799.

ON CURVILINEAR COORDINATES.

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THE present memoir is based upon Mr Warren's "Exercises in Curvilinear and Normal Coordinates," *Camb. Phil. Trans.* t. XII. (1877), pp. 455—502, and has for a principal object the establishment of the six differential equations of the second order corresponding to his six equations for normal coordinates: but the notation is different; the results are more general, inasmuch as I use throughout general curvilinear coordinates instead of his normal coordinates; and as regards my six equations for general curvilinear coordinates, the terms containing differential coefficients of the first order are presented under a different form.

If the position of a point in space is determined by the rectangular coordinates x, y, z; then p, q, r being each of them a given function of x, y, z, we have conversely x, y, z, each of them a given function of p, q, r, which are thus in effect coordinates serving to determine the position of the point, and are called curvilinear coordinates.

But it is not in the first instance necessary to regard x, y, z as rectangular coordinates, or even as Cartesian coordinates at all; we are simply concerned with the two sets of variables x, y, z and p, q, r, each variable of the one set being a given function of the variables of the other set; and, in particular, the x, y, z are regarded as being each of them a given function of the p, q, r.

Except as regards the symbols ξ, η, ζ presently mentioned, the suffixes 1, 2, 3 refer to the variables p, q, r respectively, and are used to denote differentiations in regard to these variables, viz.

$$x_1,\ x_2,\ x_3,\ x_{11},\ x_{12},\ x_{22},\ \ldots$$

are written to denote

$$\frac{dx}{dp},\ \frac{dx}{dq},\ \frac{dx}{dr},\ \frac{d^2x}{dp^2},\ \frac{d^2x}{dp\,dq},\ \frac{d^2x}{dq^2},\ \&\text{c.},$$

and so in other cases; in particular,

$$\begin{array}{lll} x_1,\ x_2,\ x_3 & \text{denote} & \dfrac{dx}{dp},\ \dfrac{dx}{dq},\ \dfrac{dx}{dr}, \\ y_1,\ y_2,\ y_3 & ,, & \dfrac{dy}{dp},\ \dfrac{dy}{dq},\ \dfrac{dy}{dr}, \\ z_1,\ z_2,\ z_3 & ,, & \dfrac{dz}{dp},\ \dfrac{dz}{dq},\ \dfrac{dz}{dr}. \end{array}$$

The minors formed with these differential coefficients are denoted by suffixed letters ξ, η, ζ, thus

$$\begin{array}{lllll} \xi_1,\ \xi_2,\ \xi_3 & \text{denote} & y_2z_3 - y_3z_2, & y_3z_1 - y_1z_3, & y_1z_2 - y_2z_1, \\ \eta_1,\ \eta_2,\ \eta_3 & ,, & z_2x_3 - z_3x_2, & z_3x_1 - z_1x_3, & z_1x_2 - z_2x_1, \\ \zeta_1,\ \zeta_2,\ \zeta_3 & ,, & x_2y_3 - x_3y_2, & x_3y_1 - x_1y_3, & x_1y_2 - x_2y_1, \end{array}$$

so that, as regards these letters ξ, η, ζ, the suffixes do *not* denote differentiations.

The determinant $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$ is put $= L$:

and the symbols (a, b, c, f, g, h), (A, B, C, F, G, H) are defined as follows:

$$\begin{array}{ll} \mathrm{a} = x_1^2 + y_1^2 + z_1^2, & A = \xi_1^2 + \eta_1^2 + \zeta_1^2, \\ \mathrm{b} = x_2^2 + y_2^2 + z_2^2, & B = \xi_2^2 + \eta_2^2 + \zeta_2^2, \\ \mathrm{c} = x_3^2 + y_3^2 + z_3^2, & C = \xi_3^2 + \eta_3^2 + \zeta_3^2, \\ \mathrm{f} = x_2x_3 + y_2y_3 + z_2z_3, & F = \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3, \\ \mathrm{g} = x_3x_1 + y_3y_1 + z_3z_1, & G = \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1, \\ \mathrm{h} = x_1x_2 + y_1y_2 + z_1z_2, & H = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2. \end{array}$$

We have then, further,

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = L, \qquad \begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = L^2,$$

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix} = L^2, \qquad \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \frac{1}{L^2},$$

$$(A, B, C, F, G, H) = (\mathrm{bc} - \mathrm{f}^2,\ \mathrm{ca} - \mathrm{g}^2,\ \mathrm{ab} - \mathrm{h}^2,\ \mathrm{gh} - \mathrm{af},\ \mathrm{hf} - \mathrm{bg},\ \mathrm{fg} - \mathrm{ch}),$$

$$L^2(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}) = (BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH),$$

which equations are at once proved, and are fundamental ones in the theory.

