888.

ON A FORM OF QUARTIC SURFACE WITH TWELVE NODES.

[From the *British Association Report*, (1886), pp. 540, 541.]

USING throughout capital letters to denote homogeneous quadric functions of the coordinates (x, y, z, w), we have as a form of quartic surface with eight nodes $\Omega = (*\between U, V, W)^2 = 0$; viz. the nodes are here the octad of points, or eight points of intersection of the quadric surfaces $U = 0$, $V = 0$, $W = 0$; the equation can, by a linear transformation on the functions U, V, W (that is, by substituting for the original functions U, V, W linear functions of these variables), be reduced to the form $\Omega = U^2 + V^2 + W^2 = 0$.

Suppose now that the function Ω can in a second manner be expressed in the like form $\Omega = P^2 + Q^2 + R^2$ (where P, Q, R are not linear functions of U, V, W); that is, suppose that we have identically $U^2 + V^2 + W^2 = P^2 + Q^2 + R^2$, this gives $U^2 - P^2 + V^2 - Q^2 + W^2 - R^2 = 0$; or, writing $U + P$, $V + Q$, $W + R = A, B, C$, and $U - P$, $V - Q$, $W - R = F, G, H$, the identity becomes $AF + BG + CH = 0$; and this identity being satisfied, the equation $\Omega = 0$ of the quartic surface may be written in the two forms

$$\Omega = (A + F)^2 + (B + G)^2 + (C + H)^2 = 0,$$

and

$$\Omega = (A - F)^2 + (B - G)^2 + (C - H)^2 = 0;$$

viz. the quartic surface has the nodes which are the intersections of the three quadric surfaces $A + F = 0$, $B + G = 0$, $C + H = 0$, and also the nodes which are the intersections of the three quadric surfaces $A - F = 0$, $B - G = 0$, $C - H = 0$. We may of course also write the equation of the surface in the form

$$\Omega = A^2 + B^2 + C^2 + F^2 + G^2 + H^2 = 0.$$

An easy way of satisfying the identity $AF+BG+CH=0$ is to assume

$$A,\ B,\ C,\ F,\ G,\ H = ayz,\ bzx,\ cxy,\ fxw,\ gyw,\ hzw,$$

where the constants a, b, c, f, g, h satisfy the condition $af+bg+ch=0$; this being so, the functions A, B, C, F, G, H, and consequently the functions $A+F$, $B+G$, $C+H$ and $A-F$, $B-G$, $C-H$ each of them vanish for the four points $(y=0,\ z=0,\ w=0)$, $(z=0,\ x=0,\ w=0)$, $(x=0,\ y=0,\ w=0)$, $(x=0,\ y=0,\ z=0)$, or say the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$. It hence appears that the quartic surface

$$\Omega = a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 + f^2x^2w^2 + g^2y^2w^2 + h^2z^2w^2 = 0$$

is a quartic surface with twelve nodes; viz. it has as nodes the last-mentioned four points, the remaining four points of intersection of the surfaces

$$ayz+fxw=0,\quad bzx+gyw=0,\quad cxy+hzw=0,$$

and the remaining four points of intersection of the surfaces

$$ayz-fxw=0,\quad bzx-gyw=0,\quad cxy-hzw=0.$$

The above is the analytical theory of one of the two forms of quartic surface with twelve nodes recently established by Dr K. Rohn in a paper in the *Berichte ü. d. Verhandlungen der K. Sächsische Gesellschaft zu Leipzig*, (1884), pp. 52—60.

889.

ON A DIFFERENTIAL EQUATION AND THE CONSTRUCTION OF MILNER'S LAMP.

[From the *Proceedings of the Edinburgh Mathematical Society*, vol. v. (1887), pp. 99—101.]

