706.

ON THE DISTRIBUTION OF ELECTRICITY ON TWO SPHERICAL SURFACES.

[From the *Philosophical Magazine*, vol. v. (1878), pp. 54—60.]

In the two memoirs "Sur la distribution de l'électricité à la surface des corps conducteurs," *Mém. de l'Inst.* 1811, Poisson considers the question of the distribution of electricity upon two spheres: viz. if the radii be a, b, and the distance of the centres be c (where $c > a + b$, the spheres being exterior to each other), and the potentials within the two spheres respectively have the constant values h and g, then—for Poisson's $f\left(\frac{x}{a}\right)$ writing $\phi(x)$, and for his $F\left(\frac{x}{b}\right)$ writing $\Phi(x)$—the question depends on the solution of the functional equations

$$a\phi(x) + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right) = h,$$

$$\frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right) + b\Phi(x) = g,$$

where of course the x of either equation may be replaced by a different variable.

It is proper to consider the meaning of these equations: for a point on the axis, at the distance x from the centre of the first sphere, or say from the point A, the potential of the electricity on this spherical surface is $a\phi x$ or $\frac{a^2}{x}\phi\left(\frac{a^2}{x}\right)$, according as the point is interior or exterior; and, similarly, if x now denote the distance from the centre of the second sphere (or, say, from the point B), then the potential of the electricity on this spherical surface is $b\Phi x$ or $\frac{b^2}{x}\Phi\left(\frac{b^2}{x}\right)$, according as the point is interior or exterior; $\phi(x)$ is thus the same function of (x, a, b) that $\Phi(x)$ is of

(x, b, a). Hence, first, for a point interior to the sphere A, if x denote the distance from A, and therefore $c-x$ the distance of the same point from B, the potential of the point in question is

$$=a\phi x+\frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right);$$

and, secondly, for a point interior to the sphere B, if x denote the distance from B and therefore $c-x$ the distance of the same point from A, the potential of the point is

$$=\frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right)+b\Phi(x).$$

The two equations thus express that the potentials of a point interior to A and of a point interior to B are $=h$ and g respectively.

It is to be added that the potential of an exterior point, distances from the points A and $B = x$ and $c-x$ respectively, is

$$=\frac{a^2}{x}\phi\left(\frac{a^2}{x}\right)+\frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right);$$

and that, by the known properties of Legendre's coefficients, when the potential upon an axial point is given, it is possible to pass at once to the expression for the potential of a point not on the axis, and also to the expression for the electrical density at a point on the two spherical surfaces respectively. The determination of the functions $\phi(x)$ and $\Phi(x)$ gives thus the complete solution of the question.

I obtain Poisson's solution by a different process as follows:—Consider the two functions

$$\frac{a^2(c-x)}{c^2-b^2-cx},\quad =\frac{\mathrm{a}x+\mathrm{b}}{\mathrm{c}x+\mathrm{d}},\text{ suppose,}$$

and

$$\frac{b^2(c-x)}{c^2-a^2-cx},\quad =\frac{\alpha x+\beta}{\gamma x+\delta},\text{ suppose;}$$

and let the nth functions be

$$\frac{\mathrm{a}_n x+\mathrm{b}_n}{\mathrm{c}_n x+\mathrm{d}_n}\text{ and }\frac{\alpha_n x+\beta_n}{\gamma_n x+\delta_n}$$

respectively.

Observing that the values of the coefficients are

$$\begin{pmatrix}\mathrm{a}, & \mathrm{b}\\ \mathrm{c}, & \mathrm{d}\end{pmatrix}=\begin{pmatrix}-a^2, & a^2c\\ -c, & c^2-b^2\end{pmatrix},\text{ and }\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}=\begin{pmatrix}-b^2, & b^2c\\ -c, & c^2-a^2\end{pmatrix},$$

so that we have

$$\mathrm{a}+\mathrm{d}=\alpha+\delta,\quad =c^2-a^2-b^2,\qquad \mathrm{ad}-\mathrm{bc}=\alpha\delta-\beta\gamma,\quad =a^2b^2,$$

and consequently that the two equations

$$\frac{(\lambda+1)^2}{\lambda}=\frac{(\mathrm{a}+\mathrm{d})^2}{\mathrm{ad}-\mathrm{bc}},\quad \frac{(\lambda+1)^2}{\lambda}=\frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma},$$

are in fact one and the same equation

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(c^2-a^2-b^2)^2}{a^2b^2}$$

for the determination of λ, then (by a theorem which [686, 687] I have recently obtained) we have the following equations for the coefficients