It is convenient to add that we have

$$\begin{vmatrix} \frac{dp}{dx}, & \frac{dp}{dy}, & \frac{dp}{dz} \\ \frac{dq}{dx}, & \frac{dq}{dy}, & \frac{dq}{dz} \\ \frac{dr}{dx}, & \frac{dr}{dy}, & \frac{dr}{dz} \end{vmatrix} = \frac{1}{L} \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix},$$

that is, $\frac{dp}{dx} = \frac{1}{L}\xi_1$, &c.; and, further,

$$dx^2 + dy^2 + dz^2 = (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h} \between dp,\ dq,\ dr)^2,$$

$$dp^2 + dq^2 + dr^2 = \frac{1}{L^2}(A,\ B,\ C,\ F,\ G,\ H \between dx,\ dy,\ dz)^2.$$

Differentiating the values of a, b, c, f, g, h, we have $\tfrac{1}{2}\mathrm{a}_1 = x_1x_{11} + y_1y_{11} + z_1z_{11}$, which may be written in the abbreviated form $\tfrac{1}{2}\mathrm{a}_1 = 1\,.\,11$; similarly

$$\mathrm{f}_1 = x_2x_{13} + y_2y_{13} + z_2z_{13} + x_3x_{12} + y_3y_{12} + z_3z_{12},$$

which in like manner may be written $\mathrm{f}_1 = 2\,.\,13 + 3\,.\,12$, and so in other cases; observe that, in the duad part of any symbol, the order of the numbers is immaterial, $2\,.\,13 = 2\,.\,31$. The whole system of equations is

$$\begin{array}{lll} \tfrac{1}{2}\mathrm{a}_1 = 1\,.\,11, & \tfrac{1}{2}\mathrm{a}_2 = 1\,.\,12, & \tfrac{1}{2}\mathrm{a}_3 = 1\,.\,13, \\ \tfrac{1}{2}\mathrm{b}_1 = 2\,.\,12, & \tfrac{1}{2}\mathrm{b}_2 = 2\,.\,22, & \tfrac{1}{2}\mathrm{b}_3 = 2\,.\,23, \\ \tfrac{1}{2}\mathrm{c}_1 = 3\,.\,13, & \tfrac{1}{2}\mathrm{c}_2 = 3\,.\,23, & \tfrac{1}{2}\mathrm{c}_3 = 3\,.\,33, \\ \mathrm{f}_1 = 2\,.\,13 + 3\,.\,12, & \mathrm{f}_2 = 2\,.\,23 + 3\,.\,22, & \mathrm{f}_3 = 2\,.\,33 + 3\,.\,23, \\ \mathrm{g}_1 = 3\,.\,11 + 1\,.\,13, & \mathrm{g}_2 = 3\,.\,12 + 1\,.\,23, & \mathrm{g}_3 = 3\,.\,13 + 1\,.\,33, \\ \mathrm{h}_1 = 1\,.\,12 + 2\,.\,11, & \mathrm{h}_2 = 1\,.\,22 + 2\,.\,12, & \mathrm{h}_3 = 1\,.\,23 + 2\,.\,13. \end{array}$$

