, if in any rational function $R(\infty)$ the symbol ∞ occurs, it implies that, taking any series of numbers $a_1, a_2, \ldots$, whose inverses in order define the number 0, the quantities $R(a_n)$ form a sequence, and the number defined by this sequence is that denoted by the symbol $R(\infty)$.

We content ourselves for the present with this definition* of the symbol ∞. It will subsequently appear that we have to specialise our ideas of infinity, or more properly, of infinite numbers, for purposes which will occur in the later chapters.

6. Limit. The number defined by a sequence is said to be the *limit of* the constituent a_n of the sequence, *when n is infinite.*

It appears from what has been said, that if $R(a, b, \ldots k)$ be any rational function of the finite set of numbers $a, b, \ldots k$ (rational or irrational), then, *provided the limit of* $R(a_n, b_{n'}, \ldots)$ *be definite,* $R(a, b, \ldots k)$ may be defined as the limit of $R(a_n, b_{n'}, \ldots)$ when $n, n', \ldots$ are infinite, and will be a number, rational or irrational. With the above restriction, then, the rational and irrational numbers form what is called a "corpus†."

7. Algebraic and transcendental numbers. An important class of numbers is that of the algebraic numbers. These are distinguished among themselves as to rank. *An algebraic number of rank m is defined as a number satisfying an irreducible equation of degree m, with rational coefficients, and satisfying no such equation of degree less than m.*

A rational number satisfies such an equation of degree 1, and any algebraic number which satisfies an algebraic equation of degree 1 is rational; thus the algebraic numbers of rank 1 are identical with the rational numbers, and all algebraic numbers of rank higher than 1 are irrational. Methods of obtaining all algebraic numbers, *i.e.* of obtaining sequences defining them, are given in all works on the Theory of Equations.

* It should be noticed that in this system of fixed numbers there is no place for the symbols $+\infty$ and $-\infty$ any more than for $+0$ and -0; see however § 15.

† A corpus is a collection of objects which reproduce themselves when subjected to the simple rules of arithmetic.

All irrational numbers which are not algebraic are classed together as *transcendental.*

It is in general a most difficult problem to determine in any special case whether a given number be rational, algebraic or transcendental. No general method and no set of necessary and sufficient conditions have at present been discovered. There are however a few isolated theorems on the subject, among which the following is one of the most important; it enables us to write down sequences defining numbers of the last of the above classes.

Liouville's Theorem. *If* $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ *be a sequence of rational fractions in their lowest terms, defining an algebraic number b of rank m, then, for every constituent* $\frac{p}{q}$ *from and after an assignable stage, we have*

$$\left|\frac{p}{q} - b\right| > \frac{1}{q^{m+1}}.$$

To prove this theorem, let the equation of degree m satisfied by b be

$$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_m = 0,$$

where the a's are integers; and let $\frac{p}{q}$ be any rational number within a certain small interval containing b, that is to say, such that the difference between that number and b, is less than a certain small positive quantity.

Then $$f(b) = 0,$$

therefore, $$f\left(\frac{p}{q}\right) = \left(\frac{p}{q} - b\right) f'(y),$$

where y is a certain number, possibly irrational, lying between $\frac{p}{q}$ and b; and f' is the first derived of f.

Now it is obviously possible to assign a finite number M, greater than any value of $|f'(y)|$, when y lies within the given interval. Hence

$$\left|f\left(\frac{p}{q}\right)\right| < M\left|\frac{p}{q} - b\right|.$$

But, $$f\left(\frac{p}{q}\right) = \frac{A}{q^m},$$

where A is some integer; and, if we choose to assign a sufficiently small interval, so that $f(x)$ vanishes for no value of x within the interval except b, A will not be zero, and therefore,

$$\left|\frac{p}{q} - b\right| > \frac{1}{Mq^m}.$$

Now there are only a finite number of rational numbers whose denominators are less than M. Hence, we can determine a second interval, lying within the former interval, and containing none of these rational numbers; in this interval for every rational number $\frac{p}{q}$,

$$\left|\frac{p}{q}-b\right|>\frac{1}{q^{m+1}},$$

so that, given any sequence having b as limit, we can determine a stage such that for all subsequent constituents the above inequality holds. Q. E. D.

The above property serves to determine whether a given sequence can represent an algebraic number, or a rational number, but it does not give a sufficient criterion to determine whether a given sequence actually does represent such a number.