WHAT sort of an equation is

$$b^3 \cos(\alpha+\theta) = a\cos\theta \int_\theta^\beta r^2 d\theta - \tfrac{2}{3}\left\{\cos\theta \int_\theta^\beta r^3 \cos\theta\, d\theta + \sin\theta \int_\theta^\beta r^3 \sin\theta\, d\theta\right\} ? \quad\ldots\ldots(1).$$

Write

$$X = \int_\theta^\beta r^2 d\theta, \quad Y = \int_\theta^\beta r^3 \cos\theta\, d\theta, \quad Z = \int_\theta^\beta r^3 \sin\theta\, d\theta \quad\ldots\ldots(2),$$

and start with the equations

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} \quad\ldots\ldots(3),$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\{a\cos\theta \,.\, X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta)\} = 0 \quad\ldots\ldots(4).$$

This last gives

$$(r - a\cos\theta)\, dr + ar\sin\theta \,.\, d\theta = 0 \quad\ldots\ldots(5),$$

and the system thus is

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} = \frac{(r-a\cos\theta)\, dr}{-ar\sin\theta} \quad\ldots\ldots(6),$$

viz. this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with four arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $=0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X=\int_\theta^\beta r^2 d\theta,\quad Y=\int_\theta^\beta r^3\cos\theta\, d\theta,\quad Z=\int_\theta^\beta r^3\sin\theta\, d\theta,$$

and further, by integration of (4),

$$L\cos\theta+M\sin\theta=a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta).$$

Here L and M denote properly determined constants: viz. the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta+M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have this value, we must have $L=b^3\cos\alpha$, $M=-b^3\sin\alpha$ (where L, M are given functions of a, β, r_0), i.e. we must have *two* given relations between a, b, α, β, r_0: or treating r_0 as a disposable constant, we must have *one* given relation between a, b, α, β.

The equation $d\theta=\dfrac{r-a\cos\theta}{-ar\sin\theta}dr$ gives $r^2-2ar\cos\theta=C$, where $C=r_0{}^2-2ar_0\cos\beta$. There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0=0$ or $r_0=2a\cos\beta$, giving $C=0$ and therefore $r=2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int\cos^2\theta\, d\theta=\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta,$$

$$\int\cos^4\theta\, d\theta=\tfrac{3}{8}\theta+\tfrac{1}{4}\sin 2\theta+\tfrac{1}{32}\sin 4\theta,$$

$$\int\cos^3\theta\sin\theta\, d\theta=-\tfrac{1}{4}\cos^4\theta.$$

And thence

$$\begin{aligned}
&a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)\\
&=\quad 4a^3\cos\theta\,\{\tfrac{1}{2}(\beta-\theta)+\tfrac{1}{4}(\sin 2\beta-\sin 2\theta)\}\\
&\quad-\tfrac{16}{3}a^3\cos\theta\,\{\tfrac{3}{8}(\beta-\theta)+\tfrac{1}{4}(\sin 2\beta-\sin 2\theta)+\tfrac{1}{32}(\sin 4\beta-\sin 4\theta)\}\\
&\quad-\tfrac{16}{3}a^3\sin\theta\,\{\qquad\qquad\qquad\qquad\qquad -\tfrac{1}{4}(\cos^4\beta-\cos^4\theta)\}\\
&=-\tfrac{1}{3}a^3\cos\theta(\sin 2\beta-\sin 2\theta)\\
&\quad-\tfrac{1}{6}a^3\cos\theta(\sin 4\beta-\sin 4\theta)\\
&\quad+\tfrac{4}{3}a^3\sin\theta(\cos^4\beta-\cos^4\theta),
\end{aligned}$$

where the terms containing β are readily reduced to $\frac{4}{3}a^3 \cos^3 \beta \sin(\theta - \beta)$; hence also the terms without β disappear of themselves: and we have

$$a \cos\theta \,.\, X - \tfrac{2}{3}(Y \cos\theta + Z \sin\theta) = \tfrac{4}{3}a^3 \cos^3\beta \,.\, \sin(\theta - \beta),$$