$$\begin{pmatrix} \mathrm{a}_n, & \mathrm{b}_n \\ \mathrm{c}_n, & \mathrm{d}_n \end{pmatrix}, \quad \begin{pmatrix} \alpha_n, & \beta_n \\ \gamma_n, & \delta_n \end{pmatrix}$$

of the nth functions; viz. these are:—

$$\mathrm{a}_n x + \mathrm{b}_n = \frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(\mathrm{a}x+\mathrm{b}) + (\lambda^n-\lambda)(-\mathrm{d}x+\mathrm{b})\},$$

$$\mathrm{c}_n x + \mathrm{d}_n = \quad ,, \quad ,, \quad \{(\lambda^{n+1}-1)(\mathrm{c}x+\mathrm{d}) + (\lambda^n-\lambda)(\ \ \mathrm{c}x-\mathrm{a})\};$$

and similarly

$$\alpha_n x + \beta_n = \frac{1}{\lambda^2-1}\left(\frac{\alpha+\delta}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(\alpha x+\beta) + (\lambda^n-\lambda)(-\delta x+\beta)\},$$

$$\gamma_n x + \delta_n = \quad ,, \quad ,, \quad \{(\lambda^{n+1}-1)(\gamma x+\delta) + (\lambda^n-\lambda)(\ \ \gamma x-\alpha)\}.$$

Observe that these equations give, as they ought to do,

$$\mathrm{a}_0 x + \mathrm{b}_0 = x, \quad \mathrm{c}_0 x + \mathrm{d}_0 = 1, \quad \mathrm{a}_1 x + \mathrm{b}_1 = \mathrm{a}x + \mathrm{b}, \quad \mathrm{c}_1 x + \mathrm{d}_1 = \mathrm{c}x + \mathrm{d};$$

and similarly

$$\alpha_0 x + \beta_0 = x, \quad \gamma_0 x + \delta_0 = 1, \quad \alpha_1 x + \beta_1 = \alpha x + \beta, \quad \gamma_1 x + \delta_1 = \gamma x + \delta.$$

Substituting in the first two equations $\frac{a^2}{c-x}$ in place of x, and in the second two equations $\frac{b^2}{c-x}$ in place of x, we obtain the following results which will be useful:—

$$\mathrm{a}_n a^2 + \mathrm{b}_n(c-x) = a^2(\gamma_n x + \delta_n),$$

$$\mathrm{c}_n a^2 + \mathrm{d}_n(c-x) = \frac{1}{b^2}(\alpha_{n+1}x + \beta_{n+1}),$$

$$\alpha_n b^2 + \beta_n(c-x) = b^2(\mathrm{c}_n x + \mathrm{d}_n),$$

$$\gamma_n b^2 + \delta_n(c-x) = \frac{1}{a^2}(\mathrm{a}_{n+1}x + \mathrm{b}_{n+1}),$$

the last two of which are obtained from the first two by a mere interchange of letters; it will therefore be sufficient to prove the first and second equations.

For the first equation we have

$$\mathrm{a}_n a^2 + \mathrm{b}_n(c-x) = \frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)[\mathrm{a}a^2+\mathrm{b}(c-x)] + (\lambda^n-\lambda)[-\mathrm{d}a^2+\mathrm{b}(c-x)]\},$$

where the term in {.} is

$$=(\lambda^{n+1}-1)[-a^4+a^2c(c-x)]+(\lambda^n-\lambda)[a^2(b^2-c^2)+a^2c(c-x)];$$

viz. this is

$$=a^2\{(\lambda^{n+1}-1)(c^2-a^2-cx)+(\lambda^n-\lambda)(b^2-cx)\};$$

or it is

$$=a^2\{(\lambda^{n+1}-1)(\gamma x+\delta)\qquad+(\lambda^n-\lambda)(\gamma x-\alpha)\},$$

whence the relation in question.

