417.

ON THE LOCUS OF THE FOCI OF THE CONICS WHICH PASS THROUGH FOUR GIVEN POINTS.

[From the *Philosophical Magazine*, vol. XXXII. (1866), pp. 362—365.]

THE curve which is the locus of the foci of the conics which pass through four given points is, as appears from a general theorem of M. Chasles, a sextic curve having a double point at each of the circular points at infinity; and Professor Sylvester, in his "Supplemental Note on the Analogues in Space to the Cartesian Ovals *in plano*" (*Phil. Mag.*, May 1866), has further remarked that the lines (eight in all) joining the circular points at infinity with any one of the four points are all of them double tangents of the curve; whence each of these points is a focus (more accurately a quadruple focus) of the curve. It is to be added that, besides the circular points at infinity, the curve has 6 double points (3 of these are the centres of the quadrangles formed by the 4 points), in all 8 double points; the class is therefore $=14$. Hence also the number of tangents to the curve from a circular point at infinity is $=10$; viz. these are the 4 double tangents each reckoned twice, and 2 single tangents; and the theoretical number of foci is $=100$; viz. we have

16 quadruple foci, or intersections of a double tangent by a double tangent . .	$16 \times 4 = 64$
16 double foci, or intersections of a double tangent by a single tangent . .	$16 \times 2 = 32$
4 single foci, or intersections of a single tangent by a single tangent . . .	$4 \times 1 = 4$
	100

To verify the foregoing results, consider any two given points I, J, and the series of conics which pass through four given points A, B, C, D; we have thus a curve the locus of the intersections of the tangents from I and the tangents from J to any conic of the series; which curve, if I, J are the circular points at infinity, is the required curve of foci. Taking $U+\lambda V=0$ for the equation of a conic of the series, the pair of tangents from I is given by an equation of the form

$$(\lambda,\ 1)^2(x,\ y,\ z)^2=0,$$

and the pair of tangents from J by an equation of the like form

$$(\lambda,\ 1)^2(x,\ y,\ z)^2=0;$$

and by eliminating λ from these equations, we obtain the equation of the required curve. This in the first instance presents itself as an equation of the eighth order; but it is to be observed that in the series of conics there are two conics each of them touching the line IJ, and that, considering the tangents drawn to either of these conics, the line IJ presents itself as part of the locus; that is, the line IJ twice repeated is part of the locus; and the residual curve is thus of the order $8-2,\ =6$; that is, the required curve is of the order 6. The consideration of the same two conics shows that each of the points I, J is a double point on the curve. Moreover, by taking for the conic any one of the line-pairs through the four points, it appears that each of the points $(AB.CD)$, $(AC.BD)$, $(AD.BC)$ is a double point on the curve: this establishes the existence of five double points. The two conics of the series which touch the line IA are a single conic taken twice, and the consideration of this conic shows that the line IA is a double tangent to the curve; similarly each of the eight lines $I(A,\ B,\ C,\ D)$ and $J(A,\ B,\ C,\ D)$ is a double tangent to the curve. Instead of seeking to establish directly the existence of the remaining three double points, the easier course is to show that, besides the four double tangents from I, the number of tangents from I to the curve is $=2$; for, this being so, the total number of tangents from I to the curve will be $(2\times 4+2=)\ 10$; that is, I being a double point, the class of the curve is $=14$; and assuming that the depression $(6\times 5-14=)\ 16$ in the class of the curve is caused by double points, the number of double points will be $=8$. But observing that in the series of conics there is one conic which passes through J, so that the tangents from J to this conic are the tangent at J twice repeated, then it is easy to see that the tangents from I to this conic, at the points where they meet the tangent at J, *touch* the required curve, and that these two tangents are in fact (besides the double tangents) the only tangents from I to the curve; that is, the number of tangents from I to the curve is $=2$.

Considering I, J as the circular points at infinity, and writing A, B, C, D to denote the squared distances of a point P from the four points A, B, C, D respectively, then, as remarked by Professor Sylvester, the equation

$$\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$$

(where λ, μ, ν, π are constants) is in general a curve of the order 8; but the ratios $\lambda:\mu:\nu:\pi$ may be so determined that the order of the curve in question shall be