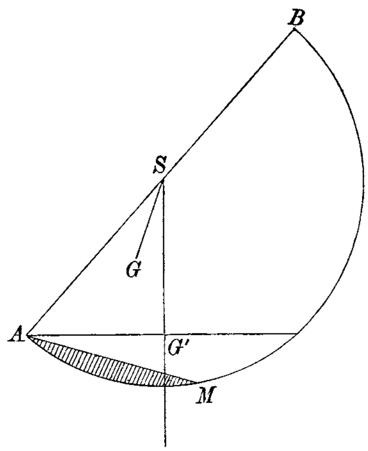
which may be put

$$= b^3 \cos(\theta + \alpha):$$

viz. this will be so if we have the *two* relations

$$\alpha = \tfrac{1}{2}\pi - \beta; \quad \text{and} \quad b^3 = -\tfrac{4}{3}a^3 \cos^3\beta.$$

I make (see figure) Milner's lamp, with a circular section, β arbitrary, but a



segment AM ($\angle SAM = \beta$) made solid. G in the line SG at right angles to AM is the C. G. of the lamp, and G' the C. G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz. in the equation $r^2 - 2ar\cos\theta = C$, we must have $C = 0$.

890.

NOTE ON THE HYDRODYNAMICAL EQUATIONS.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. xv. (1889), pp. 342—344.]

WRITING for shortness $D=\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}$, then if from the hydrodynamical equations

$$Du=\frac{d}{dx}\left(V-\frac{p}{\rho}\right),\ Dv=\frac{d}{dy}\left(V-\frac{p}{\rho}\right),\ Dw=\frac{d}{dz}\left(V-\frac{p}{\rho}\right),$$

without the aid of the equation

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

we eliminate $V-\frac{p}{\rho}$, we obtain equations not equivalent to those of Helmholtz,

$$D\xi=\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right)u,\ =\xi\frac{du}{dx}+\eta\frac{dv}{dx}+\zeta\frac{dw}{dx},\ \&c.,$$

$\left(2\xi,\ 2\eta,\ 2\zeta=\frac{dv}{dz}-\frac{dw}{dy},\ \frac{dw}{dx}-\frac{du}{dz},\ \frac{du}{dy}-\frac{dv}{dx}\right.$, as usual), but which, transforming them by means of the omitted equation, agree as they should do with his equations. But the form of the equations obtained directly by elimination as above, is an interesting one, which it is worth while to give.

We have

$$\begin{aligned}D\left(\frac{dv}{dz}-\frac{dw}{dy}\right)&=D\left(\frac{dv}{dz}-\frac{dw}{dy}\right)-\frac{d}{dz}Dv+\frac{d}{dy}Dw,\\&=\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)\left(\frac{dv}{dz}-\frac{dw}{dy}\right)\\&\quad-\frac{d}{dz}\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)\\&\quad+\frac{d}{dy}\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right),\end{aligned}$$

where the terms containing second derived functions disappear of themselves, and the expression on the right-hand is thus

$$= -\frac{du}{dz}\frac{dv}{dx} - \frac{dv}{dz}\frac{dv}{dy} - \frac{dw}{dz}\frac{dv}{dz}$$
$$+\frac{du}{dy}\frac{dw}{dx} + \frac{dv}{dy}\frac{dw}{dy} + \frac{dw}{dy}\frac{dw}{dz}.$$

Representing for shortness the Matrix

$$\left|\begin{array}{ccc} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} \end{array}\right| \text{ by } \left|\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right|, \text{ and its square by } \left|\begin{array}{ccc} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right|,$$

we have

$$\left|\begin{array}{ccc} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right| = \begin{array}{ccc|ccc} & & & \left(\frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx}\right), & \left(\frac{du}{dy}, \frac{dv}{dy}, \frac{dw}{dy}\right), & \left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}\right) \\ \hline \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} & \text{"} & \text{"} & \text{"} \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} & \text{"} & \text{"} & \text{"} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} & \text{"} & \text{"} & \text{"} \end{array}$$