These may also be written

$$\begin{array}{lll} 1\,.\,11 = \tfrac{1}{2}\mathrm{a}_1, & 1\,.\,22 = \mathrm{h}_2 - \tfrac{1}{2}\mathrm{b}_1, & 1\,.\,33 = \mathrm{g}_3 - \tfrac{1}{2}\mathrm{c}_1, \\ 2\,.\,11 = \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2, & 2\,.\,22 = \tfrac{1}{2}\mathrm{b}_2, & 2\,.\,33 = \mathrm{f}_3 - \tfrac{1}{2}\mathrm{c}_2, \\ 3\,.\,11 = \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3, & 3\,.\,22 = \mathrm{f}_2 - \tfrac{1}{2}\mathrm{b}_3, & 3\,.\,33 = \tfrac{1}{2}\mathrm{c}_3, \\ 1\,.\,23 = \tfrac{1}{2}(-\mathrm{f}_1 + \mathrm{g}_2 + \mathrm{h}_3), & 1\,.\,31 = \tfrac{1}{2}\mathrm{a}_3, & 1\,.\,12 = \tfrac{1}{2}\mathrm{a}_2, \\ 2\,.\,23 = \tfrac{1}{2}\mathrm{b}_3, & 2\,.\,31 = \tfrac{1}{2}(\mathrm{f}_1 - \mathrm{g}_2 + \mathrm{h}_3), & 2\,.\,12 = \tfrac{1}{2}\mathrm{b}_1, \\ 3\,.\,23 = \tfrac{1}{2}\mathrm{c}_2, & 3\,.\,31 = \tfrac{1}{2}\mathrm{c}_1, & 3\,.\,12 = \tfrac{1}{2}(\mathrm{f}_1 + \mathrm{g}_2 - \mathrm{h}_3). \end{array}$$

It is to be observed that we can, from each system of three equations, express a set of second differential coefficients of the x, y, z in terms of the first differential coefficients of the a, b, c, f, g, h; thus the three equations containing 11, written at length, are

$$\begin{aligned} x_1 \,.\, x_{11} + y_1 \,.\, y_{11} + z_1 \,.\, z_{11} &= \tfrac{1}{2}\mathrm{a}_1, \\ x_2 \;\;,, \;\; + y_2 \;\;,, \;\; + z_2 \;\;,, \;\; &= \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2, \\ x_3 \;\;,, \;\; + y_3 \;\;,, \;\; + z_3 \;\;,, \;\; &= \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3, \end{aligned}$$

three linear equations for the determination of x_{11}, y_{11}, z_{11}; hence for x_{11} we have

$$x_{11} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \tfrac{1}{2}\mathrm{a}_1 (y_2 z_3 - y_3 z_2) + (\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2)(y_3 z_1 - y_1 z_3) + (\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3)(y_1 z_2 - y_2 z_1),$$

or, what is the same thing,

$$L x_{11} = \tfrac{1}{2}\mathrm{a}_1 \,.\, \xi_1 + (\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2)\,\xi_2 + (\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3)\,\xi_3,$$

and so for y_{11}, z_{11}; or if (as in the sequel) we desire the value of a linear function $\alpha x_{11} + \beta y_{11} + \gamma z_{11}$, calling this for a moment $\square$, we join to the foregoing a new equation

$$\alpha x_{11} + \beta y_{11} + \gamma z_{11} = \square,$$

and then, eliminating the three quantities, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}\mathrm{a}_1 \\ x_2, & y_2, & z_2, & \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2 \\ x_3, & y_3, & z_3, & \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3 \end{vmatrix} = 0,$$

giving $L\square$ as a linear function of $\tfrac{1}{2}\mathrm{a}_1$, $\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2$, $\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3$.

We can in like manner form the expressions for the second differential coefficients of the a, b, c, f, g, h: these will of course contain third differential coefficients of the x, y, z.

Writing down only what is wanted, we have

$$\begin{aligned} \tfrac{1}{2}\mathrm{a}_{22} &= 12 \,.\, 12 + 1 \,.\, 122, \\ \tfrac{1}{2}\mathrm{b}_{11} &= 12 \,.\, 12 + 2 \,.\, 112, \\ \mathrm{h}_{12} &= 12 \,.\, 12 + 11 \,.\, 22 + 1 \,.\, 122 + 2 \,.\, 112, \end{aligned}$$

where of course $12 \,.\, 12$ denotes $x^2_{12} + y^2_{12} + z^2_{12}$, $1 \,.\, 122$ denotes $x_1 \,.\, x_{122} + y_1 \,.\, y_{122} + z_1 \,.\, z_{122}$, and so in other cases: it follows that

$$\tfrac{1}{2}(\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12}) = 12 \,.\, 12 - 11 \,.\, 22,$$

so that the third differential coefficients of x, y, z, which enter into the expression of $\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12}$ destroy each other, and this combination contains really only second differential coefficients of x, y, z.