$=6$; the resulting curve of the order 6 is (not one of a group of curves, but the very curve) the locus of the foci of the conics through the four points. And the determination of the ratios $\lambda : \mu : \nu : \pi$ is in fact quite simple; for writing

$$A = (x-a)^2 + (y-a_1)^2$$
$$= \rho^2 - 2(ax + a_1 y) + \&c.$$
$$(\text{if } \rho^2 = x^2 + y^2),$$

and therefore

$$\sqrt{A} = \rho - \frac{ax + a_1 y}{\rho} + \&c.,$$

with similar values for $\sqrt{B}$, $\sqrt{C}$, $\sqrt{D}$, it is easy to see that, considering λ, μ, ν, π as standing for $\pm\lambda$, $\pm\mu$, $\pm\nu$, $\pm\pi$ respectively, the conditions for the reduction to the order 6 are

$$\lambda + \mu + \nu + \pi = 0,$$
$$\lambda a + \mu b + \nu c + \pi d = 0,$$
$$\lambda a_1 + \mu b_1 + \nu c_1 + \pi d_1 = 0,$$

and hence that the required equation of the curve of foci is

$$\Sigma \left\{ \begin{vmatrix} 1, & 1, & 1 \\ b, & c, & d \\ b_1, & c_1, & d_1 \end{vmatrix} \sqrt{(x-a)^2 + (y-a_1)^2} \right\} = 0,$$

or, as this may also be written,

$$\Sigma \pm (B, \quad C, \quad D) \sqrt{A} = 0,$$

where (B, C, D), &c. are the areas of the triangles B, C, D, &c.

I remark, in conclusion, that the number of conditions to be satisfied in order that a curve may have for double points two given points I, J, may have besides six double points, and may have for double tangents eight given lines, is $(3+3+6+16=)\,28$; the number of constants contained in the general equation of the order 6 is $=27$. The conditions that a curve of the order 6 shall have for double points two given points I, J, shall besides have six double points, and shall have for double tangents four given lines through I and four given lines through J, are more than sufficient for the determination of the sextic curve; and the existence of a sextic curve satisfying these conditions is therefore a theorem.

In the case where the points I, J lie on a conic of the series, the consideration of this conic shows that the curve has a ninth double point, the pole of the line IJ in regard to the conic in question: in this case the sextic curve, as is known, breaks up into two cubic curves. [It need not do so, for a proper sextic curve may have nine (or indeed ten) double points.]

P.S. In general the curve $\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$ has (exclusively of multiple points at infinity) six double points; viz. these are situate at the intersections of the pairs of circles,

$$(\lambda\sqrt{A}+\mu\sqrt{B}=0,\quad \nu\sqrt{C}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\nu\sqrt{C}=0,\quad \mu\sqrt{B}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\pi\sqrt{D}=0,\quad \mu\sqrt{B}+\nu\sqrt{C}=0).$$

In the case of the curve of foci, the first, second, and third pairs of circles intersect respectively in the points $(AB\,.\,CD)$, $(AC\,.\,BD)$, $(AD\,.\,BC)$, which, as mentioned above, are double points on the curve; and they besides intersect in three other points, which are the other three double points mentioned above.

Professor Sylvester reminds me that he mentioned to me in conversation that he had himself obtained the foregoing equation $\Sigma\pm(B,\,C,\,D)\sqrt{A}=0$, for the locus of the foci of the conics which pass through the four points A, B, C, D.

Cambridge, October 10, 1866.

418.

A REMARK ON DIFFERENTIAL EQUATIONS.

[From the *Philosophical Magazine*, vol. XXXII. (1866), pp. 379—381.]

CONSIDER a differential equation $f(x, y, p) = 0$, of the first order, but of the degree n, where f is a rational and integral function of (x, y, p) not rationally decomposable into factors: the integral equation contains an arbitrary constant c, and represents therefore a system of curves, for any one of which curves the differential equation is satisfied: the differential equation is assumed to be such that the curves are algebraical curves. The curves in question may be considered as undecomposable curves; in fact, if the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ (composed of the undecomposable curves $U=0$, $V=0$, $W=0, \ldots$) satisfies the differential equation, then either the curves $U=0$, $V=0$, $W=0, \ldots$ each satisfy the differential equation, and instead of the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ we have thus the undecomposable curves $U=0$, $V=0$, $W=0, \ldots$ each satisfying the differential equation; or if any of these curves, for instance $W=0$, &c., do not satisfy the differential equation, then W^γ, &c. are mere extraneous factors which may and ought to be rejected, and instead of the original curve $U^\alpha V^\beta W^\gamma \ldots = 0$, we have the undecomposable curves $U=0$, $V=0$ satisfying the differential equation. Assuming, as above, the existence of an algebraical solution, this may always be expressed in the form $\phi(x, y, c) = 0$, where ϕ is a rational and integral function of (x, y, c), of the degree n as regards the arbitrary constant c: this appears by the consideration that for given values (x_0, y_0) of (x, y) the differential equation and the integral equation must each of them give the same number of values of p. It is to be observed that ϕ regarded as a function of (x, y, c) cannot be rationally decomposable into factors; for if the equation were $\phi = \Phi\Psi \ldots = 0$, Φ, Ψ, &c. being each of them rational and integral functions of (x, y, c), then the differential equation would be satisfied by at least one of the equations $\Phi = 0$, $\Psi = 0, \ldots$ that is, by an equation of a degree less than n in the arbitrary constant c.