The proof of the second equation is a little more complicated. We have

$$\mathrm{c}_n a^2+\mathrm{d}_n(c-x)=\frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)[ca^2+\mathrm{d}(c-x)]+(\lambda^n-\lambda)[ca^2-\mathrm{a}(c-x)]\},$$

where the term in { } is

$$=(\lambda^{n+1}-1)[-ca^2+(c^2-b^2)(c-x)]+(\lambda^n-\lambda)[-ca^2+a^2(c-x)].$$

Comparing this with

$$\alpha_{n+1}x+\beta_{n+1}=\frac{1}{\lambda^2+1}\left(\frac{\alpha+\delta}{\lambda+1}\right)^n\{(\lambda^{n+2}-1)(\alpha x+\beta)+(\lambda^{n+1}-\lambda)(-\delta x+\beta)\},$$

where the term in { } is

$$=(\lambda^{n+2}-1)[b^2(c-x)]+(\lambda^{n+1}-\lambda)[-c(c^2-a^2-b^2)+(c^2-a^2)(c-x)],$$

it is to be observed that the quotient of the two terms in { } is in fact a constant; this is most easily verified as follows. Dividing the first of them by the second, we have a quotient which when $x=c$ is

$$\frac{(\lambda^{n+1}-1)(-ca^2)+(\lambda^n-\lambda)(-ca^2)}{(\lambda^{n+1}-\lambda)\{-c(c^2-a^2-b^2)\}},\quad=\frac{a^2(\lambda^{n+1}-1+\lambda^n-\lambda)}{(\lambda^{n+1}-\lambda)(c^2-a^2-b^2)},\quad=\frac{a^2(\lambda+1)}{(c^2-a^2-b^2)\lambda},$$

and when $x=0$ is

$$\frac{(\lambda^{n+1}-1)c(c^2-a^2-b^2)}{(\lambda^{n+2}-1)b^2c+(\lambda^{n+1}-\lambda)b^2c},\quad=\frac{(\lambda^{n+1}-1)(c^2-a^2-b^2)}{(\lambda^{n+2}-1+\lambda^{n+1}-\lambda)b^2},\quad=\frac{c^2-a^2-b^2}{b^2(\lambda+1)}:$$

these two values are equal by virtue of the equation which defines λ; and hence the quotient of the two linear functions having equal values for $x=c$ and $x=0$, has always the same value; say it is $=\dfrac{c^2-a^2-b^2}{b^2(\lambda+1)}$. Hence, observing that $\mathrm{a}+\mathrm{d}=\alpha+\delta$, $=c^2-a^2-b^2$, the quotient, $\mathrm{c}_n a^2+\mathrm{d}_n(c-x)$ divided by $\alpha_{n+1}x+\beta_{n+1}$, is

$$=\frac{\lambda+1}{c^2-a^2-b^2}\cdot\frac{c^2-a^2-b^2}{b^2(\lambda+1)},\quad=\frac{1}{b^2};$$

or we have the required equation

$$\mathrm{c}_n a^2+\mathrm{d}_n(c-x)=\frac{1}{b^2}(\alpha_{n+1}x+\beta_{n+1}).$$

Considering now the functional equations, suppose for the moment that g is $=0$; the two equations may be satisfied by assuming

$$\phi(x) = \quad h\left\{\frac{1}{\mathrm{c}_0 x + \mathrm{d}_0} + \frac{\omega}{\mathrm{c}_1 x + \mathrm{d}_1} + \dots\right\} L,$$

$$\Phi(x) = -h\left\{\frac{\omega}{\alpha_1 x + \beta_1} + \frac{\omega^2}{\alpha_2 x + \beta_2} + \dots\right\} M.$$

We in fact, from the foregoing relations, at once obtain

$$\frac{a^2}{c-x}\phi\frac{a^2}{c-x} = \quad h\left\{\frac{\omega}{\alpha_1 x + \beta_1} + \frac{\omega^2}{\alpha_2 x + \beta_2}\dots\right\}\frac{a^2b^2L}{\omega},$$

$$\frac{b^2}{c-x}\Phi\frac{b^2}{c-x} = -h\left\{\frac{\omega}{\mathrm{c}_1 x + \mathrm{d}_1} + \frac{\omega^2}{\mathrm{c}_2 x + \mathrm{d}_2}\dots\right\} M.$$

To satisfy the first equation we must have $M = aL$; viz. this being so, the equation becomes

$$a\phi x + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right) = \frac{aLh}{\mathrm{c}_0 x + \mathrm{d}_0};$$

or, since $\mathrm{c}_0 x + \mathrm{d}_0 = 1$, the equation will be satisfied if only $aL = 1$, whence also $M = 1$. And the second equation will be satisfied if only $\frac{a^2b^2L}{\omega} = bM$; viz. substituting for L, M their value, we find $\omega = ab$.