Thus in the case of the sequence of fractions, expressed in the decimal or any other scale, such that each is got from the preceding by appending one more figure on the right, we know that the number b defined can only be rational if the figures ultimately recur in some cycle. This property is quite independent of the above, and cannot be proved from the above inequality. All that we can deduce from the inequality is that, if b be rational, then, when n is sufficiently large, n successive figures after the nth figure cannot all be noughts; while, in the more general case, when b is an algebraic number of rank m, mn successive figures after the nth cannot all be noughts, n being sufficiently large.

This property serves to define a class of transcendental numbers discovered by Liouville and called after his name, which were historically the first numbers to be proved transcendental. These are the numbers

$$\frac{1}{10}+\frac{1}{10^{1.2}}+\frac{1}{10^{1.2.3}}+\frac{1}{10^{1.2.3.4}}+\ldots=\cdot 110001000000000000000000010\ldots,$$

and the decimal fractions got by replacing any 1 by any other figure.

Such numbers may of course be constructed in any other scale, and will still be transcendental.

The best known transcendental numbers are π and e: these do not belong to the class of Liouville numbers.

CHAPTER II.

REPRESENTATION OF NUMBERS ON THE STRAIGHT LINE.

8. One of the most fundamental properties of the set of rational numbers is their *order*. We shall find in the sequel that the idea of order is one of the most essential to the understanding of sets of points, and that we habitually use the order of some or all of the rational numbers as a standard of comparison.

The order of the rational numbers as a whole is such that we cannot say which is the next rational number in order of magnitude after any given one a, or before a given one c; indeed, if a and c be any two rational numbers, we can always insert a rational number b between them.

It is of assistance to the imagination that we can set up a (1, 1)-correspondence between the rational numbers and certain points of the straight line, in such a way that the order is maintained, that is to say if A_p, A_q, A_r are three of the points, corresponding to the rational numbers p, q, r, A_q lies between A_p and A_r if, and only if, q lies between p and r, and *vice versa.*

We shall now discuss shortly, how and under what assumptions with respect to the nature of the straight line, this correspondence can be extended to the irrational numbers.

In setting up the (1, 1)-correspondence referred to, measurement may be entirely avoided; in this way various difficulties which have nothing to do with the subject in hand do not come into the discussion.

The principle of the correspondence which we here choose and which is commonly referred to as the *projective scale,* is that if a, b, c, d be any four harmonic rational numbers, that is if

$$(a, b, c, d) \equiv \frac{a-b}{b-c}\frac{c-d}{d-a} = -1,$$

or, which is the same thing, ...(1)

$$\frac{1}{b-a} + \frac{1}{d-a} = \frac{2}{c-a},$$

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the points corresponding to these four rational numbers shall form a harmonic range.

Now the equation (1) always defines a rational number d, if a, b, c be rational, unless

$$2b = a + c;$$

in the latter case there is no rational number satisfying the equation (1); but if d describe any sequence of positive or negative rational numbers, whose absolute magnitude increases without limit, $\left(a, \frac{a+c}{2}, c, d\right)$ approaches the limit -1. Hence in accordance with the meaning attached by us to the symbol ∞, we shall write

$$\left(a, \frac{a+c}{2}, c, \infty\right) = -1. \quad \text{.....................(2)}$$

It follows that, p being any positive integer,

$$(p-1, p, p+1, \infty) = (-p, 0, p, \infty) = \left(0, \frac{1}{p+1}, \frac{1}{p}, \frac{1}{p-1}\right) = -1. \quad \text{.....................(3)}$$

Now we start with any two points P, Q, and any point between them to which we attach the integer 1, or, as we shall express this more concisely, we choose this third point as the point 1. We shall see subsequently that we shall come to attach the symbols 0 and ∞ to the points P, Q, in consequence of the equations (2) and (3).

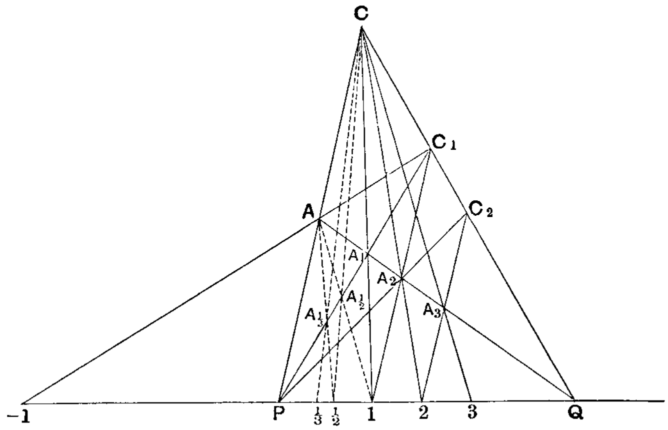

Fig. 1.

Outside the line PQ take any point C. Join CP, and on this line, between C and P, take any point A, and join AQ.