Similarly

$$\begin{aligned}
\mathrm{f}_{31} &= \quad 31\,.\,23 + 33\,.\,12 + \quad 3\,.\,123 \qquad\qquad + 2\,.\,133,\\
\mathrm{g}_{23} &= \quad 31\,.\,23 + 33\,.\,12 + \quad 3\,.\,123 + 1\,.\,233\\
\mathrm{c}_{12} &= 2\,.\,31\,.\,23 \qquad\qquad + 2\,.\,3\,.\,123\\
\mathrm{h}_{33} &= 2\,.\,31\,.\,23 \qquad\qquad\qquad\qquad + 1\,.\,233 + 2\,.\,133,
\end{aligned}$$

and thence

$$\mathrm{f}_{31} + \mathrm{g}_{23} - \mathrm{c}_{12} - \mathrm{h}_{33} = -2\,(31\,.\,23 - 33\,.\,12),$$

so that here again we have a combination containing only second differential coefficients of x, y, z.

There are thus, in all, the six combinations

$$\begin{aligned}
&\mathrm{b}_{33} + \mathrm{c}_{22} - 2\mathrm{f}_{23} \ldots\ldots\ldots\ldots(\mathfrak{A}),\\
&\mathrm{c}_{11} + \mathrm{a}_{33} - 2\mathrm{g}_{31} \ldots\ldots\ldots\ldots(\mathfrak{B}),\\
&\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12} \ldots\ldots\ldots\ldots(\mathfrak{C}),\\
&\mathrm{g}_{12} + \mathrm{h}_{31} - \mathrm{a}_{23} - \mathrm{f}_{11} \ldots\ldots\ldots\ldots(\mathfrak{F}),\\
&\mathrm{h}_{23} + \mathrm{f}_{12} - \mathrm{b}_{31} - \mathrm{g}_{22} \ldots\ldots\ldots\ldots(\mathfrak{G}),\\
&\mathrm{f}_{31} + \mathrm{g}_{23} - \mathrm{c}_{12} - \mathrm{h}_{33} \ldots\ldots\ldots\ldots(\mathfrak{H}),
\end{aligned}$$

each really containing only second differential coefficients of x, y, z; and we thus understand how each of these combinations may be expressible in terms of the first differential coefficients of (a, b, c, f, g, h). We have, in fact, for thus expressing these combinations the six equations called $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$, $(\mathfrak{F})$, $(\mathfrak{G})$, $(\mathfrak{H})$ about to be obtained, and which are the generalisations of Warren's six equations for normal coordinates.

I consider the several determinants of the form

$$\begin{vmatrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

in all 18, since the suffixes for the top row may be 11, 22, 33, 23, 31, 12, and those for the second and third rows 2, 3; 3, 1; or 1, 2. Each determinant is a linear function of second differential coefficients of x, y, z (thus the determinant written down is $=\xi_3 x_{11} + \eta_3 y_{11} + \zeta_3 z_{11}$), and as such it can, by what precedes, be expressed by means of the first differential coefficients of the a, b, c, f, g, h. Thus, if the determinant above written down be called $\square$, writing ξ_3, η_3, ζ_3 for α, β, γ, we have

$$\begin{vmatrix} \xi_3, & \eta_3, & \zeta_3, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}\mathrm{a}_1 \\ x_2, & y_2, & z_2, & \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2 \\ x_3, & y_3, & z_3, & \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3 \end{vmatrix} = 0,$$

that is,

$$L\,.\,\square = -\begin{vmatrix} & \xi_3, & \eta_3, & \zeta_3 \\ \tfrac{1}{2}\mathrm{a}_1, & x_1, & y_1, & z_1 \\ \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2, & x_2, & y_2, & z_2 \\ \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3, & x_3, & y_3, & z_3 \end{vmatrix},$$

where the right-hand side is

$$= \tfrac{1}{2}a_1 \,.\, \xi_1\xi_3 + \eta_1\eta_3 + \zeta_1\zeta_3$$
$$- (h_1 - \tfrac{1}{2}a_2) \,.\, -(\xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3)$$
$$+ (g_1 - \tfrac{1}{2}a_3) \,.\, \xi_3^2 + \eta_3^2 + \zeta_3^2,$$

which is

$$= G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

or, as this might be written,

$$= (G,\ F,\ C \between a_1,\ h_1,\ g_1) - \tfrac{1}{2}(G,\ F,\ C \between a_1,\ a_2,\ a_3).$$