viz. the combinations which enter into the foregoing formula are

$$C' = \frac{dv}{dx}\frac{du}{dz} + \frac{dv}{dy}\frac{dv}{dz} + \frac{dv}{dz}\frac{dw}{dz},$$

and

$$B'' = \frac{dw}{dx}\frac{du}{dy} + \frac{dw}{dy}\frac{dv}{dy} + \frac{dw}{dz}\frac{dw}{dy},$$

and the equation thus is $D(c' - b'') + C' - B'' = 0$; viz. the three equations are

$$D(c' - b'') + C' - B'' = 0,$$
$$D(a'' - c) + A'' - C = 0,$$
$$D(b - a') + B - A' = 0,$$

which are the equations in question.

Observe that we have

$$C' - B'' = (a', b', c'')(c, c', c'') - (a'', b'', c'')(b, b', b'')$$
$$= a'c' + b'c' + c'c'' \qquad - a''b - b'b'' - b''c'',$$

and thence, writing

$$\rho = a(c' - b'') + b(a'' - c) + c(b - a'),$$
$$= ac' - ab'' + a''b - a'c,$$

we have

$$C' - B'' + \rho = (a + b' + c'')(c' - b'') = 0,$$

if $a + b' + c'' = 0$; viz. this being so, $C' - B'' = -\rho$, or the first equation is

$$D(c' - b'') = \rho, \; = a(c' - b'') + b(a'' - c) + c(b - a'),$$

that is, $D\xi = \xi \frac{du}{dx} + \eta \frac{du}{dy} + \zeta \frac{du}{dz}$, the first equation of Helmholtz, and we thus have the equations of Helmholtz, if $a - b' + c'' = 0$, that is, if $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$.

The foregoing three equations $D(c' - b'') + C' - B'' = 0$, &c., are the quaternion equation ($\sigma = iu + jv + kw$, $\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$, $\frac{d}{dt} = D$, denotes a complete differentiation),

$$\frac{d}{dt} V\nabla\sigma = V\nabla_1\sigma_2 S\sigma_1\nabla_2$$

of Mr M^{c}Aulay's paper "Some General Theorems in Quaternion Integration," *Messenger of Mathematics*, vol. XIV. (1884), pp. 26—37; see p. 34.

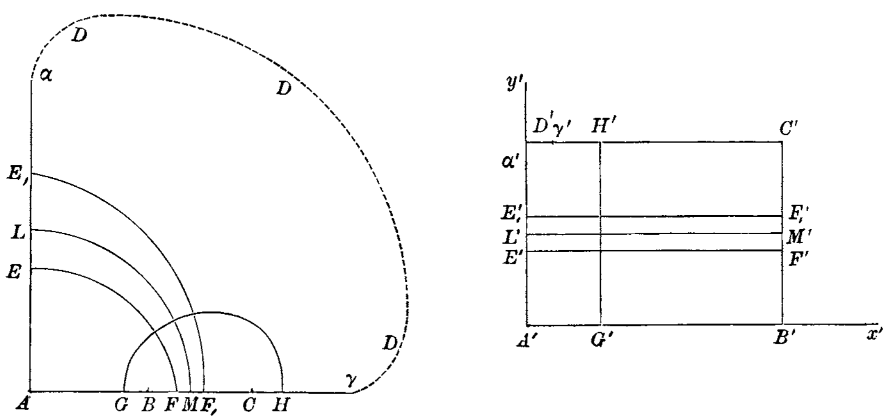
891.

ON THE BINODAL QUARTIC AND THE GRAPHICAL REPRESENTATION OF THE ELLIPTIC FUNCTIONS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XIV. (1889), pp. 484—494. Read May 6, 1889.]