Supposing, in like manner, that $h = 0$, g retaining its proper value, we find a like solution for the two equations; and by simply adding the solutions thus obtained, we have a solution of the original two equations

$$a\phi(x) + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right) = h,$$

$$\frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right) + b\Phi(x) = g;$$

viz. the solution is

$$\phi(x) = \quad \frac{h}{a}\left\{\frac{1}{\mathrm{c}_0 x + \mathrm{d}_0} + \frac{ab}{\mathrm{c}_1 x + \mathrm{d}_1} + \dots\right\} - g\left\{\frac{ab}{\mathrm{a}_1 x + \mathrm{b}_1} + \frac{(ab)^2}{\mathrm{a}_2 x + \mathrm{b}_2} + \dots\right\}$$

$$\Phi(x) = -h\left\{\frac{ab}{\alpha_1 x + \beta_1} + \frac{(ab)^2}{\alpha_2 x + \beta_2} + \dots\right\} + \frac{g}{b}\left\{\frac{1}{\gamma_0 x + \delta_0} + \frac{ab}{\gamma_1 x + \delta_1} + \dots\right\}.$$

We have a general solution containing an arbitrary constant P by adding to the foregoing values for ϕx a term

$$= \frac{Pb(a-b)}{\sqrt{a^2(c-x) - x(c^2 - b^2 - cx)}},$$

and for Φx a term

$$= \frac{Pa(b-a)}{\sqrt{b^2(c-x) - x(c^2 - a^2 - cx)}},$$

as may be easily verified if we observe that the function

$$a^2(c-x)-x(c^2-b^2-cx),$$

writing therein $\dfrac{a^2}{c-x}$ for x, becomes

$$=\frac{a^2}{(c-x)^2}\{b^2(c-x)-x(c^2-a^2-cx)\}:$$

and similarly that

$$b^2(c-x)-x(c^2-a^2-cx),$$

writing therein $\dfrac{b^2}{c-x}$ for x, becomes

$$=\frac{b^2}{(c-x)^2}\{a^2(c-x)-x(c^2-b^2-cx)\}.$$

More generally, the terms to be added are for ϕx a term as above, where P denotes a function of x which remains unaltered when x is changed into $\dfrac{a^2(c-x)}{c^2-b^2-cx}$, and for Φx a term as above with P' instead of P, where P' denotes what P becomes when x is changed into $\dfrac{a^2}{c-x}$. But these additional terms vanish for the electrical problem, and the correct values of ϕx, Φx are the particular values given above.

It is to be remarked that the function

$$\frac{a^2(c-x)}{c^2-b^2-cx} \text{ is } =\frac{a^2}{c-\dfrac{b^2}{c-x}};$$

viz. considering x as the distance of a point X from A, then taking the image of X in regard to the sphere B, and again the image of this image in regard to the sphere A, the function in question is the distance of this second image from A. And similarly the function

$$\frac{b^2(c-x)}{c^2-a^2-cx} \text{ is } =\frac{b^2}{c-\dfrac{a^2}{c-x}};$$

viz. considering here x as the distance of the point X from B, then taking the image of X in regard to the sphere A, and again the image of this image in regard to the sphere B, the function in question is the distance of this second image from B. It thus appears that Poisson's solution depends upon the successive images of X in regard to the spheres B and A alternately, and also on the successive images of X in regard to the spheres A and B alternately. This method of images is in fact employed in Sir W. Thomson's paper "On the Mutual Attraction or Repulsion between two Electrified Spherical Conductors," *Phil. Mag.*, April and August, 1853.

707.

ON THE COLOURING OF MAPS.

[From the *Proceedings of the Royal Geographical Society*, vol. I., no. 4 (1879), pp. 259—261.]

THE theorem that four colours are sufficient for any map, is mentioned somewhere by the late Professor De Morgan, who refers to it as a theorem known to map-makers. To state the theorem in a precise form, let the term "area" be understood to mean a simply or multiply connected* area: and let two areas, if they touch along a line, be said to be "attached" to each other; but if they touch only at a point or points, let them be said to be "appointed" to each other. For instance, if a circular area be divided by radii into sectors, then each sector is attached to the two contiguous sectors, but it is appointed to the several other sectors. The theorem then is, that if an area be partitioned in any manner into areas, these can be, with four colours only, coloured in such wise that in every case two attached areas have distinct colours; appointed areas may have the same colour. Detached areas may in a map represent parts of the same country, but this relation is not in anywise attended to: the colours of such detached areas will be the same, or different, as the theorem may require.