Retaining the former form, the result is

$$L \,.\, \square = G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

and it is now very easy to write down the complete system of the 18 equations; viz. if the determinant above written down be called 11 . 1 . 2, and so in other cases, then we have

	2 . 3	A	H	G
	3 . 1	H	B	F
	1 . 2	G	F	C
L . 11	„ =	$\tfrac{1}{2}a_1$	$h_1 - \tfrac{1}{2}a_2$	$g_1 - \tfrac{1}{2}a_3$
„ 22	„ =	$h_2 - \tfrac{1}{2}b_1$	$\tfrac{1}{2}b_2$	$f_2 - \tfrac{1}{2}b_3$
„ 33	„ =	$g_3 - \tfrac{1}{2}c_1$	$f_3 - \tfrac{1}{2}c_2$	$\tfrac{1}{2}c_3$
„ 23	„ =	$\tfrac{1}{2}(-f_1 + g_2 + h_3)$	$\tfrac{1}{2}b_3$	$\tfrac{1}{2}c_2$
„ 31	„ =	$\tfrac{1}{2}a_3$	$\tfrac{1}{2}(f_1 - g_2 + h_3)$	$\tfrac{1}{2}c_1$
„ 12	„ =	$\tfrac{1}{2}a_2$	$\tfrac{1}{2}b_1$	$\tfrac{1}{2}(f_1 + g_2 - h_3)$:

read for instance

$$L \,.\, 11 \,.\, 1 \,.\, 2 = G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

the equation obtained above.

There are eighteen functions as shown by the following diagram:

	2 . 3	3 . 1	1 . 2
$(22)(33) - (23)^2$	2 . 3		
$(33)(11) - (31)^2$	3 . 1		
$(11)(22) - (12)^2$	1 . 2		
$(31)(12) - (11)\ (23)$	2 . 3		
$(12)(23) - (22)\ (31)$	3 . 1		
$(23)(31) - (33)\ (12)$	1 . 2,		

viz. in any line of the diagram the bracketed duads may belong to any one at pleasure, but all to the same, of the three pairs 2.3, 3.1, 1.2; thus the first line might be

$$(22.3.1)(33.3.1)-(23.3.1)^2,$$

or, instead of the 3.1, we might have 2.3 or 1.2. But of the 18 functions I distinguish 6, viz. those in which for the six lines respectively the pairs are 2.3, 3.1, 1.2, 2.3, 3.1, 1.2, as shown in the diagram. Each of these six functions can be obtained under two different forms, and by equating these we have the equations ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$), ($\mathfrak{F}$), ($\mathfrak{G}$), ($\mathfrak{H}$), before referred to; thus ($\mathfrak{C}$) is obtained by equating two different forms of the function $(11.1.2)(22.1.2)-(12.1.2)^2$; and ($\mathfrak{H}$) is obtained by equating two different forms of the function $(23.1.2)(31.1.2)-(33.1.2)(12.1.2)$.

The determinants 11.1.2, &c., may be denoted by accented letters a, b, c, f, g, h, as follows:

$$\left|\begin{matrix} x_{11}, & x_{22}, & x_{33}, & x_{23}, & x_{31}, & x_{12} \\ y_{11}, & y_{22}, & y_{33}, & y_{23}, & y_{31}, & y_{12} \\ z_{11}, & z_{22}, & z_{33}, & z_{23}, & z_{31}, & z_{12} \end{matrix}\right| \begin{matrix} x_2, & x_3 \\ y_2, & y_3 \\ z_2, & z_3 \end{matrix} = \text{a}', \text{b}', \text{c}', \text{f}', \text{g}', \text{h}',$$

$$\left|\begin{matrix} \quad ,, \quad & \quad ,, \quad \\ & \\ & \end{matrix}\right| \begin{matrix} x_3, & x_1 \\ y_3, & y_1 \\ z_3, & z_1 \end{matrix} = \text{a}'', \text{b}'', \text{c}'', \text{f}'', \text{g}'', \text{h}'',$$

$$\left|\begin{matrix} \quad ,, \quad & \quad ,, \quad \\ & \\ & \end{matrix}\right| \begin{matrix} x_1, & x_2 \\ y_1, & y_2 \\ z_1, & z_2 \end{matrix} = \text{a}''', \text{b}''', \text{c}''', \text{f}''', \text{g}''', \text{h}''';$$

viz.