But the equation $\phi(x, y, c) = 0$ is not of necessity the equation of an undecomposable curve, and the undecomposable curve which constitutes the proper solution of the differential equation cannot always be represented by an equation of the form in question. For although ϕ regarded as a function of (x, y, c) is not rationally decomposable into factors, yet it may very well happen that ϕ regarded as a function of (x, y) is rationally decomposable into factors (geometrically the sections by the planes $z = c$ of the undecomposable surface $\phi(x, y, z) = 0$ may each of them be composed of two or more distinct curves); and assuming that the function ϕ is thus decomposed into its prime factors, then each factor equated to 0 gives an undecomposable curve satisfying the differential equation, and constituting the proper solution thereof.

It may be observed that, by the foregoing process of decomposition, we sometimes reduce the original equation $\phi(x, y, c) = 0$ into a like equation $\phi_1(x, y, c_1) = 0$ of a more simple form. Thus, for instance, if we have $\phi(x, y, c) = U^2 - c = 0$, U being a rational and integral function of (x, y), then instead of $\phi = U^2 - c = 0$ we have the equations $U + \sqrt{c} = 0$, $U - \sqrt{c} = 0$, each of which is an equation of the form $U - c_1 = 0$, or we pass from the original equation $\phi(x, y, c) = U^2 - c = 0$ to the simplified equation

$$\phi_1(x, y, c_1) = U - c_1 = 0.$$

Again, to take a somewhat more complicated instance, if the given integral equation be

$$\phi(x, y, c) = U^4 + c^2V^4 + (c+1)^2 W^2 - 2cU^2V^2 - 2(c+1)U^2W^2 - 2c(c+1)V^2W^2 = 0,$$

then the equation $U + V\sqrt{c} + W\sqrt{c+1} = 0$, writing therein $\sqrt{c} = \dfrac{2c_1}{c_1^2 - 1}$, and therefore $\sqrt{c+1} = \dfrac{c_1^2+1}{c_1^2-1}$, becomes

$$U(c_1^2 - 1) + V.2c_1 + W(c_1^2 + 1) = 0;$$

so that we pass from the original equation $\phi(x, y, c) = 0$ to the simplified equation

$$\phi_1(x, y, c_1) = U(c_1^2 - 1) + V.2c_1 + W(c_1^2 + 1) = 0.$$

But observe that the possibility of the rationalization depends on the form of the radicals $\sqrt{c}$ and $\sqrt{c+1}$; if we had had $\sqrt{c}$ and $\sqrt{c^2+1}$ (or c and $\sqrt{c^4+1}$), the rationalization could not have been effected.

Returning to the case of an integral equation $\phi(x, y, c) = 0$, where ϕ regarded as a function of (x, y) is decomposable into factors, then equating to zero any one of the prime factors of ϕ, we obtain an integral equation $\psi(x, y, c_1, c_2, \ldots c_k) = 0$, where $c_1, c_2 \ldots c_k$ are irrational functions (not of necessity representable by radicals, and without any superior limit to the number of these functions) of c: here ψ regarded as a function of (x, y) is of course undecomposable, and the equation $\psi(x, y, c_1, c_2, \ldots c_k) = 0$ belongs to the undecomposable curve which is the proper solution of the differential equation. The result may be stated under a quasi-geometrical form; viz. regarding $c_1, c_2, \ldots c_k$ as the coordinates of a point in k-dimensional space, then as these are functions of the single parameter c, the point to which they belong is an arbitrary point on a certain curve or $(k-1)$fold locus C in the k-dimensional space. And this curve must be such that to given values of (x, y) there shall correspond n points on the curve; that is, treating (x, y) as constants, the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)=0$, and the curve or $(k-1)$fold locus C, shall meet in n points. The conclusion stated in the foregoing quasi-geometrical form is, that the solution of the differential equation may be exhibited in the form $\psi(x, y, c_1, c_2 \ldots c_k)=0$; viz. ψ is a rational and integral function of $(x, y, c_1, c_2 \ldots c_k)$, where $(c_1, c_2 \ldots c_k)$ are the coordinates of an arbitrary or variable point on a curve or $(k-1)$fold locus C in a k-dimensional space, which curve meets the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)$ in n points, and where ψ regarded as a function of (x, y) is not rationally decomposable into factors.

Cambridge, October 13, 1866.

419.

A THEOREM ON DIFFERENTIAL OPERATORS.

[From a *paper by* PROF. SYLVESTER, "*Note on the Test Operators which occur in the Calculus of Invariants, &c.,*" *Philosophical Magazine,* vol. XXXII. (1866), pp. 461—472, see p. 471.]