It is easy to see that four colours are wanted; for instance, we have a circle divided into three sectors, the whole circle forming an *enclave* in another area; then we require three colours for the three sectors, and a fourth colour for the surrounding area: if the circle were divided into four sectors, then for these two colours would

* An area is "connected" when every two points of the area can be joined by a continuous line lying wholly within the area; the area within a non-intersecting closed curve, or say an area having a single boundary, is "simply connected"; but if besides the exterior boundary there is one or more than one interior boundary (that is, if there is within the exterior boundary one or more than one *enclave* not belonging to the area), then the area is "multiply connected." The theorem extends to multiply connected areas, but there is no real loss of generality in taking, and we may for convenience take the areas of the theorem to be each of them a simply connected area.

be sufficient, and taking a third colour for the surrounding area, three colours only would be wanted; and so in general according as the number of sectors is even or odd, three colours or four colours are wanted. And in any tolerably simple case it can be seen that four colours are sufficient. But I have not succeeded in obtaining a general proof: and it is worth while to explain wherein the difficulty consists. Supposing a system of n areas coloured according to the theorem with four colours only, if we add an $(n+1)$th area, it by no means follows that we can *without altering the original colouring* colour this with one of the four colours. For instance, if the original colouring be such that the four colours all present themselves in the exterior boundary of the n areas, and if the new area be an area enclosing the n areas, then there is not any one of the four colours available for the new area.

The theorem, if it is true at all, is true under more stringent conditions. For instance, if in any case the figure includes four or more areas meeting in a point (such as the sectors of a circle), then if (introducing a new area) we place at the point a small circular area, cut out from and attaching itself to each of the original sectorial areas, it must according to the theorem be possible with four colours only to colour the new figure; and this implies that it must be possible to colour the original figure so that only three colours (or it may be two) are used for the sectorial areas. And in precisely the same way (the theorem is in fact really the same) it must be possible to colour the original figure in such wise that only three colours (or it may be two) present themselves in the exterior boundary of the figure.

But now suppose that the theorem *under these more stringent conditions* is true for n areas: say that it is possible with four colours only, to colour the n areas in such wise that not more than three colours present themselves in the external boundary: then it might be easy to prove that the $n+1$ areas could be coloured with four colours only: but this would be insufficient for the purpose of a general proof; it would be necessary to show further that the $n+1$ areas could be with the four colours only coloured *in accordance with the foregoing boundary condition*; for without this we cannot from the case of the $n+1$ areas pass to the next case of $n+2$ areas. And so in general, whatever more stringent conditions we import into the theorem as regards the n areas, it is necessary to show not only that the $n+1$ areas can be coloured with four colours only, but that they can be coloured in accordance with the more stringent conditions. As already mentioned, I have failed to obtain a proof.

708.

NOTE SUR LA THÉORIE DES COURBES DE L'ESPACE.

[From the *Compte Rendu de l'Association Française pour l'Avancement des Sciences* (1880), pp. 135—139.]

En considérant dans l'espace une courbe d'espèce donnée, déterminée au moyen d'un nombre suffisant de points, la courbe n'est pas déterminée uniquement; mais on a par les points un certain nombre de telles courbes. Par exemple, la courbe unicursale d'ordre $2p$ dépend, comme on voit sans peine, de $8p$ constantes et sera ainsi déterminée par $4p$ points (le cas $p=1$ est une exception): on ne connaît pas, je pense, le nombre des courbes par les $4p$ points; mais pour le cas particulier $p=2$ (c'est-à-dire pour une courbe quartique de seconde espèce, ou autrement dit, une courbe excubo-quartique) ce nombre est $=4$: théorème démontré par moi depuis longtemps par des considérations géométriques. (Voir Salmon, *Geometry of three dimensions*, 3^e éd. 1874, p. 319.) Ce n'est que dernièrement que j'ai considéré la question analytique, de trouver les équations d'une courbe excubo-quartique qui passe par 8 points donnés; et même j'ai pris pour les 8 points une disposition qui n'est pas tout à fait générale: l'investigation elle-même, et la forme du résultat, m'ont paru assez intéressantes pour que je les soumette à l'Association.

En considérant sur une courbe excubo-quartique 4 points donnés, le plan passant par 3 quelconques de ces points rencontre la courbe dans un seul point; et l'on obtient ainsi encore 4 points sur la courbe: voilà mon système de 8 points donnés, savoir en partant de 4 points quelconques, je prends un point quelconque dans chacun des plans qui passent par 3 de ces points, et j'obtiens ainsi les autres 4 points. Et par un tel système de 8 points, je cherche à faire passer une courbe de l'espèce dont il s'agit.