$$\left|\begin{matrix} x_{11}, & x_2, & x_3 \\ y_{11}, & y_2, & y_3 \\ z_{11}, & z_2, & z_3 \end{matrix}\right| = \text{a}', \text{\&c.}$$

In this notation, ($\mathfrak{C}$) is obtained by equating two values of $\text{a}'''\text{b}'''-(\text{h}''')^2$, and ($\mathfrak{H}$) by equating two values of $\text{f}'''\text{g}'''-\text{c}'''\text{h}'''$.

The forms which I call the second are those given by the immediate substitution of the foregoing values of (11.1.2), &c.; to obtain the first forms, I proceed to calculate those of ($\mathfrak{C}$) and ($\mathfrak{H}$).

Forming by the ordinary rule the product of the determinants (11.1.2) and (22.1.2), which are

$$\left|\begin{matrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right| \text{ and } \left|\begin{matrix} x_{22}, & y_{22}, & z_{22} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right|,$$

this is

$$\begin{vmatrix} 11.22, & 1.11, & 2.11 \\ 1.22, & 1.1\ , & 1.2 \\ 2.22, & 1.2\ , & 2.2 \end{vmatrix},$$

where 11.22 denotes $x_{11}x_{22} + y_{11}y_{22} + z_{11}z_{22}$, and the like for the other symbols. In like manner, the square of the determinant $(12.1.2)$, that is, of

$$\begin{vmatrix} x_{12}, & y_{12}, & z_{12} \\ x_1\ , & y_1\ , & z_1 \\ x_2\ , & y_2\ , & z_2 \end{vmatrix},$$

is

$$\begin{vmatrix} 12.12, & 1.12, & 2.12 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 2.1\ , & 2.2 \end{vmatrix},$$

or, observing that in the two resulting determinants the terms 11.22 and 12.12 are multiplied by the same factor, the expression for the difference gives

$$(11.1.2)(22.1.2) - (12.1.2)^2$$

$$= (11.22 - 12.12) \begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix} + \begin{vmatrix} 0\ , & 1.11, & 2.11 \\ 1.22, & 1.1\ , & 1.2 \\ 2.22, & 1.2\ , & 2.2 \end{vmatrix} - \begin{vmatrix} 0\ , & 1.12, & 2.12 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 1.2\ , & 2.2 \end{vmatrix},$$

containing $11.22 - 12.12$, which, by what precedes, is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12});$$

the other terms are also known, viz. the whole value is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2) + \begin{vmatrix} 0\ , & \tfrac{1}{2}a_1, & h_1 - \tfrac{1}{2}a_2 \\ h_2 - \tfrac{1}{2}b_1, & a, & h \\ \tfrac{1}{2}b_2, & h, & b \end{vmatrix} - \begin{vmatrix} 0\ , & \tfrac{1}{2}a_2, & \tfrac{1}{2}b_1 \\ \tfrac{1}{2}a_2, & a\ , & h \\ \tfrac{1}{2}b_1, & h\ , & b \end{vmatrix},$$

which is

$$\begin{aligned} = &-\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2) \\ &+ a\,(\tfrac{1}{4}b_1^2 - \tfrac{1}{2}b_2h_1 + \tfrac{1}{4}a_2b_2) \\ &+ b\,(\tfrac{1}{4}a_2^2 - \tfrac{1}{2}a_1h_2 + \tfrac{1}{4}a_1b_1) \\ &+ h\,(\tfrac{1}{4}a_1b_2 - \tfrac{1}{4}a_2b_1 + h_1h_2 - \tfrac{1}{2}a_2h_2 - \tfrac{1}{2}b_1h_1), \end{aligned}$$

$= (ab - h^2)\,k$, where k is the measure of curvature.