THE paper concludes with an Observation from Professor Cayley as follows:

"In the case of two variables, if

$$P_1 = (ax + by)\frac{d}{dx} + (cx + dy)\frac{d}{dy},$$

then in the notation of matrices,

$$P_1 = \begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix} (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_2 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^2 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_3 = \tfrac{1}{6}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right);$$

whence also

$$P * P_2 = P_2 * P_1 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right) = 3P_3,$$

which accords with your theorem,

$$E_1 * E_2 * = E_2 * E_1 * = E_1E_2 * + 3E_3 *."$$

I have taken the liberty of writing in the above $\frac{d}{dx}, \frac{d}{dy}$ for δ_x, δ_y, and P for δ in the original. It will be useful to bear in mind that in any operator such as $E_1 *$ or $E_2 *$, *the asterisk forms an integral part* of the symbol. Thus $E_1 * E_2 *$, if we choose, may be written under the form of $E_1 *$ multiplied by $E_2 *$, i.e. $(E_1 *) \times (E_2 *)$, where the cross is the sign of ordinary algebraical multiplication.

420.

ON RICCATI'S EQUATION.

[From the *Philosophical Magazine*, vol. XXXVI. (1868), pp. 348—351.]

THE following is, it appears to me, the proper form in which to present the solution of Riccati's equation.

The equation may be written

$$\frac{dy}{dx} + y^2 = x^{2q-2},$$

which is integrable by algebraic and exponential functions if $(2i+1)q = \pm 1$, i being zero, or a positive integer. To effect the integration, writing $y = \frac{1}{u}\frac{du}{dx}$, we have

$$\frac{d^2u}{dx^2} = x^{2q-2}u.$$

The peculiar advantage of this well-known transformation has not (so far as I am aware) been explicitly stated; it puts in evidence the form under which the sought-for function y contains the constant of integration. In fact if $u = P$, $u = Q$ be two particular solutions of the equation in u, then the general solution is $u = CP + DQ$; and denoting by P', Q' the derived functions, the value of y is

$$y = \frac{CP' + DQ'}{CP + DQ},$$

showing the form under which the constant of integration $C \div D$ is contained in y. To complete the solution, assume

$$u = ze^{\frac{1}{q}x^q};$$

we find

$$\frac{d^2z}{dx^2} + 2x^{q-1}\frac{dz}{dx} + (q-1)\,x^{q-2}\,z = 0:$$

considering first the particular integral of the form

$$z = A + Bx^q + Cx^{2q} + Dx^{3q} + \&c.,$$

we find that the equation will be satisfied if

$$\begin{aligned}
(\ q-1)\,A + \ q\,(\ q-1)\,B &= 0,\\
(3q-1)\,B + 2q\,(2q-1)\,C &= 0,\\
(5q-1)\,C + 3q\,(3q-1)\,D &= 0,\\
(7q-1)\,D + 4q\,(4q-1)\,E &= 0,\\
\&c.&
\end{aligned}$$

If $A = 1$, this is

$$\begin{aligned}
A &= \ 1,\\
B &= -\frac{q-1}{q\,(q-1)},\\
C &= +\frac{(q-1)(3q-1)}{q\,(q-1)\,2q\,(2q-1)},\\
D &= -\frac{(q-1)(3q-1)(5q-1)}{q\,(q-1)\,2q\,(2q-1)\,3q\,(3q-1)},\\
&\&c.,
\end{aligned}$$

where it is to be noticed that the series may be considered to stop so soon as there is in the numerator a factor $= 0$. For instance, if $5q-1=0$, then if the particular integral had been assumed to be $z = A + Bx^q + Cx^{2q}$, the only conditions to be satisfied by the coefficients are the first and second equations giving the foregoing values of A, B, C. It is immaterial that the analytical expressions of F and the subsequent coefficients contain in the denominators the evanescent factor $5q-1$; the coefficients after C do not ever come into consideration.

Thus if $(2i+1)\,q = +1$, the series terminates, and we have for u the finite particular solution

$$u = P = \left(1 - \frac{q-1}{q\,(q-1)}\,x^q + \frac{(q-1)(3q-1)}{q\,(q-1)\,2q\,(2q-1)}\,x^{2q} - \&c.\right) e^{\frac{1}{q}x^q}:$$

and it is easy to see that we may herein change the sign of x^q, thereby obtaining another finite particular solution,

$$u = Q = \left(1 + \frac{q-1}{q\,(q-1)}\,x^q + \frac{(q-1)(3q-1)}{q\,(q-1)\,2q\,(2q-1)}\,x^{2q} + \&c.\right) e^{-\frac{1}{q}x^q}.$$