En prenant $x=0$, $y=0$, $z=0$, $w=0$, pour les équations des plans du tétraèdre formé par les 4 premiers points, les coordonnées de ces points seront $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$: et pour les coordonnées des 4 autres points, je prends $(0, y_1, z_1, w_1)$, $(x_2, 0, z_2, w_2)$, $(x_3, y_3, 0, w_3)$, $(x_4, y_4, z_4, 0)$.

Les équations de la courbe sont $x : y : z : w = P : Q : R : S$, où P, Q, R, S sont des fonctions $(*)(\theta, 1)^4$ d'un paramètre variable θ; il s'agit de faire passer une telle courbe par les 8 points.

Je prends $\alpha, \beta, \gamma, \delta, a, b, c, d$ pour les valeurs du paramètre θ qui correspondent aux 8 points respectivement.

Pour que la courbe passe par les premiers 4 points, il faut et il suffit que les équations soient de la forme

$$x : y : z : w = A\frac{\theta - a}{\theta - \alpha} : B\frac{\theta - b}{\theta - \beta} : C\frac{\theta - c}{\theta - \gamma} : D\frac{\theta - d}{\theta - \delta};$$

les conditions pour les autres 4 points seront alors

$$\begin{aligned}
\cdot\ y_1 : z_1 : w_1 &= \quad\cdot\quad B\frac{a-b}{a-\beta} : C\frac{a-c}{a-\gamma} : D\frac{a-d}{a-\delta},\\
x_2\ \cdot : z_2 : w_2 &= A\frac{b-a}{b-\alpha} \quad\cdot\quad : C\frac{b-c}{b-\gamma} : D\frac{b-d}{b-\delta},\\
x_3 : y_3\ \cdot : w_3 &= A\frac{c-a}{c-\alpha} : B\frac{c-b}{c-\beta} \quad\cdot\quad : D\frac{c-d}{c-\delta},\\
x_4 : y_4 : z_4\ \cdot &= A\frac{d-a}{d-\alpha} : B\frac{d-b}{d-\beta} : C\frac{d-c}{d-\gamma} \quad\cdot\quad\cdot
\end{aligned}$$

Évidemment il y a deux équations qui donnent la valeur de $B : C$, et qui servent ainsi pour éliminer cette quantité. De cette manière on obtient six équations que j'écris comme voici:

$$\begin{aligned}
\lambda &= \frac{y_1z_4}{y_4z_1} = \frac{a-b\,.\,d-c}{a-c\,.\,d-b}\cdot\frac{a-\gamma\,.\,d-\beta}{a-\beta\,.\,d-\gamma},\\
\mu &= \frac{w_1y_3}{y_1w_3} = \frac{a-d\,.\,c-b}{a-b\,.\,c-d}\cdot\frac{a-\beta\,.\,c-\delta}{a-\delta\,.\,c-\beta},\\
\nu &= \frac{z_1w_2}{z_2w_1} = \frac{a-c\,.\,b-d}{a-d\,.\,b-c}\cdot\frac{a-\delta\,.\,b-\gamma}{a-\gamma\,.\,b-\delta},\\
\varpi &= \frac{z_2x_4}{zx_2} = \frac{b-c\,.\,d-a}{b-a\,.\,d-c}\cdot\frac{b-\alpha\,.\,d-\gamma}{b-\gamma\,.\,d-\alpha},\\
\kappa &= \frac{x_2w_3}{x_3w_2} = \frac{b-a\,.\,c-d}{b-d\,.\,c-a}\cdot\frac{b-\delta\,.\,c-\alpha}{b-\alpha\,.\,c-\delta},\\
\rho &= \frac{x_3y_4}{x_4y_3} = \frac{c-a\,.\,d-b}{c-b\,.\,d-a}\cdot\frac{c-\beta\,.\,d-\alpha}{c-\alpha\,.\,d-\beta};
\end{aligned}$$

savoir $\lambda, \mu, \nu, \varpi, \kappa, \rho$ dénotent ici les quantités données $\lambda = \frac{y_1z_4}{y_4z_1}$, etc. Le nombre des équations indépendantes est 5, car l'on a identiquement $\lambda\mu\nu\varpi\kappa\rho = 1$. Je remarque que l'on peut faire sur le paramètre θ une transformation linéaire quelconque $(h\theta + i) : (j\theta + k)$, et introduire ainsi 3 constantes arbitraires; on peut donc prendre à