In the same way, we have

$$(23.1.2)(31.1.2) - (33.1.2)(12.1.2)$$

$$= \begin{vmatrix} 23.31, & 1.31, & 2.31 \\ 1.23, & 1.1\ , & 1.2 \\ 2.23, & 1.2\ , & 2.2 \end{vmatrix} - \begin{vmatrix} 12.33, & 1.33, & 2.33 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 1.2\ , & 2.2 \end{vmatrix},$$

which is

$$=(23.31-12.33)\begin{vmatrix}1.1, & 1.2\\ 1.2, & 2.2\end{vmatrix}+\begin{vmatrix}0\;, & 1.13, & 2.13\\ 1.23, & 1.1\;, & 1.2\\ 2.23, & 1.2\;, & 2.2\end{vmatrix}-\begin{vmatrix}0\;, & 1.33, & 2.33\\ 1.12, & 1.1\;, & 1.2\\ 2.12, & 1.2\;, & 2.2\end{vmatrix},$$

containing $23.31-12.33$, which by what precedes is

$$=-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33});$$

the other terms are also known, and the value is

$$=-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33})(ab-h^2)$$
$$+\begin{vmatrix}0 & , \tfrac{1}{2}a_3, & \tfrac{1}{2}(f_1-g_2+h_3)\\ \tfrac{1}{2}(-f_1+g_2+h_3), & a\;, & h\\ \tfrac{1}{2}b_3 & , h\;, & b\end{vmatrix}-\begin{vmatrix}0\;, & g_3-\tfrac{1}{2}c_1, & f_3-\tfrac{1}{2}c_2\\ \tfrac{1}{2}a_2\;, & a\;, & h\\ \tfrac{1}{2}b_1, & h\;, & b\end{vmatrix},$$

or finally this is

$$\begin{aligned}=&-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33})(ab-h^2)\\ &+a\,(\tfrac{1}{2}b_1f_3-\tfrac{1}{4}b_3f_1-\tfrac{1}{4}b_1c_2+\tfrac{1}{4}b_3g_2-\tfrac{1}{4}b_3h_3)\\ &+b\,(\tfrac{1}{2}a_2g_3-\tfrac{1}{4}a_3g_2-\tfrac{1}{4}a_2c_1+\tfrac{1}{4}a_3f_1-\tfrac{1}{4}a_3h_3)\\ &+h\,\{\tfrac{1}{4}a_3b_3+\tfrac{1}{4}b_1c_1+\tfrac{1}{4}a_2c_2-\tfrac{1}{2}b_1g_3-\tfrac{1}{2}a_2f_3+\tfrac{1}{4}h_3{}^2-\tfrac{1}{4}(f_1-g_2)^2\}.\end{aligned}$$

The remaining four of the six functions can of course be obtained from the two just found by a cyclical interchange of the letters and suffix-numbers 1, 2, 3, and it is not worth while to write down the values.

The two values of $(11.1.2)(22.1.2)-(12.1.2)^2$ are

$$\begin{aligned}-(a_{22}+b_{11}-2h_{12})(ab-h^2)&+a\,(\tfrac{1}{4}b_1{}^2-\tfrac{1}{2}b_2h_1+\tfrac{1}{4}a_2b_2)\\ &+b\,(\tfrac{1}{4}a_2{}^2-\tfrac{1}{2}a_1h_2+\tfrac{1}{4}a_1b_1)\\ &+h\,\{\tfrac{1}{4}(a_1b_2-a_2b_1)+h_1h_2-\tfrac{1}{2}b_1h_1-\tfrac{1}{2}a_2h_2),\end{aligned}$$

and

$$\begin{aligned}&\{G.\tfrac{1}{2}a_1+F(h_1-\tfrac{1}{2}a_2)+C(g_1-\tfrac{1}{2}a_3)\}.\{G(h_2-\tfrac{1}{2}b_1)+F.\tfrac{1}{2}b_2+C(f_2-\tfrac{1}{2}b_3)\}\\ &\qquad-\{G.\tfrac{1}{2}a_2+F.\tfrac{1}{2}b_1+C.\tfrac{1}{2}(f_1+g_2-h_3)\}^2,\end{aligned}$$

where, in the first value, for a, b, h, $ab-h^2$ we must write $L^{-2}(BC-F^2)$, $L^{-2}(CA-G^2)$, $L^{-2}(FG-CH)$, and L^2C; making this change, multiplying by 4 and equating, we obtain

$$\begin{aligned}&-2L^2C(a_{22}+b_{11}-2h_{12})\\ &+(BC-F^2)(b_1{}^2-2b_2h_1+a_2b_2)\\ &+(CA-G^2)(a_2{}^2-2a_1h_2+a_1b_1)\\ &+(FG-CH)(a_1b_2-a_2b_1+4h_1h_2-2b_1h_1-2a_2h_2)\\ &-\{Ga_1+F(2h_1-a_2)+C(2g_1-a_3)\}\{G(2h_2-b_1)+Fb_2+C(2f_2-b_3)\}\\ &+\{Ga_2+Fb_1+C(f_1+g_2-h_3)\}^2 \qquad =0.\end{aligned}$$