I APPROACH the subject from the question of the graphical representation of the elliptic functions: assuming as usual that the modulus is real, positive, and less than unity, and to fix the ideas considering the function sn (but the like considerations are applicable to the functions cn and dn), then the equation $x + iy = \text{sn}(x' + iy')$ establishes a (1, 1) correspondence between the xy infinite quarter plane, and the $x'y'$ rectangle (sides K and K'): viz. to any given point $x + iy$, x and y each positive, there corresponds a single point $x' + iy'$, x', y' each positive and less than K, K' respectively: and conversely to any such point $x' + iy'$, there corresponds a single point $x + iy$, x and y each positive.

I draw in the $x'y'$-figure the rectangle $A'B'C'D'$ (sides K and K'), and in the xy-figure, I take on the axis of x, the points B, C where $AB = 1$, $AC = \frac{1}{k}$: and the



point D at infinity. We have thus in the $x'y'$-figure the closed curve or contour

$A'B'C'D'A'$: and corresponding hereto we have in the xy-figure the closed curve or contour $ABCDA$, viz. here D is the point at infinity considered as a line always at infinity, extending from the point at infinity on the positive part of the axis of x, to the point at infinity on the positive part of the axis of y, the contour being thus AB, BC, CD (D at infinity on the axis of x); and then D (at infinity on the axis of y) A. And thus to a point P' describing successively the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$ there corresponds a point P describing successively the lines AB, BC, CD, DA: to P' at D' there corresponds P at D, viz. this is any point at infinity from D on the axis of x to D on the axis of y. There is no real breach of continuity: in further illustration, suppose that P', instead of actually coming to D', just cuts off the corner, viz. that it passes from a point γ' on $C'D'$ to a point α' on $D'A'$ (γ', α' each of them very near to D'): then P passes from a point γ very near D on the axis of x (that is, at a great distance from A) to a point α very near D on the axis of y (that is, at a great distance from A): and to the indefinitely small arc $\gamma'\alpha'$ described by P' there corresponds the indefinitely large arc $\gamma\alpha$ described by P.

We thus see that, if P' describe any arc $E'F'$ passing from a point E' of $A'D'$ to a point F' of $B'C'$, then P will describe an arc EF passing from a point E of AD to a point F of BC: and similarly, if P' describe any arc $G'H'$ passing from a point G' of $A'B'$ to a point H' of $C'D'$, then P will describe an arc GH passing from a point G of AB to a point H of CD.

Supposing $E'F'$ is a straight line parallel to $A'x'$, that is, cutting $A'D'$ and $B'C'$ each at right angles, then EF will be an arc cutting AD and BC each at right angles: and so if $G'H'$ is a straight line parallel to $A'y'$, that is, cutting $A'B'$ and $C'D'$ each at right angles, then GH will be an arc cutting AB and CD each at right angles: and moreover, since $E'F'$ and $G'H'$ cut each other at right angles, then also EF and GH cut each other at right angles.

Supposing, as above, that $E'F'$ and $G'H'$ are straight lines, we have EF and GH each of them the arc of a special bicircular quartic: the theory was in fact established in a very elegant manner in a memoir by Siebeck, "Ueber eine Gattung von Curven vierten Grades, welche mit den elliptischen Functionen zusammenhängen," *Crelle*, t. LVII. (1860), pp. 359—370, and t. LIX. (1861), pp. 173—184.

In particular, if P' describe the line $L'M'$ lying halfway between $A'B'$ and $D'C'$ (that is, if $A'L' = \frac{1}{2}K'$), then P will describe the circular quadrant LM, radius $\frac{1}{\sqrt{k}}$, viz. in this case the bicircular quartic degenerates into a circle twice repeated: and so if P' describe successively the lines $E'F'$ and $E_1'F_1'$ equidistant from $L'M'$ ($AE' = \frac{1}{2}K' - \eta$, $AE_1' = \frac{1}{2}K' + \eta$), then P will describe the arcs EF and E_1F_1 which are the images of each other in regard to the centre A and circular quadrant LM, and which together constitute the quadrant of one and the same bicircular quartic.

A bicircular quartic is of course a binodal quartic with the circular points at infinity for the two nodes: there is no real gain of generality in considering the