Developing the fourth and fifth lines, it appears that in this expression the coefficients of F^2, G^2 and FG each of them vanish; the whole equation is thus divisible by C, and omitting this factor throughout, the equation becomes

$$\begin{aligned}0 = &- 2L^2(a_{22} + b_{11} - 2h_{12})\\ &+ A\,(a_2{}^2 - 2a_1h_2 + a_1b_1)\\ &+ B\,(b_1{}^2 - 2b_2h_1 + a_2b_2)\\ &+ C\,\{-a_3b_3 + 2a_3f_2 + 2b_3g_1 - 4f_2g_1 + (f_1 + g_2 - h_3)^2\}\\ &+ F\,\{(a_3b_2 - a_2b_3) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) + 2(a_2f_2 + b_1f_1 - 2h_1f_2)\}\\ &+ G\,\{a_1b_3 - a_3b_1 + 2(a_2f_1 - a_1f_2) + 2(a_3h_2 - a_2h_3) + 2(a_2g_2 + b_1g_1 - 2g_1h_2)\}\\ &+ H\,\{2(a_2h_2 + b_1h_1 - 2h_1h_2)\};\end{aligned}$$

this is substantially the required equation ($\mathfrak{C}$), but the form of it may be greatly simplified.

Forming the identity,

$$\begin{aligned}0 = 2L[(a_2 - h_1)L_2 + (b_1 - h_2)L_1] = &- A\,(a_1b_1 - a_1h_2 - a_2h_1 + a_2{}^2)\\ &- B\,(b_1{}^2 - b_1h_2 - b_2h_1 + a_2b_2)\\ &- C\,(b_1c_1 + a_2c_2 - c_1h_2 - c_2h_1)\\ &- 2F\,(b_1f_1 + a_2f_2 - f_1h_2 - f_2h_1)\\ &- 2G\,(b_1g_1 + a_2g_2 - g_1h_2 - g_2h_1)\\ &- 2H\,(a_2h_2 + b_1h_1 - 2h_1h_2),\end{aligned}$$

we add hereto the last preceding equation; the coefficients of A, B, F, G, H thus assume new and simple forms, but the coefficient of C requires a further transformation.

Assume

$$\begin{aligned}\Omega = &(b_1c_1 - f_1{}^2) + (c_2a_2 - g_2{}^2) + (a_3b_3 - h_3{}^2)\\ &+ (g_2h_3 + g_3h_2 - a_2f_3 - a_3f_2) + (h_3f_1 + h_1f_3 - b_1g_3 - b_3g_1) + (f_1g_2 + f_2g_1 - c_1h_2 - c_2h_1);\end{aligned}$$

then, if we add to the equation C multiplied by this value and subtract $C\Omega$, the coefficient of C takes its proper form, and the equation is

$$\begin{aligned}&- 2L^2(a_{22} + b_{11} - 2h_{12})\\ &+ 2L[(b_1 - h_2)L_1 + (a_2 - h_1)L_2] - C\Omega\\ &+ A\,\{-(ah_2 - a_2h)\}\\ &+ B\,\{(b_1h_2 - b_2h_1)\}\\ &+ C\,\{-(a_2f_3 - a_3f_2) + (b_3g_1 - b_1g_3) - (g_2h_3 - g_3h_2) - (h_3f_1 - h_1f_3) + 3(f_1g_2 - f_2g_1)\}\\ &+ F\,\{-(a_2b_3 - a_3b_2) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) - 2(h_1f_2 - h_2f_1)\}\\ &+ G\,\{-(a_3b_1 - a_1b_3) - 2(a_1f_2 - a_2f_1) - 2(a_2h_3 - a_3h_2) - 2(g_1h_2 - g_2h_1)\}\\ &+ H\,\{-(a_1b_2 - a_2b_1)\}\\ &= 0;\end{aligned}$$